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MEAN SQUARE STABILITY FOR DISCRETE BOUNDED LINEAR SYSTEMS IN HILBERT SPACE*

C. S. KUBRUSLY†

Abstract. The asymptotic behaviour for infinite-dimensional discrete linear systems driven by white noise is considered in this paper. Both the evolution and convergence of the state correlation operators sequence are analysed. Mean square stability conditions are investigated, including a comparison with the deterministic stability problem. The particular case of compact operators is considered in some detail.

Key words. asymptotic stability, infinite-dimensional systems, linear dynamical systems, discrete-time systems, stochastic systems

1. Introduction. Conditions for asymptotic stability of finite-dimensional discrete linear system operating either in a deterministic or stochastic environment are by now well established (cf. [10], [9]). On the other hand the same problem in an infinite-dimensional setting, which is endowed with a much richer structure, still presents some unsolved questions.

As far as the asymptotic stability problem for infinite-dimensional discrete linear deterministic systems is concerned, there is available in the current literature a fairly complete collection of results (cf. § 3). This does not seem to be the case for discrete stochastic systems, although some few results have already been investigated by using different approaches and under different motivations. For instance, the convergence analysis of stochastic approximation algorithms in Hilbert space considered in [13] and [8] actually gives asymptotic stability conditions for infinite-dimensional dynamical systems. Questions related to optimal stochastic control problems have also motivated some partial results in this direction (cf. [6], [16] and [17]).

In this paper we consider the mean square stability problem, by analysing both the evolution and asymptotic behaviour of state correlation operators, for discrete linear systems in Hilbert space. The paper is organized as follows. Notational preliminaries and basic concepts, which will be needed along the text, are considered in § 2. These comprise bounded linear transformations, positive and nuclear operators, correlation operators, and approximate controllability. A brief review on asymptotic stability for deterministic discrete systems is presented in § 3, including the auxiliary results which will be used in the sequel. The central theme of the paper appears in § 4. There it is analysed the evolution and convergence of the state correlation sequence $\{Q_i; i \geqq 0\}$ for discrete linear systems driven by white noise. The main results (cf. Lemma 2, Theorems 1, 2 and Corollary 1) deal with the relationship between convergence of $\{Q_i; i \geqq 0\}$ and the spectral radius $r_\sigma(A)$ of the system operator A. It is shown that $r_\sigma(A) < 1$ (i.e. uniform asymptotic stability for the free system) is sufficient to ensure uniform convergence of $\{Q_i; i \geqq 0\}$ to a correlation operator (i.e. mean square stability for the disturbed system). Necessary and sufficient conditions for uniform convergence of $\{Q_i; i \geqq 0\}$ to a positive correlation operator are also given, for the case of a compact system operator A.

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† Department of Research and Development, Scientific Computation Laboratory—LCC/CNPq, R. Lauro Müller 455, Rio de Janeiro, RJ, 22290, Brazil and Department of Electrical Engineering, Catholic University—PUC/RJ, R. Marquês de S. Vicente 209, Rio de Janeiro, RJ, 22453, Brazil.

2. Notational and conceptual preliminaries. In this section we pose the notation and some basic concepts which will be used in the sequel. Throughout this paper we assume that U and H are separable nontrivial Hilbert spaces. $\langle\ ;\ \rangle$ and $\|\ \ \|$ will stand for inner product and norm, respectively.

Bounded linear transformations. Let X and Y be Banach spaces. $\mathscr{B}[X, Y]$ will denote the Banach space of all bounded linear transformations of X into Y. For notational simplicity we write $\mathscr{B}[X]$ for $\mathscr{B}[X, X]$. $\mathscr{N}(T)$ and $\mathscr{R}(T)$ will stand for the null space and range space of $T \in \mathscr{B}[X, Y]$, respectively. The spectrum of $T \in \mathscr{B}[X]$ will be denoted by $\sigma(T)$. $P\sigma(T) \subset \sigma(T)$ will denote the point spectrum (i.e. the set of all eigenvalues) of $T \in \mathscr{B}[X]$. $r_\sigma(T) = \sup\{|\lambda|: \lambda \in \sigma(T)\}$ is the spectral radius of $T \in \mathscr{B}[X]$. $T^* \in \mathscr{B}[H, U]$ is the adjoint of $T \in \mathscr{B}[U, H]$. We shall write $T_i \to^w T$, $T_i \to^s T$, or $T_i \to^u T$ if a sequence $\{T_i; i \geqq 0\}$ of operators in $\mathscr{B}[H]$ converges weakly, strongly, or uniformly to $T \in \mathscr{B}[H]$ as $i \to \infty$, respectively.

Positive and nuclear operators. A self-adjoint operator $T = T^* \in \mathscr{B}[H]$ will be called nonnegative ($T \geqq 0$), positive ($T > 0$), or strictly positive ($T \succ 0$) according to the following standard definitions:

$$T \geqq 0 \iff \langle Tx; x\rangle \geqq 0 \quad \forall x \in H,$$

$$T > 0 \iff \langle Tx; x\rangle > 0 \quad \forall x \neq 0 \in H,$$

$$T \succ 0 \iff \exists \gamma > 0 \text{ such that } \langle Tx; x\rangle \geqq \gamma\|x\|^2 \quad \forall x \in H.$$

