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# ON STOCHASTIC MODELLING FOR DISCRETE BILINEAR SYSTEMS IN HILBERT SPACE

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Infinite-dimensional discrete-time bilinear models driven by Hilbert-space-valued random sequences can be rigorously defined as the uniform limit of finite-dimensional bilinear models. Existence and uniqueness of solutions for such infinite-dimensional models can be established by assuming only independence and structural similarity for the stochastic environment under consideration. Uniform structure equiconvergence implies uniform state convergence under suitable stability-like conditions.

## 1. Introduction

Mathematical modelling is usually understood as the problem of building a mathematical model, for instance in terms of differential or difference equations, for a given real, say physical, system. This is closely related to the system identification problem (see, e.g., [2]). Generally, the model candidates are well-defined mathematical objects such as, for instance, nonlinear ordinary differential equations whose existence and uniqueness of solutions have been previously established. In this paper we shall be dealing with the problem of properly defining a class of such mathematical objects, rather than with the problem of exhibiting a well-known mathematical equation which may reasonably describe the behaviour of some real process.

The class of mathematical objects that we shall be considering here is the one which describes the behaviour of an important subclass of nonlinear dynamical systems, namely bilinear systems (see, e.g., [1,6–8,11]). The major part of the available literature on bilinear systems is related to deterministic, continuous-time, and finite-dimensional models, although some important contributions outside the above category have already been published (see, e.g., the references in [3,5]). Here we shall be considering discrete-time bilinear systems operating in a stochastic environment, whose model is formally given by the following difference equation:

$$x_{i+1} = \left[A_0 + \sum_{k\geqslant 1} A_k\langle w_i\,;\, e_k\rangle\right]x_i + u_i,$$

where $\{A_k\,;\, k \geqslant 0\}$ is a sequence of bounded linear operators on some separable Hilbert space $H$, $\{e_k\,;\, k\geqslant 1\}$ is an orthonormal basis for $H$, and $\{u_i\,;\, i\geqslant 0\}$, $\{w_i\,;\, i\geqslant 0\}$, and $\{x_i\,;\, i\geqslant 0\}$ are $H$-valued random sequences. Suppose $H$ is infinite-dimensional, so that the above series may be infinite. The purpose of the present paper is to give a rigorous definition of the above infinite-dimensional stochastic discrete bilinear model, and to investigate the problem of approximating its state sequence by state sequences generated by finite-dimensional models.

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The paper is organized as follows. $H$-valued second-order random variables are briefly reviewed in Section 2, since they comprise the basic support upon which the main results will be built. In Section 3 it is shown that the infinite-dimensional structure under consideration can be properly defined as the uniform limit of finite-dimensional structures on the Hilbert space $\mathscr{H}$ of all second-order $H$-valued random variables. Existence and uniqueness of the state sequence generated by the underlying limiting model are also verified in Section 3. Finally, the reverse problem of approximating the resulting state sequence by using finite-dimensional structures is considered in Section 4.

## 2. Notational and conceptual preliminaries

Throughout this paper we shall assume that $H$ is a separable nontrivial Hilbert space. $\|\cdot\|$ and $\langle\cdot\,;\cdot\rangle$ will stand for norm and inner product in $H$, respectively. Let $B[H]$ denote the Banach algebra of all bounded linear transformations of $H$ into itself. We shall use the same symbol $\|\cdot\|$ to denote the uniform induced norm in $B[H]$. Let $T^* \in B[H]$ be the adjoint of $T\in B[H]$, and set $B[H]^+ = \{T\in B[H]: 0 \leqslant T = T^*\}$, the closed convex cone of all self-adjoint nonnegative (i.e., $0\leqslant\langle Th\,;h\rangle$ $\forall h\in H$) operators on $H$. Let $B_\infty[H]$ denote the class of all compact operators from $B[H]$, and set $B_\infty[H]^+ = B_\infty[H]\cap B[H]^+$. For $T\in B_\infty[H]^+$ we define the trace of $T$ as usual,

$$\operatorname{tr}(T) \overset{\text{def}}{=} \sum_{k=1}^{\infty}\langle Te_k\,; e_k\rangle = \sum_{k=1}^{\infty}\lambda_k,$$

where $\{e_k\,; k\geqslant 1\}$ is any orthonormal basis for $H$, and $\{\lambda_k\geqslant 0\,; k\geqslant 1\}$ is the set of all eigenvalues of $T$, each of them counted according to its multiplicity. Now set $|T| = (T^*T)^{1/2} \in B[H]^+$ for any $T\in B[H]$, and recall that $T\in B[H]^+ \Leftrightarrow T = |T|$ and $T\in B_\infty[H] \Leftrightarrow |T| \in B_\infty[H]$. Let $B_1[H] = \{T\in B_\infty[H]: \operatorname{tr}(|T|)<\infty\}$ denote the class of all nuclear operators on $H$. In particular, set $B_1[H]^+ = B_1[H]\cap B[H]^+ = \{T\in B_\infty[H]^+ : \operatorname{tr}(T)<\infty\}$. Finally, for any $f$, $g\in H$ define the outer product operator $(f\circ g)\in B_1[H]$ as follows: $(f\circ g)h = \langle h\,; g\rangle f$ for all $h\in H$, so that $(f\circ f)\in B_1[H]^+$. For a brief presentation on nuclear (or trace-class) operators, the reader is referred to [12].