If $T \geqq 0 \in \mathscr{B}[H]$ ($T > 0$, $T \succ 0$), then there exists a unique $T^{1/2} \geqq 0 \in \mathscr{B}[H]$ ($T^{1/2} > 0$, $T^{1/2} \succ 0$) such that $(T^{1/2})^2 = T$. For $T \geqq 0 \in \mathscr{B}[H]$ we define the trace of T as usual:

$$\operatorname{tr}(T) \overset{\text{def.}}{=} \sum_k \langle Te_k; e_k\rangle = \sum_k \lambda_k,$$

where $\{e_k; k \geqq 1\}$ is any orthonormal basis for H, and $\{\lambda_k \geqq 0; k \geqq 1\}$ is the set of all $\lambda \in P\sigma(T)$, each of them counted according to its multiplicity. $T \geqq 0 \in \mathscr{B}[H]$ is nuclear (or trace-class) if $\operatorname{tr}(T) < \infty$. Let $\mathscr{B}_1[H]$ denote the class of all nuclear operators on H, and recall that $\mathscr{B}_1[H] \subset \mathscr{B}_\infty[H] \subset \mathscr{B}[H]$, where $\mathscr{B}_\infty[X]$ denotes the class of all compact linear operators on a Banach space X. The following well-known result will be needed in the sequel.

Remark 1. If $T \in \mathscr{B}[H]$ has a bounded inverse (in particular, is strictly positive) and it is compact (in particular, nuclear), then H is necessarily finite-dimensional.

Correlation operators. For arbitrary $x, y \in H$ define the operator $x \circ y \in \mathscr{B}[H]$ as follows [3]:

$$(x \circ y)z = x\langle z; y\rangle,$$

for every $z \in H$. Now let u and v be H-valued second order random variables,[1] and define the following sesquilinear form:

$$E\{\langle (u \circ v)x; y\rangle\} \overset{\text{def.}}{=} E\{\langle x; v\rangle\langle u; y\rangle\}$$

[1] Let $(\Omega, \mathscr{A}, p)$ be a probability space where $\mathscr{A}$ is a σ-algebra of subsets of a nonempty basic set Ω, and p is a probability measure defined on $\mathscr{A}$. An H-valued second order random variable is a p-measurable map $u: \Omega \to H$ such that

$$E\{\|u\|^2\} = \int_\Omega \|u(\omega)\|^2\, dp < \infty$$

(i.e. $u \in L_2(\Omega, p; H)$). Here E denotes the expectation operator. An H-valued second order random sequence $\{u_i; i \geqq 0\}$ is a family of H-valued second order random variables. For an introduction to the theory of H-valued random variables see, for instance, [1].

on $H \times H$, which is bounded. The symbol E on the right hand side denotes expectation in the usual way. Then (cf. [14, p. 120]) there exists a unique operator in $\mathcal{B}[H]$, say $E\{u \circ v\}$, defined by

$$\langle E\{u \circ v\}x; y\rangle = E\{\langle (u \circ v)x; y\rangle\}$$

for every $x, y \in H$. We call $E\{u \circ v\} \in \mathcal{B}[H]$ the correlation of u and v.

Remark 2. The following auxiliary results are readily verified.

$$E\{u \circ v\} = E\{v \circ u\}^*,$$

$$E\{(u+v)\circ(u+v)\} = E\{u \circ u\} + E\{v \circ v\} + E\{u \circ v\} + E\{v \circ u\},$$

$$E\{Au \circ Bv\} = A\, E\{u \circ v\}B^* \quad \forall\, A, B \in \mathcal{B}[H].$$

Moreover, the correlation of u is self-adjoint nonnegative and nuclear; that is

$$0 \leqq E\{u \circ u\} = E\{u \circ u\}^* \in \mathcal{B}_1[H],$$

since

$$\langle E\{u \circ u\}x; x\rangle = E\{|\langle x; u\rangle|^2\} \quad \forall x \in H,$$

$$\operatorname{tr}(E\{u \circ u\}) = E\{\|u\|^2\}.$$

H-valued second order random variables u and v are said to be uncorrelated if $E\{u \circ v\} = E\{u\} \circ E\{v\}$. An H-valued second order random sequence $\{u_i; i \geqq 0\}$ is wide sense stationary if $E\{u_i \circ u_j\}$ depends only on the difference $i-j$ for all $i, j \geqq 0$. It is a white noise if $E\{u_i \circ u_j\} = 0$ for all $i \neq j$.

Approximate controllability. A pair of operators $A \in \mathcal{B}[H]$ and $B \in \mathcal{B}[U, H]$ is approximate controllable [2], briefly (A, B) is A-C (also called weakly reachable [4]), if

$$\bigcap_{j=0}^{\infty} \mathcal{N}(B^* A^{*j}) = \{0\}.$$

We shall be particularly interested in the approximate controllability for the pair $(A, BR^{1/2})$, for some $R = R^* \geqq 0 \in \mathcal{B}[U]$[2]. Notice that

$$(A, BR^{1/2}) \text{ is A-C} \quad \Rightarrow \quad (A, B) \text{ is A-C},$$

since $\mathcal{N}(B^* A^{*j}) \subset \mathcal{N}(R^{1/2} B^* A^{*j})$, and

$$R > 0 \text{ and } (A, B) \text{ is A-C} \quad \Rightarrow \quad (A, BR^{1/2}) \text{ is A-C},$$

since $R > 0 \Rightarrow \mathcal{N}(R^{1/2}) = \{0\} \Rightarrow \mathcal{N}(R^{1/2} B^* A^{*j}) \subset \mathcal{N}(B^* A^{*j})$. Also notice that the reverses of the above statements are not generally true.

3. Deterministic asymptotic stability. Asymptotic stability for discrete deterministic infinite-dimensional linear systems has been investigated by several authors (e.g. see [5], [15], [7], [12]). In this section we present some basic concepts and auxiliary results which will be used in § 4.