Let $(\Omega, \Sigma, \mu)$ be a probability space, where $\Sigma$ is a $\sigma$-algebra of subsets of a nomempty basic set $\Omega$, and $\mu$ is a probability measure on $\Sigma$. Let $\mathscr{H}$ be the set of equivalence classes of $H$-valued measurable maps $x$ defined almost everywhere on $\Omega$, such that

$$\|x\|_{\mathscr{H}}^2 \overset{\text{def}}{=} \varepsilon\{\|x\|^2\} = \int_\Omega \|x(\omega)\|^2\, \mathrm{d}\mu(\omega) < \infty,$$

where $\varepsilon$ stands for the expectation operator for scalar-valued random variables. The above is the so-called second-order property. Now, set the following inner product in $\mathscr{H}$:

$$\langle x\,; y\rangle_{\mathscr{H}} \overset{\text{def}}{=} \varepsilon\{\langle x\,; y\rangle\} = \int_\Omega \langle x(\omega)\,; y(\omega)\rangle\, \mathrm{d}\mu(\omega)$$

for all $x$, $y\in\mathscr{H}$, which induces the above norm in $\mathscr{H}$. Thus, $\mathscr{H} = L_2(\Omega, \mu\,; H)$: the Hilbert space of all second-order $H$-valued random variables. For any $x$, $y\in\mathscr{H}$, consider the sesquilinear functional $\varepsilon\{\langle \cdot\,; y\rangle \langle x; \cdot\rangle\}: H^2 \to \mathbb{C}$, which is bounded. Then (cf. [12, p. 120]) there exists a unique operator in $B[H]$, say $\mathscr{E}\{x \circ y\}$, referred to as the *correlation* of $x \in \mathscr{H}$ and $y \in \mathscr{H}$, such that

$$\langle \mathscr{E}\{x \circ y\} f; g\rangle = \varepsilon\{\langle f; y\rangle\langle x; g\rangle\} \quad \forall f,\, g \in H.$$

**2.1. Remark.** It is a simple matter to show that, for every $x, y \in \mathscr{H}$ and $T \in B[H]$,

(a) $\mathscr{E}\{x \circ y\} \in B_1[H]$ and $\mathscr{E}\{x \circ x\} \in B_1[H]^+$,

(b) $\operatorname{tr}(\mathscr{E}\{x \circ x\}) = \|x\|^2_{\mathscr{H}}$,

(c) $\|Tx\|_{\mathscr{H}} \leqslant \|T\| \cdot \|x\|_{\mathscr{H}}$.

Now consider a family $\{x_\xi \in \mathscr{H};\, \xi \in \Xi \neq \emptyset\}$ of random variables. For each $\xi \in \Xi$ let $\{e_{\xi,k};\, k \geqslant 1\}$ be an orthonormal basis for $H$ made up of all eigenvectors of $\mathscr{E}\{x_\xi \circ x_\xi\} \in B_1[H]^+$, whose existence is ensured by the Spectral Theorem (see, e.g., [9, p. 460]). Such a family is said to be *structurally similar* if there exists an orthonormal basis for $H$, say $\{e_k;\, k \geqslant 1\}$, such that $\{e_{\xi,k};\, k \geqslant 1\} = \{e_k;\, k \geqslant 1\}$ for every $\xi \in \Xi$. $\{e_k;\, k \geqslant 1\}$ is referred to as the *common orthonormal basis* for $H$ of $\{x_\xi \in \mathscr{H};\, \xi \in \Xi\}$. Note that structural similarity may be thought of as a generalization of correlation stationarity. Actually, a family $\{x_\xi \in \mathscr{H};\, \xi \in \Xi \neq \emptyset\}$ is *correlation stationary* if there exists a $Q \in B_1[H]^+$ such that $\mathscr{E}\{x_\xi \circ x_\xi\} = Q$ for every $\xi \in \Xi$. For any family $\{x_\xi \in \mathscr{H};\, \xi \in \Xi \neq \emptyset\}$ we set

$$\mathscr{I}_{\{x_\xi;\, \xi \in \Xi\}} = \left\{y \in \mathscr{H}\colon\ y \text{ is independent of } \left\{x_\xi \in \mathscr{H};\, \xi \in \Xi\right\}\right\}.$$

In particular, for any $x \in \mathscr{H}$, $\mathscr{I}_x = \{y \in \mathscr{H}\colon\ y$ is independent of $x \in \mathscr{H}\}$.

**2.2. Remark.** Note that $y \in \mathscr{I}_x \Leftrightarrow x \in \mathscr{I}_y$. The following well-known independence properties (see, e.g., [10]) will be needed in the sequel:

(a) If $x \in \mathscr{I}_y$, then, for every measurable functionals $\phi,\, \psi: H \to \mathbb{C}$,

$$\varepsilon\{\phi(x)\psi(y)\} = \varepsilon\{\phi(x)\}\,\varepsilon\{\psi(y)\}.$$

(b) If $\{y_\upsilon \in \mathscr{H};\, \upsilon \in \Upsilon \neq \emptyset\}$ is independent of $\{x_\xi \in \mathscr{H};\, \xi \in \Xi \neq \emptyset\}$, then, for any finite subset $\{x_{\xi_k};\, 1 \leqslant k \leqslant m\}$ of $\{x_\xi;\, \xi \in \Xi\}$ and for every measurable map $N: H^m \to H$,

$$N(x_{\xi_1}, \ldots, x_{\xi_m}) \in \mathscr{I}_{\{y_\upsilon;\, \upsilon \in \Upsilon\}}.$$

**2.3. Remark.** For any $x \in \mathscr{H}$ consider the linear functional $\varepsilon\{\langle \cdot\,; x\rangle\}: H \to \mathbb{C}$, which is bounded. Then, by the Riez Representation Theorem (see, e.g., [9, p. 345]), there exists a unique element in $H$, say $E\{x\}$, referred to as the *expectation* of $x \in \mathscr{H}$, such that

$$\langle E\{x\};\, h\rangle = \varepsilon\{\langle x; h\rangle\} \quad \forall h \in H.$$