DEFINITION 1. Let X be a Banach space, $A \in \mathcal{B}[X]$, and define an X-valued sequence $\{x_i; i \geqq 0\}$ as follows:

(1) $$x_{i+1} = Ax_i, \qquad x_0 \in X.$$

[2] If R is thought of as a correlation operator for an input disturbance sequence, then approximate controllability for the pair $(A, BR^{1/2})$ is sometimes termed stochastic approximate controllability.

The free linear system given in (1) (or equivalently, the operator $A \in \mathcal{B}[X]$) is:

(a) *uniformly asymptotically stable* if $A^i \to^u 0$. That is,

$$\|A^i\| \to 0 \quad \text{as } i \to \infty.$$

(b) *strongly asymptotically stable* if $A^i \to^s 0$. That is,

$$\|A^i x\| \to 0 \quad \text{as } i \to \infty \quad \forall x \in X.$$

Remark 3. By the Banach–Steinhaus theorem [14] it is immediate to verify that

$$\sup_i \|A^i x\| < \infty \quad \forall x \in X \quad \Rightarrow \quad r_\sigma(A) \leqq 1,$$

since $r_\sigma(A)^j = r_\sigma(A^j) \leqq \|A^j\| \leqq \sup_i \|A^i\| < \infty$, $\forall j \geqq 0$ (the reverse is clearly not true, for take any operator $A \in \mathcal{B}[\mathbb{R}^2]$ such that $r_\sigma(A) = 1$ and $\|A^i\| \to \infty$ as $i \to \infty$). Moreover it is also readily verified by contradiction that

$$A^i \overset{s}{\to} 0 \quad \Rightarrow \quad P\sigma(A) \subset \{\lambda \in \mathbb{C}: |\lambda| < 1\}.$$

However even the combined reverse is not true, that is

$$r_\sigma(A) \leqq 1 \text{ and } P\sigma(A) \subset \{\lambda \in \mathbb{C}: |\lambda| < 1\} \quad \not\Rightarrow \quad A^i \overset{s}{\to} 0,$$

since by setting $X = l_2$ and letting $A \in \mathcal{B}[l_2]$ be the right shift operator (i.e. $A(\xi_1, \xi_2, \cdots) = (0, \xi_1, \xi_2, \cdots)$ for all $x = (\xi_1, \xi_2, \cdots) \in l_2$), it follows [11] that $r_\sigma(A) = 1$, $P\sigma(A) = \emptyset$, but $\|A^i x\| = \|x\|$ $\forall i \geqq 0$, for an arbitrary $x \in l_2$.

On the other hand there are several equivalent ways of stating uniform asymptotic stability.

LEMMA 1. *Let X be a complex*[3] *Banach space and $A \in \mathcal{B}[X]$. The following properties are equivalent*:

(a) $\|A^i\| \to 0$ as $i \to \infty$.

(b) $r_\sigma(A) < 1$.

(c) *There exist real constants* $\gamma \geqq 1$ *and* $\rho \in (0, 1)$, *such that*

$$\|A^i\| \leqq \gamma \rho^i \quad \forall\, i \geqq 0.$$

(d) $\|A^{i_0}\| < 1$ *for some* $i_0 \geqq 0$.

(e) $\sum_{i=0}^{\infty} \|A^i\|^k < \infty$ *for any* $k > 0$.

(f) $\sum_{i=0}^{\infty} \|A^i\|^{k_0} < \infty$ *for some* $k_0 > 0$.

(g) $\sum_{i=0}^{\infty} \|A^i x\|^k < \infty$, $\forall x \in X$, *for any* $k \geqq 1$.

(h) $\sum_{i=0}^{\infty} \|A^i x\|^{k_0} < \infty$, $\forall \in X$, *for some* $k_0 \geqq 1$.

Proof. It is trivially verified that (c)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(a). Since $r_\sigma(A)^i = r_\sigma(A^i) \leqq \|A^i\|$, $\forall i \geqq 0$, one gets (a)$\Rightarrow$(b). By the well-known Gelfand formula, $\|A^i\|^{1/i} \to r_\sigma(A)$ as $i \to \infty$, and by the radical test for infinite series, it follows that (b) $\Rightarrow \|A^i\| < \rho^i$, $\forall i \geqq i_0$,

[3] For a real Banach space X the lemma still holds if $r_\sigma(A)$ is changed to $r_\sigma(A^+)$, where $A^+ \in \mathcal{B}[X^+]$ is defined by $A^+(x + \sqrt{-1}y) = Ax + \sqrt{-1}Ay$, for all $x, y \in X$, with the complex Banach space X^+ denoting the complexification of X (cf. [14]). Notice that $\|A^{+i}\|_{\mathcal{B}[X^+]} = \|A^i\|_{\mathcal{B}[X]}$, $\forall i \geqq 0$.

for some integer $i_0 \geqq 0$ and any $\rho \in (r_\sigma(A), 1)$; which implies (c) with $\gamma = \max\{\|A^j\|: 0 \leqq j \leqq i_0\}\rho^{-i_0} \geqq 1$. Since $\|A^i x\|^k \leqq \|A^i\|^k \|x\|^k$, for all $x \in X$, it is immediate to verify that (e) $\Rightarrow$(g). That (g)$\Rightarrow$(h) is trivial. It has been proved in [15] that (h)$\Rightarrow$(b). Finally it is clear that (c)$\Rightarrow$(d), and (d)$\Rightarrow$(a) since $\|A^{ji_0}\| \leqq \|A^{i_0}\|^j, \forall j \geqq 0$. □