Two random variables $x, y \in \mathscr{H}$ are said to be *uncorrelated* if

$$\mathscr{E}\{x \circ y\} = E\{x\} \circ E\{y\}.$$

By the Riez Representation Theorem (and according to the definition of expectation, correlation, and outer product operator), uncorrelatedness turns out to be equivalent to

$$\varepsilon\{f(x)\ \bar{g}(y)\} = \varepsilon\{f(x)\}\ \varepsilon\{\bar{g}(y)\}$$

for every bounded linear functionals $f, g: H \to \mathbb{C}$, with the upper bar denoting complex conjugate. Hence, independence obviously implies uncorrelatedness, according to Remark 2.2(a). However, only uncorrelatedness will not suffice our needs in Section 3. We shall really need the separation property in Remark 2.2(a) for quadratic functionals, which is not generally true for uncorrelated random variables. When uncorrelatedness is enough (e.g., for linear models), a sharper and more elegant approach can be developed. To see this, let us first recall the following particular case of Remark 2.2(b). Take any finite subset $\{x_{\xi_k}; 1 \leqslant k \leqslant m\}$ of a family $\{x_\xi; \xi \in \Xi \neq \emptyset\}$. If $x \in \mathscr{I}_{\{x_\xi; \xi \in \Xi\}}$, then, for every measurable map $N: H^m \to H$,

$$N(x_{\xi_1}, \ldots, x_{\xi_m}) \in \mathscr{I}_x.$$

However, the above is not generally true if we assume $x \in \bigcap_{\xi \in \Xi} \mathscr{I}_{x_\xi}$ instead of $x \in \mathscr{I}_{\{x_\xi; \xi \in \Xi\}}$. Indeed, it is possible that $(y+z) \notin \mathscr{I}_x$ even if $x, y, z \in \mathscr{H}$ are pairwise independent. Thus, $\mathscr{I}_x$ is not a linear subspace of $\mathscr{H}$, opposite to $\mathscr{U}_x = \{y \in \mathscr{H}: y \text{ is uncorrelated with } x \in \mathscr{H}\}$, which is a closed linear subspace of $\mathscr{H}$. Therefore, if uncorrelatedness was sufficient, we could replace the set $\mathscr{I}_x$ by the Hilbert space $\mathscr{U}_x$ throughout the next section, which would certainly supply a nicer set-up. But, this is not the case for nonlinear models.

## 3. Infinite-dimensional stochastic bilinear model

Consider the infinite-dimensional stochastic discrete bilinear model that has formally been introduced in Section 1. The purpose of this section is to give a rigorous definition for such a model. This will be achieved in Lemma 3.4 below. We begin by establishing two auxiliary results that will suffice our needs.

**3.1. Proposition.** *Let $\{w_\xi \in \mathscr{H}; \xi \in \Xi \neq \emptyset\}$ be a structurally similar family with a common orthonormal basis $\{e_k; k \geqslant 1\}$ for $H$. For each $\xi \in \Xi$ and for every $n \geqslant 1$, set*

$$\mathscr{A}_{w_\xi}(n) = A_0 + \sum_{k=1}^{n} A_k \langle w_\xi; e_k\rangle : \mathscr{I}_{w_\xi} \to \mathscr{H},$$

*where $\{A_k \in B[H]; k \geqslant 0\}$ is a uniformly bounded sequence of operators. We claim that, for each $\xi \in \Xi$, the sequence of maps $\{\mathscr{A}_{w_\xi}(n): \mathscr{I}_{w_\xi} \to \mathscr{H}; n \geqslant 1\}$ converges uniformly or, equivalently, for each $\xi \in \Xi$ there exists a map $\mathscr{A}_{w_\xi}: \mathscr{I}_{w_\xi} \to \mathscr{H}$ such that*

$$\sup_{0 \neq v \in \mathscr{I}_{w_\xi}} \frac{\|\mathscr{A}_{w_\xi}(n)v - \mathscr{A}_{w_\xi}v\|_{\mathscr{H}}}{\|v\|_{\mathscr{H}}} \to 0 \quad \text{as } n \to \infty.$$

*Such a map has the following properties*:

$$\|\mathscr{A}_{w_\xi}v\|_{\mathscr{H}} \leqslant \Big(\|A_0\| + \sup_{k \geqslant 1}\|A_k\|\, \|w_\xi\|_{\mathscr{H}}\Big)\|v\|_{\mathscr{H}}, \qquad \mathscr{A}_{w_\xi}\alpha v = \alpha \mathscr{A}_{w_\xi} v$$

*for every $\alpha \in \mathbb{C}$ and $v \in \mathscr{I}_{w_\xi}$, and, for each $i \geqslant 1$,*

$$\mathscr{A}_{w_\xi}\Bigg(\sum_{j=0}^{i} v_j\Bigg) = \sum_{j=0}^{i} \mathscr{A}_{w_\xi} v_j,$$

*whenever* $v_j \in \mathscr{I}_{w_\xi}$ *for every* $j = 0, 1, \dots, i$ *and* $(\sum_{j=0}^{k} v_j) \in \mathscr{I}_{w_\xi}$ *for every* $k = 1, \dots, i$, *which happens whenever* $w_\xi \in \mathscr{I}_{\{v_0, v_1, \dots, v_i\}}$ *according to Remark* 2.2(*b*).