Remark 4. Obviously uniform asymptotic stability implies strong asymptotic stability. The fundamental difference between finite- and infinite-dimensional formulations relies upon the reverse of the above statement, which is not generally true for infinite-dimensional spaces. For instance, set $X = l_2$ and let $A \in \mathcal{B}[l_2]$ be the left shift operator (i.e. $A(\xi_1, \xi_2, \cdots) = (\xi_2, \xi_3, \cdots)$ for all $x = (\xi_1, \xi_2, \cdots) \in l_2$). It is easy to show that $\|A^i x\| \to 0$ as $i \to \infty$ for all $x \in l_2$, but $\|A^i\| = 1$ $\forall i \geqq 0$. However, if $A \in \mathcal{B}_\infty[X]$ (in particular, if $\dim(X) < \infty$), then strong and uniform asymptotic stability are equivalent concepts. Indeed, for $A \in \mathcal{B}_\infty[X]$, $\sigma(A) - \{0\} = P\sigma(A) - \{0\}$. Hence, if $A \in \mathcal{B}_\infty[X]$ is strongly asymptotically stable, then the compact set $\sigma(A)$ is contained in the unit open ball, according to Remark 3, and so $r_\sigma(A) < 1$.

4. State correlation evolution and mean square stability. Consider a discrete linear dynamical system evolving in a stochastic environment, and modelled by the following autonomous difference equation.

$$x_{i+1} = Ax_i + Bu_{i+1}, \qquad x_0 = Bu_0, \tag{2}$$

where $A \in \mathcal{B}[H]$ and $B \in \mathcal{B}[U, H]$. Here $\{x_i; i \geqq 0\}$ denotes an H-valued state sequence such that x_0 is an $\mathcal{R}(B) \subset H$-valued second order random variable. The input disturbance sequence $\{u_i; i \geqq 0\}$ is assumed to be an U-valued second order wide sense stationary white noise, with correlation operator

$$R = R^* = E\{u_i \circ u_i\} \geqq 0 \in \mathcal{B}_1[U] \quad \forall i \geqq 0.$$

Now define the following self-adjoint nonnegative operator.

$$Q_i = Q_i^* = \sum_{j=0}^{i} A^j Q_0 A^{*j} \geqq 0 \in \mathcal{B}_1[H], \qquad Q_0 = BRB^*,$$

for every $i \geqq 0$. Notice that Q_i is actually nuclear since R is nuclear, A and B are bounded, and $\mathcal{B}_1[H]$ is a two-sided ideal of $\mathcal{B}[H]$ (cf. [14, p. 173]). On iterating (2) from x_0 onwards, and using Remark 2, it is a simple matter to show that Q_i is the state correlation operator; that is,

$$Q_i = E\{x_i \circ x_i\} \quad \forall i \geqq 0,$$

which has the following further properties.

PROPOSITION 1.

(a) $Q_j = A^{i+1} Q_{j-i-1} A^{*i+1} + Q_i \quad \forall j > i \geqq 0.$

In particular,

$$Q_{i+1} = AQ_iA^* + Q_0 = A^{i+1}Q_0A^{*i+1} + Q_i \quad \forall i \geqq 0.$$

Therefore, for every $i \geqq 0$,

(b) $Q_i \leqq Q_{i+1}$,

thus $\mathrm{tr}(Q_i) \leqq \mathrm{tr}(Q_{i+1})$ *and* $\|Q_i\| \leqq \|Q_{i+1}\|$. *Moreover,*

(c) $Q_i > 0 \quad \Leftrightarrow \quad \bigcap_{j=0}^{i} \mathcal{N}(R^{1/2}B^*A^{*j}) = \{0\}.$

Proof. Let i, j be any integers such that $j > i \geqq 0$. Then

$$Q_j = \sum_{l=0}^{i} A^l Q_0 A^{*l} + \sum_{l=i+1}^{j} A^l Q_0 A^{*l} = Q_i + \sum_{l=0}^{j-i-1} A^{l+i+1} Q_0 A^{*l+i+1},$$

thus following the result in (a). The particular cases are trivially obtained by setting $i = 0$ and $j = i + 1$, respectively. The result in (b) is then readily verified since

$$\langle Q_i x; x \rangle = \sum_{j=0}^{i} \|R^{1/2} B^* A^{*j} x\|^2 \quad \forall x \in H.$$

Therefore $\{\operatorname{tr}(Q_i)\}$ and $\{\|Q_i\|\}$ are nondecreasing sequences. Since $Q_i \geqq 0$ one gets $Q_i > 0 \Leftrightarrow \{\langle Q_i x; x \rangle = 0 \Rightarrow x = 0\}$. But

$$\langle Q_i x; x \rangle = 0 \quad \Leftrightarrow \quad x \in \bigcap_{j=0}^{i} \mathcal{N}(R^{1/2} B^* A^{*j}),$$

thus following the result in (c). □

We shall be particularly interested in the asymptotic behaviour of the sequence $\{Q_i; i \geqq 0\}$.

LEMMA 2. (a) *If* $Q_i \to^w Q \in \mathcal{B}[H]$, *then* $Q_i \to^s Q$, *and the limit has the following properties*: $0 \leqq Q_i \leqq Q = Q^*$, $\|Q_i\| \nearrow \|Q\|$, *and*

$$Q = A^{i+1} Q A^{*i+1} + Q_i \quad \forall i \geqq 0.$$

Moreover,

$$(A, BR^{1/2}) \text{ is A-C} \quad \Leftrightarrow \quad Q > 0 \quad \Rightarrow \quad P\sigma(A^*) \subset \{\lambda \in \mathbb{C}: |\lambda| < 1\}.$$

(b) *If* $Q_i \to^w Q \in \mathcal{B}_1[H]$, *then* $\operatorname{tr}(Q_i) \nearrow \operatorname{tr}(Q)$, *and* $Q_i \to^u Q$.