**Proof.** For each $\xi \in \Xi$ we get from Remark 2.2(a) that

$$\begin{aligned}
&\langle A_k \langle w_\xi ; e_k \rangle v ; A_l \langle w_\xi ; e_l \rangle v \rangle_{\mathscr{H}} \\
&\quad = \varepsilon \{ \langle w_\xi ; e_k \rangle \langle e_l ; w_\xi \rangle \langle A_k v ; A_l v \rangle \} \\
&\quad = \varepsilon \{ \langle w_\xi ; e_k \rangle \langle e_l ; w_\xi \rangle \} \varepsilon \{ \langle A_k v ; A_l v \rangle \} = \langle \mathscr{E} \{ w_\xi \circ w_\xi \} e_l ; e_k \rangle \langle A_k v ; A_l v \rangle_{\mathscr{H}}
\end{aligned}$$

for every $k$, $l \geqslant 1$, whenever $v \in \mathscr{I}_{w_\xi}$. Moreover, since $\{ w_\xi \in \mathscr{H} ; \xi \in \Xi \}$ is structurally similar, $\mathscr{E}\{ w_\xi \circ w_\xi \} e_k = \lambda_{\xi,k} e_k$, so that $\sum_{k=1}^{\infty} \lambda_{\xi,k} = \operatorname{tr}(\mathscr{E}\{ w_\xi \circ w_\xi \}) = \| w_\xi \|_{\mathscr{H}}^2 < \infty$, for each $\xi \in \Xi$ and every $k \geqslant 1$, where $\lambda_{\xi,k} \geqslant 0$ is the eigenvalue of $\mathscr{E}\{ w_\xi \circ w_\xi \} \in B_1[H]^+$ associated with the common eigenvector $e_k$ for each $k \geqslant 1$ (cf. Remark 2.1(b)). Hence, for each $\xi \in \Xi$,

$$\begin{aligned}
&\left\| \sum_{k=m}^{p} A_k \langle w_\xi ; e_k \rangle v \right\|_{\mathscr{H}}^2 \\
&\quad = \sum_{k,l=m}^{p} \langle A_k \langle w_\xi ; e_k \rangle v ; A_l \langle w_\xi ; e_l \rangle v \rangle_{\mathscr{H}} \\
&\quad = \sum_{k,l=m}^{p} \lambda_{\xi,l} \langle e_l ; e_k \rangle \langle A_k v ; A_l v \rangle_{\mathscr{H}} \leqslant \sup_{m \leqslant k \leqslant p} \| A_k \|^2 \| v \|_{\mathscr{H}}^2 \sum_{k=m}^{p} \lambda_{\xi,k}
\end{aligned}$$

for all $v \in \mathscr{I}_{w_\xi}$ and for any $1 \leqslant m \leqslant p$, according to Remark 2.1(c). Therefore, uniform convergence follows for each $\xi \in \Xi$, since $\mathscr{H}$ is complete, and

$$\sup_{\nu \geqslant 1} \sup_{0 \neq v \in \mathscr{I}_{w_\xi}} \frac{\| \mathscr{A}_{w_\xi}(n+\nu) v - \mathscr{A}_{w_\xi}(n) v \|_{\mathscr{H}}}{\| v \|_{\mathscr{H}}} \leqslant \sup_{k \geqslant 1} \| A_k \| \left( \sum_{k=n+1}^{\infty} \lambda_{\xi,k} \right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

Let us finally verify the bounded linear-like properties of the map $\mathscr{A}_{w_\xi}$ (which just fails to be a bounded linear one because its domain $\mathscr{I}_{w_\xi}$ is not a linear subspace of $\mathscr{H}$). Boundedness is readily verified since, for each $\xi \in \Xi$ and for any $v \in \mathscr{I}_{w_\xi}$,

$$\| \mathscr{A}_{w_\xi}(n) v \|_{\mathscr{H}} \leqslant \left( \| A_0 \| + \sup_{k \geqslant 1} \| A_k \| \, \| w_\xi \|_{\mathscr{H}} \right) \| v \|_{\mathscr{H}}$$

for every $n \geqslant 1$. Homogeneity is trivial. The additivity property can be verified as follows. For an arbitrary $\xi \in \Xi$, take $u, v \in \mathscr{I}_{w_\xi}$ for which $(u+v) \in \mathscr{I}_{w_\xi}$. Then, in such a case, $\mathscr{A}_{w_\xi}(n)(u+v) = \mathscr{A}_{w_\xi}(n) u + \mathscr{A}_{w_\xi}(n) v$ for every $n \geqslant 1$, so that

$$\begin{aligned}
&\| \mathscr{A}_{w_\xi}(u+v) - \left( \mathscr{A}_{w_\xi} u + \mathscr{A}_{w_\xi} v \right) \|_{\mathscr{H}} \\
&\quad \leqslant \| \mathscr{A}_{w_\xi}(u+v) - \mathscr{A}_{w_\xi}(n)(u+v) \|_{\mathscr{H}} + \| \mathscr{A}_{w_\xi}(n) u - \mathscr{A}_{w_\xi} u \|_{\mathscr{H}} \\
&\qquad + \| \mathscr{A}_{w_\xi}(n) v - \mathscr{A}_{w_\xi} v \|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, our additivity property holds true for $i = 1$. Now let $v_j \in \mathscr{I}_{w_\xi}$ for every $j = 0, 1, \dots, i, i+1$

and $(\Sigma_{j=0}^{k} v_j) \in \mathscr{I}_{w_\xi}$ for every $k = 1, 2, \ldots, i, i+1$, and suppose the desired additivity property holds for some $i \geqslant 1$. Then, since it holds for $i = 1$,

$$\mathscr{A}_{w_\xi}\left(\sum_{j=0}^{i} v_j + v_{i+1}\right) = \mathscr{A}_{w_\xi} \sum_{j=0}^{i} v_j + \mathscr{A}_{w_\xi} v_{i+1} = \sum_{j=0}^{i} \mathscr{A}_{w_\xi} v_j + \mathscr{A}_{w_\xi} v_{i+1},$$

which concludes the proof by induction. □

**3.2. Remark.** In the preceding proposition we got uniform pointwise convergence (i.e., uniform on $\mathscr{I}_{w_\xi}$ for each $\xi \in \Xi$), but not uniform equiconvergence (i.e., uniform on $\mathscr{I}_{w_\xi}$ and uniform in $\xi \in \Xi$). Now, suppose the structurally similar family $\{w_\xi \in \mathscr{H}; \xi \in \Xi\}$ is *correlation dominated* in the following sense: there exists a $Q \in B_1[H]^+$ such that

$$\mathscr{E}\{w_\xi \circ w_\xi\} \leqslant Q \quad \forall \xi \in \Xi.$$

For instance, correlation stationarity for $\{w_\xi \in \mathscr{H}; \xi \in \Xi\}$ characterizes a particular case for which the above assumptions obviously hold true. By setting $\lambda_k = \langle Qe_k; e_k \rangle \geqslant 0$ for each $k \geqslant 1$, we get $\lambda_{\xi,k} = \langle \mathscr{E}\{w_\xi \circ w_\xi\} e_k; e_k \rangle \leqslant \lambda_k$ for all $\xi \in \Xi$ and every $k \geqslant 1$, so that

$$\sup_{\xi \in \Xi} \sum_{k=m}^{p} \lambda_{\xi,k} \leqslant \sum_{k=m}^{p} \sup_{\xi \in \Xi} \lambda_{\xi,k} \leqslant \sum_{k=m}^{p} \lambda_k$$

for any $1 \leqslant m \leqslant p$, where $\Sigma_{k=1}^{\infty} \lambda_k = \operatorname{tr}(Q) < \infty$, Hence,

$$\sup_{\xi \in \Xi} \sup_{0 \neq v \in \mathscr{I}_{w_\xi}} \frac{\|(\mathscr{A}_{w_\xi}(n) - \mathscr{A}_{w_\xi})v\|_{\mathscr{H}}}{\|v\|_{\mathscr{H}}} \leqslant \sup_{k \geqslant 1} \|A_k\| \left(\sum_{k=n+1}^{\infty} \lambda_k\right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

Thus, the sequence of maps $\{\mathscr{A}_{w_\xi}(n): \mathscr{I}_{w_\xi} \to \mathscr{H}; n \geqslant 1\}$ converges to $\mathscr{A}_{w_\xi}: \mathscr{I}_{w_\xi} \to \mathscr{H}$ uniformly on $\mathscr{I}_{w_\xi} \subset \mathscr{H}$ and uniformly in $\xi \in \Xi$ (i.e., uniform equiconvergence), whenever the structurally similar family $\{w_\xi \in \mathscr{H}; \xi \in \Xi\}$ is correlation dominated.

**3.3. Proposition.** *Consider a sequence $\{v_i \in \mathscr{H}; i \geqslant 0\}$. Let $\{w_i \in \mathscr{H}; i \geqslant 0\}$ be a structurally similar sequence such that $w_0 \in \mathscr{I}_{v_0}$ and $w_j \in \mathscr{I}_{\{w_0, \ldots, w_{j-1}, v_0, v_1, \ldots, v_j\}}$ for every $j \geqslant 1$. Then*

$$v_i \in \mathscr{I}_{w_i}, \qquad \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} v_j \in \mathscr{I}_{w_i} \quad \forall j = 0, \ldots, i-1,$$

$$\sum_{j=0}^{k} \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} v_j \in \mathscr{I}_{w_i} \quad \forall k = 0, \ldots, i-1,$$

$$\sum_{j=0}^{i-1} \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} v_j + v_i \in \mathscr{I}_{w_i},$$

$$\mathscr{A}_{w_i}\left[\sum_{j=0}^{i-1} \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} v_j + v_i\right] = \sum_{j=0}^{i} \mathscr{A}_{w_i} \cdots \mathscr{A}_{w_j} v_j,$$

*for every $i \geqslant 1$, with $\mathscr{A}_{w_j}: \mathscr{I}_{w_j} \to \mathscr{H}$ defined as in Proposition* 3.1.

**Proof.** Since $v_0 \in \mathscr{I}_{w_0}$ we get $(\mathscr{A}_{w_0} v_0) \in \mathscr{H}$ by Proposition 3.1. Hence, according to Remark 2.2(b), $w_1 \in \mathscr{I}_{\{w_0, v_0, v_1\}}$ implies that $v_1$, $\mathscr{A}_{w_0} v_0$, and $\mathscr{A}_{w_0} v_0 + v_1$ are in $\mathscr{I}_{w_1}$. Then, by the additivity property of $\mathscr{A}_{w_1}$ in Proposition 3.1,

$$\mathscr{A}_{w_1}\left[\mathscr{A}_{w_0} v_0 + v_1\right] = \mathscr{A}_{w_1}\mathscr{A}_{w_0} v_0 + \mathscr{A}_{w_1} v_1.$$