Proof. If $Q_i \to^w Q \in \mathcal{B}[H]$, then by the Banach–Steinhaus theorem $\{Q_i\}$ is uniformly bounded (cf. [14, p. 78]). Therefore since $\{Q_i\}$ is a nondecreasing sequence (according to Proposition 1(b)) of self-adjoint operators, it follows that $Q_i \to^s Q$, and $Q = Q^*$ (cf. [14, p. 79]). Actually $0 \leqq Q_i \leqq Q$ for every $i \geqq 0$, since

$$\langle Q_i x; x \rangle = \sum_{j=0}^{i} \|R^{1/2} B^* A^{*j} x\|^2 \leqq \sum_{j=0}^{\infty} \|R^{1/2} B^* A^{*j} x\|^2 = \langle Qx; x \rangle$$

for all $x \in H$. Thus $\|Q_i\| \leqq \|Q\|$. Hence the nondecreasing sequence $\{\|Q_i\|\}$ converges, and $\|Q\| = \sup_{\|x\|=1} \lim_{i\to\infty} \langle Q_i x; x \rangle \leqq \lim_{i\to\infty} \|Q_i\|$. Then $\|Q_i\| \nearrow \|Q\|$. By Proposition 1(a) it follows that

$$Q_j - (A^{i+1} Q A^{*i+1} + Q_i) = A^{i+1}(Q_{j-i-1} - Q) A^{*i+1}$$

for every $j > i \geqq 0$. Therefore, since $Q_j \to^s Q$,

$$\|[Q_j - (A^{i+1} Q A^{*i+1} + Q_i)]x\| \leqq \|A^{i+1}\| \, \|(Q_{j-i-1} - Q) A^{*i+1} x\| \to 0$$

as $j \to \infty$, for all $x \in H$ and every $i \geqq 0$. Then, by uniqueness of the strong limit, $Q = A^{i+1} Q A^{*i+1} + Q_i$, $\forall i \geqq 0$. Moreover,

$$\langle Qx; x \rangle = 0 \quad \Leftrightarrow \quad x \in \bigcap_{j=0}^{\infty} \mathcal{N}(R^{1/2} B^* A^{*j}).$$

So, recalling that $Q \geqq 0$, one has

$$Q > 0 \quad \Leftrightarrow \quad \{\langle Qx; x \rangle = 0 \Rightarrow x = 0\} \quad \Leftrightarrow \bigcap_{j=0}^{\infty} \mathcal{N}(R^{1/2} B^* A^{*j}) = \{0\}.$$

Finally take any $\lambda \in P\sigma(A^*)$ (if $P\sigma(A^*)=\emptyset$ the result is trivial), and let $x\neq 0$ be an eigenvalue associated to λ. Then

$$\langle (Q-Q_{i-1})x; x\rangle = \langle A^i Q A^{*i} x; x\rangle = |\lambda|^{2i}\langle Qx; x\rangle \quad \forall i \geqq 1.$$

Hence $|\lambda|<1$ whenever $Q_i \to^w Q>0$, which completes the proof of part (a). Now assume that $Q\in\mathscr{B}_1[H]$. Then $\operatorname{tr}(Q_i)\leqq \operatorname{tr}(Q)$, since $Q_i\leqq Q$, and the nondecreasing sequence $\{\operatorname{tr}(Q_i)\}$ converges. Thus, for any orthonormal basis $\{e_k\}$ and for every $n\geqq 1$,

$$\operatorname{tr}(Q)=\lim_{i\to\infty}\sum_{k=1}^{n}\langle Q_i e_k; e_k\rangle + \sum_{k=n+1}^{\infty}\langle Qe_k; e_k\rangle$$

$$\leqq \lim_{i\to\infty}\operatorname{tr}(Q_i) + \sum_{k=n+1}^{\infty}\langle Qe_k; e_k\rangle \searrow \lim_{i\to\infty}\operatorname{tr}(Q_i) \quad \text{as } n\to\infty.$$

Then $\operatorname{tr}(Q_i)\nearrow \operatorname{tr}(Q)$. Therefore

$$\|Q-Q_i\| \leqq \operatorname{tr}(Q-Q_i) = \operatorname{tr}(Q)-\operatorname{tr}(Q_i)\to 0 \quad \text{as } i\to\infty. \qquad \square$$

Remark 5. Concerning the final statement of Lemma 2(a) it is worth mentioning that positivity (which is sufficient) is not necessary, but nonnegativity is not sufficient (i.e. $Q\geqq 0 \not\Rightarrow P\sigma(A^*)\subset\{\lambda\in\mathbb{C}: |\lambda|<1\} \not\Rightarrow Q>0$). It is also easy to show that $\mathcal{N}(Q)\subset \mathcal{N}(Q_{i+1})\subset\mathcal{N}(Q_i)$, $\forall i\geqq 0$.

We shall say that the linear system in (2) is mean square stable if the state correlation sequence $\{Q_i; i\geqq 0\}$ converges to a correlation operator Q (i.e. $E\{x_i\circ x_i\}\to E\{x\circ x\}$ as $i\to\infty$ for some second order H-valued random variable x), such that the Lyapunov equation $Q=AQA^*+Q_0$ in Lemma 2(a) has a solution $Q\geqq 0\in\mathscr{B}_1[H]$. However, by Lemma 2(b), the above convergence has to be uniform. So we define as follows.

DEFINITION 2. The linear system in (2) is *mean square stable* if

$$Q_i \overset{u}{\to} Q\in\mathscr{B}_1[H].$$

We now investigate the connection between mean square and uniform asymptotic stability concepts.