Thus, the desired result holds for $i = 1$. Now suppose it holds for some $i \geqslant 1$, so that $(\mathscr{A}_{w_i} \ldots \mathscr{A}_{w_j} v_j) \in \mathscr{H}$ for every $j = 0, \ldots, i$ by Proposition 3.1. Hence, according to Remark 2.2(b), $w_{i+1} \in \mathscr{I}_{\{w_0, \ldots, w_i, v_0, v_1, \ldots, v_{i+1}\}}$ implies that

$$v_{i+1} \in \mathscr{I}_{w_{i+1}}, \qquad \mathscr{A}_{w_i} \ldots \mathscr{A}_{w_j} v_j \in \mathscr{I}_{w_{i+1}} \quad \forall j = 0, \ldots, i,$$

$$\sum_{j=0}^{k} \mathscr{A}_{w_i} \ldots \mathscr{A}_{w_j} v_j \in \mathscr{I}_{w_{i+1}} \quad \forall k = 0, \ldots, i,$$

$$\sum_{j=0}^{i} \mathscr{A}_{w_i} \ldots \mathscr{A}_{w_j} v_j + v_{i+1} \in \mathscr{I}_{w_{i+1}}.$$

Then, by the additivity property of $\mathscr{A}_{w_{i+1}}$ in Proposition 3.1,

$$\mathscr{A}_{w_{i+1}}\left[\sum_{j=0}^{i} \mathscr{A}_{w_i} \ldots \mathscr{A}_{w_j} v_j + v_{i+1}\right] = \sum_{j=0}^{i+1} \mathscr{A}_{w_{i+1}} \ldots \mathscr{A}_{w_j} v_j.$$

Thus, the result holds for $i + 1$, which concludes the proof by induction. □

**3.4. Lemma.** *Let $\{w_i \in \mathscr{H};\, i \geqslant 0\}$ be a structurally similar sequence with a common orthonormal basis $\{e_k;\, k \geqslant 1\}$ for H. Set*

$$\mathscr{A}_{w_i} = A_0 + \sum_{k=1}^{\infty} A_k \langle w_i;\, e_k \rangle : \mathscr{I}_{w_i} \to \mathscr{H}$$

*for every $i \geqslant 0$, as defined in Proposition* 3.1, *where $\{A_k \in B[H];\, k \geqslant 0\}$ is uniformly bounded. Given $x_0 \in \mathscr{H}$ and $\{u_i \in \mathscr{H};\, i \geqslant 0\}$, assume further that $w_0 \in \mathscr{I}_{x_0}$ and $w_j \in \mathscr{I}_{\{x_0, u_0, \ldots, u_{j-1}, w_0, \ldots, w_{j-1}\}}$ for every $j \geqslant 1$. Then the difference equation in $\mathscr{H}$,*

$$x_{i+1} = \left[A_0 + \sum_{k=1}^{\infty} A_k \langle w_i;\, e_k \rangle\right] x_i + u_i,$$

*has a unique solution, which lies in $\mathscr{I}_{w_i}$ for every $i \geqslant 0$, given by $x_1 = \mathscr{A}_{w_0} x_0 + u_0$ and, for every $i \geqslant 2$,*

$$x_i = \mathscr{A}_{w_{i-1}} \ldots \mathscr{A}_{w_0} x_0 + \sum_{j=1}^{i-1} \mathscr{A}_{w_{i-1}} \ldots \mathscr{A}_{w_j} u_{j-1} + u_{i-1}.$$

**Proof.** Set $v_0 = x_0$ and $v_{j+1} = u_j$ for every $j \geqslant 0$, so that $w_0 \in \mathscr{I}_{v_0}$ and $w_j \in \mathscr{I}_{\{w_0, \ldots, w_{j-1}, v_0, v_1, \ldots, v_j\}}$ for every $j \geqslant 1$. Then, by Proposition 3.3,

$$(a) \qquad \sum_{j=0}^{i-1} \mathscr{A}_{w_{i-1}} \ldots \mathscr{A}_{w_j} v_j + v_i \in \mathscr{I}_{w_i}, \qquad \mathscr{A}_{w_i}\left[\sum_{j=0}^{i-1} \mathscr{A}_{w_{i-1}} \ldots \mathscr{A}_{w_j} v_j + v_i\right] = \sum_{j=0}^{i} \mathscr{A}_{w_i} \ldots \mathscr{A}_{w_j} v_j$$

for every $i \geqslant 1$. Therefore, by setting

$$(b) \qquad x_i = \sum_{j=0}^{i-1} \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} v_j + v_i \in \mathscr{I}_{w_i}$$

for every $i \geqslant 1$, we get

$$(c) \qquad x_{i+1} = \mathscr{A}_{w_i} x_i + v_{i+1}, \qquad x_0 = v_0,$$

for every $i \geqslant 0$. On the other hand, if $\{x_i \in \mathscr{H};\, i \geqslant 0\}$ solves the difference equation (c) then, from (a), it is readily verified by induction that (b) holds true for every $i \geqslant 1$. □

## 4. Finite-dimensional approximations

Under the assumptions of Lemma 3.4, consider the state sequence $\{x_i \in \mathscr{I}_{w_i};\, i \geqslant 0\}$, so that

$$(1) \qquad x_{i+1} = \mathscr{A}_{w_i} x_i + u_i = \left[ A_0 + \sum_{k=1}^{\infty} A_k \langle w_i ; e_k \rangle \right] x_i + u_i$$

for every $i \geqslant 0$. Now, for each $n \geqslant 1$, consider an approximate state sequence $\{x_i(n) \in \mathscr{I}_{w_i};\, i \geqslant 0\}$, such that

$$(2) \qquad x_{i+1}(n) = \mathscr{A}_{w_i}(n) x_i(n) + u_i = \left[ A_0 + \sum_{k=1}^{n} A_k \langle w_i ; e_k \rangle \right] x_i(n) + u_i$$