THEOREM 1.

a) $Q_i \overset{u}{\to} Q>0\in\mathscr{B}[H] \;\Rightarrow\; r_\sigma(A)<1.$

b) $Q_i \overset{s}{\to} Q>0\in\mathscr{B}[H] \;\Rightarrow\; r_\sigma(A)\leqq 1.$

Proof. Since $Q>0$, $\exists Q^{-1}\in\mathscr{B}[H]$. By Lemma 2(a) $Q-Q_i=A^{i+1}QA^{*i+1}$, $\forall i\geqq 0$. Hence

$$\text{(a)}\quad \|A^{i+1}\|^2 = \|A^{i+1}(Q^{1/2})(Q^{1/2})^{-1}\|^2 \leqq \|A^{i+1}(Q^{1/2})\|^2\|(Q^{1/2})^{-1}\|^2$$
$$= \|A^{i+1}QA^{*i+1}\|\,\|Q^{-1}\| = \|Q-Q_i\|\,\|Q^{-1}\|\to 0 \quad \text{as } i\to\infty,$$

thus following the desired result by Lemma 1.

$$\text{(b)}\quad \|A^{*i+1}x\| = \|(Q^{1/2})^{-1}(Q^{1/2})A^{*i+1}x\|^2 \leqq \|(Q^{1/2})^{-1}\|^2\|(Q^{1/2})A^{*i+1}x\|^2$$
$$= \|Q^{-1}\|\,\langle A^{i+1}QA^{*i+1}x; x\rangle = \|Q^{-1}\|\,\langle (Q-Q_i)x; x\rangle \to 0 \quad \text{as } i\to\infty,$$

for all $x\in H$, thus following part (b) by Remark 3, since $r_\sigma(A^*)=r_\sigma(A)$. $\square$

THEOREM 2.

$$r_\sigma(A)<1 \;\Rightarrow\; Q_i \overset{u}{\to} Q\in\mathscr{B}_1[H].$$

Proof. Suppose $r_\sigma(A)<1$. Since

$$\|Q_i\| \leqq \sum_{j=0}^{i} \|A^j Q_0 A^{*j}\| \leqq \|Q_0\| \sum_{j=0}^{i} \|A^j\|^2 \quad \forall i \geqq 0,$$

it follows by Lemma 1 that $\{Q_i; i\geqq 0\}$ is uniformly bounded. Therefore, using Proposition 1(a), we get for every $j>i\geqq 0$,

$$\|Q_j - Q_i\| \leqq \|Q_{j-i-1}\| \, \|A^{i+1}\|^2 \leqq \sup_k \|Q_k\| \, \|A^{i+1}\|^2 \to 0 \quad \text{as } i\to\infty,$$

by Lemma 1. Then $Q_i \xrightarrow{u} Q \in \mathcal{B}[H]$, since $\{Q_i\}$ is a Cauchy sequence and $\mathcal{B}[H]$ is a Banach space. Finally, since A^j and B are bounded and R is nuclear, it can be shown [14, p. 173] that $\operatorname{tr}(A^jBRB^*A^{*j}) \leqq \|A^j\|^2\|B\|^2 \operatorname{tr}(R)$. Therefore, using Lemma 1 again,

$$\operatorname{tr}(Q_i) = \sum_{j=0}^{i} \operatorname{tr}(A^jBRB^*A^{*j}) \leqq \operatorname{tr}(R)\, \|B\|^2 \sum_{j=0}^{\infty} \|A^j\|^2.$$

Hence $\{\operatorname{tr}(Q_i)\}$ is a bounded sequence, and so (cf. [14, p. 179]) the uniform limit Q of the nuclear sequence $\{Q_i\}$ must be nuclear. □

Remark 6. We notice from Lemma 2 that $Q_i \xrightarrow{w} Q \Leftrightarrow Q_i \xrightarrow{s} Q$, and $Q_i \xrightarrow{w} Q \in \mathcal{B}_1[H] \Rightarrow Q_i \xrightarrow{u} Q$. However it can be shown that

(a) $Q_i \xrightarrow{s} Q \in \mathcal{B}[H] \quad \not\Rightarrow \quad Q_i \xrightarrow{u} Q \in \mathcal{B}[H]$,

(b) $Q_i \xrightarrow{u} Q \in \mathcal{B}[H] \quad \not\Rightarrow \quad Q \in \mathcal{B}_1[H]$.

Moreover, it can also be verified that both strong convergence and positivity are not sufficient in Theorem 1(a). That is,

(c) $Q_i \xrightarrow{s} Q > 0 \in \mathcal{B}[H] \quad \not\Rightarrow \quad r_\sigma(A) < 1$,

(d) $Q_i \xrightarrow{u} Q > 0 \in \mathcal{B}_1[H] \quad \not\Rightarrow \quad r_\sigma(A) \leqq 1$.

To illustrate the above statements we consider the following examples.

Example 1. First we show that the statements (a) and (c) in Remark 6 hold true. Set $H=l_2$ and $U=\mathbb{R}^1$. Let $A\in\mathcal{B}[l_2]$ be the right shift operator, $A(\xi_1, \xi_2, \cdots) = (0, \xi_1, \xi_2, \cdots)$ for all $x=(\xi_1, \xi_2, \cdots)\in l_2$. Let $B\in\mathcal{B}[\mathbb{R}^1, l_2]$ be given by $Bu=(u, 0, \cdots)$ for all $u\in\mathbb{R}^1$, and set $R=1$, the identity operator in $\mathbb{R}^1$. It is a simple matter to verify that

$$Q_i = \sum_{j=0}^{i} A^jBB^*A^{*j} = \operatorname{diag}(1, \cdots, 1, 0, \cdots) \geqq 0 \in \mathcal{B}_1[l_2] \quad \forall i\geqq 0,$$

with the nonzero entries at the first $i+1$ positions, such that $\operatorname{tr}(Q_i)=i+1$. Hence

$$Q_i \xrightarrow{s} Q = I > 0 \in \mathcal{B}[l_2],$$

since $\|(I-Q_i)x\|^2 = \sum_{k=i+2}^{\infty} |\xi_k|^2 \to 0$ as $i\to\infty$ for all $x=(\xi_1, \xi_2, \cdots)\in l_2$, although $\{Q_i\}$ does not converge uniformly since $\|I-Q_i\|=1$, $\forall i\geqq 0$. This supports the statement (a) in Remark 6. However, as it is well known [11], $r_\sigma(A)=1$, thus confirming the statement (c) in Remark 6.

Example 2. Now we illustrate the statements (b) and (d) in Remark 6. Let $\{\varepsilon_k; k\geqq 1\}$ be a real positive sequence in l_1, and define a real positive strictly decreasing null sequence $\{\lambda_k; k\geqq 1\}$ as follows.

$$\lambda_{k+1} = \lambda_k - \varepsilon_k, \qquad \lambda_1 = \sum_{k=1}^{\infty} \varepsilon_k.$$

Set $H=U=l_2$. Let $A\in\mathscr{B}[l_2]$ be a constantly weighted left shift operator, $A(\xi_1,\xi_2,\cdots)=\rho^{1/2}(\xi_2,\xi_3,\cdots)$ for all $x=(\xi_1,\xi_2,\cdots)\in l_2$, such that [11] $r_\sigma(A)=\rho^{1/2}>0$. Let $B=I\in\mathscr{B}[l_2]$, the identity operator, and $R=\operatorname{diag}(\varepsilon_1,\varepsilon_2,\cdots)>0\in\mathscr{B}_1[l_2]$, with $\operatorname{tr}(R)=\lambda_1$. It is readily verified that

$$Q_i=\sum_{j=0}^{i}A^jRA^{*j}=\operatorname{diag}\left(\sum_{j=0}^{i}\rho^j\varepsilon_{j+1},\ \sum_{j=0}^{i}\rho^j\varepsilon_{j+2},\cdots\right)>0\in\mathscr{B}_1[l_2],$$

with $\operatorname{tr}(Q_i)=\sum_{j=0}^{i}\rho^j\lambda_{j+1}$, $\forall i\geqq 0$. In particular, with $r_\sigma(A)=\rho=1$ it follows that

$$Q_i=\operatorname{diag}(\lambda_1-\lambda_{i+2},\ \lambda_2-\lambda_{i+3},\cdots)>0\in\mathscr{B}_1[l_2],$$

with $\operatorname{tr}(Q_i)=\sum_{k=1}^{i+1}\lambda_k$. Thus

$$Q_i\xrightarrow{u}Q=\operatorname{diag}(\lambda_1,\lambda_2,\cdots)>0\in\mathscr{B}[l_2],$$

since $\|Q-Q_i\|=\lambda_{i+2}\to 0$ as $i\to\infty$. However

$$Q\in\mathscr{B}_1[l_2]\quad\Leftrightarrow\quad\operatorname{tr}(Q)=\sum_{k=1}^{\infty}\lambda_k<\infty,$$

which does not necessarily happen. For instance,

$$\varepsilon_k=\frac{1}{k(k+1)}\quad\forall k\geqq 1\quad\Rightarrow\quad\lambda_k=\frac{1}{k}\quad\forall k\geqq 1\quad\Rightarrow\quad Q\notin\mathscr{B}_1[l_2],$$

$$\varepsilon_k=\frac{2k+1}{k^2(k+1)^2}\quad\forall k\geqq 1\quad\Rightarrow\quad\lambda_k=\frac{1}{k^2}\quad\forall k\geqq 1\quad\Rightarrow\quad Q\in\mathscr{B}_1[l_2].$$

This confirms the statement (b) in Remark 6. Now let $r_\sigma(A)^2=\rho>1$ and set $\varepsilon_k=\alpha^{k-1}$, $\forall k\geqq 1$, with $0<\alpha<\rho^{-1}<1$. Then $R=\operatorname{diag}(1,\alpha,\alpha^2,\cdots)>0\in\mathscr{B}_1[l_2]$, with $\operatorname{tr}(R)=(1-\alpha)^{-1}$, and

$$Q_i=\frac{1-(\alpha\rho)^{i+1}}{1-\alpha\rho}R>0\in\mathscr{B}_1[l_2]\quad\forall i\geqq 0,$$

with $\operatorname{tr}(Q_i)=[1-(\alpha\rho)^{i+1}][(1-\alpha\rho)(1-\alpha)]^{-1}$. Thus

$$Q_i\xrightarrow{u}Q=(1-\alpha\rho)^{-1}R>0\in\mathscr{B}_1[l_2],$$

with $\operatorname{tr}(Q)=[(1-\alpha\rho)(1-\alpha)]^{-1}$, since $\|Q-Q_i\|=(1-\alpha\rho)^{-1}(\alpha\rho)^{i+1}\to 0$ as $i\to\infty$. However $r_\sigma(A)>1$, thus supporting the statement (d) in Remark 6.