for every $i \geqslant 0$. Here, $x_0(n)$ is supposed to be endowed with independence properties similar to those imposed to $x_0$. Precisely, we assume that $x_0(n) \in \mathscr{H}$ is such that $w_0 \in \mathscr{I}_{\{x_0, x_0(n)\}}$ and $w_j \in \mathscr{I}_{\{x_0, x_0(n), u_0, \ldots, u_{j-1}, w_0, \ldots, w_{j-1}\}}$ for every $j \geqslant 1$, for each $n \geqslant 1$. Note that this includes the usual particular case of $x_0(n)$ being actually indentified with $x_0$ (i.e., $x_0(n) = x_0 \in \mathscr{H}$ for every $n \geqslant 1$). We refer to the models (1) and (2) as infinite-dimensional and finite-dimensional, respectively. However, note that the state in (1) and the approximate state in (2) are both $H$-valued random sequences, hence infinite-dimensional. What is really finite-dimensional in (2) is the sequence $\{(\langle w_i ; e_1 \rangle, \ldots, \langle w_i ; e_n \rangle);\, i \geqslant 0\}$, in the sense that it is a $\mathbb{C}^n$-valued random sequence. Since, by Proposition 3.1,

$$(3) \qquad \mathscr{A}_{w_i}(n) \to \mathscr{A}_{w_i} \quad \text{as } n \to \infty$$

uniformly on $\mathscr{I}_{w_i} \subset \mathscr{H}$ and pointwise in $i$ (which will be referred to as uniform pointwise structure convergence—cf. Remark 3.2), it is natural to enquire about an expected state convergence,

$$(4) \qquad x_i(n) \to x_i \quad \text{as } n \to \infty$$

in $\mathscr{H}$, for each $i \geqslant 0$. Such a pointwise state convergence actually follows in a natural and straightforward way from the uniform pointwise structure convergence. For, from (1) and (2) we may write

$$x_{i+1} - x_{i+1}(n) = \left[ A_0 + \sum_{k=1}^{n} A_k \langle w_i ; e_k \rangle \right] (x_i - x_i(n)) + \sum_{k=n+1}^{\infty} A_k \langle w_i ; e_k \rangle x_i$$

for every $n \geqslant 1$ and $i \geqslant 0$. Note that, by the independence condition under consideration and from Remark 2.2(b), we actually have $x_i(n) \in \mathscr{I}_{w_i}$ and $(x_i - x_i(n)) \in \mathscr{I}_{w_i}$ for every $n \geqslant 1$ and $i \geqslant 0$. By the above equation we get

$$\|x_{i+1}(n) - x_{i+1}\|_{\mathscr{H}} \leqslant \Big(\|A_0\| + \sup_{k\geqslant 1}\|A_k\|\|w_i\|_{\mathscr{H}}\Big)\|x_i(n) - x_i\|_{\mathscr{H}} + \left\| \sum_{k=n+1}^{\infty} A_k\langle w_i\,;\, e_k\rangle x_i \right\|_{\mathscr{H}},$$

where, for each $i \geqslant 0$,

$$\left\| \sum_{k=n+1}^{\infty} A_k\langle w_i\,;\, e_k\rangle x_i \right\|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty,$$

according to Proposition 3.1. Thus, $\|x_{i+1}(n) - x_{i+1}\|_{\mathscr{H}} \to 0$ as $n \to \infty$ whenever $\|x_i(n) - x_i\|_{\mathscr{H}} \to 0$ as $n \to \infty$. If $\|x_0(n) - x_0\|_{\mathscr{H}} \to 0$ as $n \to \infty$, we have confirmed by induction that the following holds.

**4.1. Corollary.** *Consider the assumption of Lemma* 3.4, *and the models in* (1) *and* (2). *If*

$$\|x_0(n) - x_0\|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty,$$

*then, for each* $i \geqslant 0$,

$$\|x_i(n) - x_i\|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty.$$

*Equivalently, the state sequence generated by the infinite-dimensional model in* (1) *can always be pointwise approximated by state sequences generated by finite-dimensional models as in* (2), *whenever uniform pointwise structure convergence holds true.*

On the other hand, if (3) holds uniformly on $\mathscr{I}_{w_i} \subset \mathscr{H}$ and uniformly in $i$ (which will be referred to as uniform structure equiconvergence—cf. Remark 3.2), it is also natural to enquire whether (4) holds in $\mathscr{H}$ uniformly in $i$. However, such a uniform state convergence does not necessarily follow from uniform structure equiconvergence. To verify this, let us consider the simplest class of counterexamples where the bilinear models are reduced to linear ones. Thus, assume that $A_k = 0$ for every $k \geqslant 1$, so that

$$x_i - x_i(n) = A_0^i(x_0 - x_0(n))$$

for every $n \geqslant 1$ and $i \geqslant 0$. Now suppose $x_0(n) = (1 - \beta_n)x_0 \in \mathscr{H}$ for every $n \geqslant 1$, for some complex sequence $\{\beta_n \neq 0 \in \mathbb{C}\,;\, n \geqslant 1\}$ such that $|\beta_n| \to 0$ as $n \to \infty$, where the random variable $x_0 \in \mathscr{H}$ degenerates to $h_0 \in H$ (i.e., $\|x_0 - h_0\|_{\mathscr{H}} = 0$ for some $h_0 \in H$, so that $x_0 = h_0 \in H$ with probability one). Then,

$$\|x_i(n) - x_i\|_{\mathscr{H}} = |\beta_n| \left\| A_0^i h_0 \right\|$$

for every $n \geqslant 1$ and $i \geqslant 0$. Moreover, suppose $A_0 \in B[H]$ is power unbounded (i.e., $\sup_{i \geqslant 0}\|A_0^i\| = \infty$), so that, by the Banach–Steinhaus Theorem (see, e.g. [12, p. 74]), there exists an $h \in H$ such that $\sup_{i \geqslant 0}\|A_0^i h\| = \infty$. By setting $h_0 = h$ we get

$$\sup_{i\geqslant 0}\|x_i(n) - x_i\|_{\mathscr{H}} = |\beta_n| \sup_{i\geqslant 0} \left\| A_0^i h_0 \right\| = \infty$$

for every $n \geqslant 1$, even though $\|x_0(n) - x_0\|_{\mathscr{H}} \to 0$ as $n \to \infty$ and $\|(\mathscr{A}_{w_i}(n) - \mathscr{A}_{w_i})v\|_{\mathscr{H}} = 0$ for all $v \in \mathscr{I}_{w_i} \subset \mathscr{H}$, for every $n \geqslant 1$ and $i \geqslant 0$. Summing up leads to the following remark.