Remark 7. By Theorem 1(a), Theorem 2, and Remark 1 one has

$$Q_i\xrightarrow{u}Q>0\in\mathscr{B}[H]\quad\Rightarrow\quad\dim(H)<\infty,$$

although (cf. Example 1)

$$Q_i\xrightarrow{s}Q>0\in\mathscr{B}[H]\quad\not\Rightarrow\quad\dim(H)<\infty.$$

If $\dim(H)<\infty$, then $\mathscr{B}_1[H]=\mathscr{B}[H]$, $P\sigma(A)=\sigma(A)$, strict positivity is equivalent to positivity, and uniform convergence is equivalent to strong convergence. Therefore, in such a case, it follows by Theorem 1 and Theorem 2 that

$$Q_i\to Q>0\in\mathscr{B}[H]\quad\Leftrightarrow\quad r_\sigma(A)<1\text{ and }Q>0.$$

However the assumption $Q > 0$, which appears in both sides of the above statement, may not be dismissed. That is, even for finite-dimensional spaces,

$$Q_i \to Q \in \mathcal{B}[H] \quad \not\Rightarrow \quad r_\sigma(A) \leqq 1, \qquad r_\sigma(A) < 1 \quad \not\Rightarrow \quad Q > 0,$$

as it is readily verified. These finite-dimensional results can be extended to infinite-dimensional spaces, whenever A is compact, as follows.

COROLLARY 1. *If* $A \in \mathcal{B}_\infty[H]$, *then the following properties are equivalent*:

(a) $r_\sigma(A) < 1$ *and* $(A, BR^{1/2})$ *is* A-C.

(b) $Q_i \overset{u}{\to} Q > 0 \in \mathcal{B}_1[H]$.

(c) $Q_i \overset{s}{\to} Q > 0 \in \mathcal{B}[H]$.

Proof. (a)$\Rightarrow$(b) by Lemma 2(a) and Theorem 2, and (b)$\Rightarrow$(c) trivially, for any $A \in \mathcal{B}[H]$. Now assume that $A \in \mathcal{B}_\infty[H]$. Then (c)$\Rightarrow$(a), since $r_\sigma(A) = r_\sigma(A^*) = \max\{|\lambda|: \lambda \in P\sigma(A^*) \cup \{0\}\} < 1$, by Lemma 2(a). □

5. Concluding remarks. In this paper we have considered mean square stability for discrete bounded linear systems in Hilbert space driven by white noise. The evolution and convergence of the state correlation operators sequence were investigated in Proposition 1 and Lemma 2. It has been shown in Theorem 2 that uniform asymptotic stability is a sufficient condition for mean square stability, although the reverse is not necessarily true (cf. Remark 6), as it occurs in a finite-dimensional setting whenever $Q > 0$ (cf. Theorem 1 and Remark 7).

For compact operators the discrete-time stability problem is quite clear, being a straightforward generalization of the finite-dimensional case. Indeed, as recalled in Remark 4, for deterministic systems strong and uniform asymptotic stability are equivalent concepts whenever A is compact. Comparing Remark 7 with Corollary 1 it is readily verified that a similar situation actually happens for stochastic systems with a compact operator A.

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REFERENCES

[1] A. V. BALAKRISHNAN, *Applied Functional Analysis*, 2nd ed., Springer-Verlag, Berlin, 1980.

[2] R. F. CURTAIN AND A. J. PRITCHARD, *Infinite Dimensional Linear Systems Theory*, Lecture Notes in Control and Information Sciences, 8, Springer-Verlag, Berlin, 1978.

[3] P. L. FALB, *Infinite-dimensional filtering: the Kalman–Bucy filter in Hilbert space*, Inform. Control, 11 (1967), pp. 102–137.

[4] P. A. FUHRMANN, *On weak and strong reachability and controllability of infinite-dimensional linear systems*, J. Optim. Theory Appl., 9 (1972), pp. 77–89.

[5] ———, *On observability and stability in infinite-dimensional linear systems*, J. Optim. Theory Appl., 12 (1973), pp. 173–181.

[6] W. W. HAGER AND L. L. HOROWITZ, *Convergence and stability properties of the discrete Riccati operator equation and the associated optimal control and filtering problems*, this Journal, 14 (1976), pp. 295–312.

[7] E. W. KAMEN AND W. L. GREEN, *Asymptotic stability of linear difference equations defined over a commutative Banach algebra*, J. Math. Anal. Appl., 75 (1980), pp. 584-601; *Addendum*, 84 (1981), pp. 295-298.

[8] C. S. KUBRUSLY, *Applied stochastic approximation algorithms in Hilbert space*, Int. J. Control, 28 (1978), pp. 23–31.

[9] H. J. KUSHNER, *Introduction to Stochastic Control Theory*, Holt, Rinehart & Winston, New York, 1971.

[10] J. P. LA SALLE, *The Stability of Dynamical Systems*, CBMS Regional Conference Series in Applied Mathematics, 25, Society for Industrial and Applied Mathematics, Philadelphia, 1976.

[11] A. W. NAYLOR AND G. R. SELL, *Linear Operator Theory in Engineering and Science*, 2nd ed., Springer-Verlag, Berlin, 1982.

[12] K. M. PRZYLUSKI, *The Lyapunov equation and the problem of stability for linear bounded discrete-time systems in Hilbert space*, Appl. Math. Optim., 6 (1980), pp. 97–112.

[13] J. H. VENTER, *On Dvoretzky stochastic approximation theorems*, Ann. Math. Statist. 37 (1966), pp. 1534–1544.

[14] J. WEIDMANN, *Linear Operators in Hilbert Spaces*, Springer-Verlag, Berlin, 1980.

[15] J. ZABCZYK, *Remarks on the control of discrete-time distributed parameter systems*, this Journal, 12 (1974), pp. 721–735.

[16] ———, *On optimal stochastic control systems in Hilbert space*, this Journal, 13 (1975), pp. 1217–1234.

[17] ———, *Stability properties of the discrete Riccati operator equation*, Kybernetica, 13 (1977), pp. 1-10.