**4.2. Remark.** Consider the assumptions of Lemma 3.4, and the models in (1) and (2). The additional conditions

$$\|x_0(n) - x_0\|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty$$

and

$$\sup_{i \geqslant 0} \sup_{0 \neq v \in \mathscr{I}_{w_i}} \frac{\|(\mathscr{A}_{w_i}(n) - \mathscr{A}_{w_i})v\|_{\mathscr{H}}}{\|v\|_{\mathscr{H}}} \to 0 \quad \text{as } n \to \infty$$

do not guarantee that

$$\sup_{i \geqslant 0} \|x_i(n) - x_i\|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty.$$

Equivalently, the state sequence generated by the infinite-dimensional model in (1) may not be uniformly approximated by a state sequence generated by a finite-dimensional model as in (2), even under the assumption of uniform structure equiconvergence.

Now we shall give a sufficient condition for uniform structure equiconvergence to imply uniform state convergence. Assume that $\{w_i \in \mathscr{H}; i \geqslant 0\}$ is correlation dominated (so that uniform structure equiconvergence holds according to Remark 3.2) and $\{u_i \in \mathscr{H}; i \geqslant 0\}$ is uniformly bounded (i.e., $\sup_{i \geqslant 0}\|u_i\|^2_{\mathscr{H}} = \sup_{i \geqslant 0}\text{tr}(\mathscr{E}\{u_i \circ u_i\}) < \infty$; note that correlation dominance implies uniform boundedness, but the converse is not necessarily true). Set

$$\alpha = \|A_0\| + \sup_{k \geqslant 1}\|A_k\| \sup_{i \geqslant 0}\|w_i\|_{\mathscr{H}},$$

and suppose $\alpha < 1$. From Proposition 3.1 we have

$$\|x_{i+1}\|_{\mathscr{H}} \leqslant \|A_{w_i}x_i\|_{\mathscr{H}} + \|u_i\|_{\mathscr{H}} \leqslant \alpha\|x_i\|_{\mathscr{H}} + \sup_{i \geqslant 0}\|u_i\|_{\mathscr{H}},$$

so that, by induction,

$$\|x_i\|_{\mathscr{H}} \leqslant \alpha^i\|x_0\|_{\mathscr{H}} + \sup_{i \geqslant 0}\|u_i\|_{\mathscr{H}} \sum_{j=0}^{i-1} \alpha^j < \|x_0\|_{\mathscr{H}} + \sup_{i \geqslant 0}\|u_i\|_{\mathscr{H}}(1-\alpha)^{-1} < \infty$$

for every $i \geqslant 1$, whenever $\alpha < 1$. Since $\sup_{i \geqslant 0}\|x_i\|_{\mathscr{H}} < \infty$ we get (cf. the proof of Proposition 3.1 and Remark 3.2)

$$0 \leqslant \gamma_n \overset{\text{def}}{=} \sup_{i \geqslant 0} \left\| \sum_{k=n+1}^{\infty} A_k \langle w_i; e_k \rangle x_i \right\|_{\mathscr{H}}$$
$$\leqslant \sup_{k \geqslant 1}\|A_k\| \sup_{i \geqslant 0}\|x_i\|_{\mathscr{H}} \left( \sum_{k=n+1}^{\infty} \lambda_k \right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\|x_i(n) - x_i\|_{\mathscr{H}} \leqslant \alpha^i\|x_0(n) - x_0\|_{\mathscr{H}} + \gamma_n \sum_{j=0}^{i-1} \alpha^j < \|x_0(n) - x_0\|_{\mathscr{H}} + \gamma_n(1-\alpha)^{-1}$$

for every $i, n \geqslant 1$. Thus we have shown that the following holds.

**4.3. Theorem.** *Consider the assumptions of Lemma* 3.4, *and the models in* (1) *and* (2). *Assume further that* $\{w_i \in \mathscr{H};\ i \geqslant 0\}$ *is correlation dominated and* $\{u_i \in \mathscr{H};\ i \geqslant 0\}$ *is uniformly bounded. If*

$$\|x_0(n) - x_0\|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty$$

*and*

$$\alpha = \|A_0\| + \sup_{k \geqslant 1} \|A_k\| \Big( \sup_{i \geqslant 0} \operatorname{tr}(\mathscr{E}\{w_i \circ w_i\}) \Big)^{1/2} < 1,$$

*then*

$$\sup_{i \geqslant 0} \|x_i(n) - x_i\|_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty.$$

*Equivalently, the state sequence generated by the infinite-dimensional model in* (1) *can be uniformly approximated by state sequences generated by finite-dimensional models as in* (2), *whenever* $\alpha < 1$.

## 5. Concluding remarks

This paper dealt with modelling of infinite-dimensional discrete bilinear systems driven by $H$-valued second-order random sequences. The stochastic environment, under which the system is supposed to operate, was characterized by independence and structural similarity only. No assumption on stationarity was required, and the probability distributions involved were allowed to be arbitrary and unknown. Actually, stationarity was replaced by the less stringent assumption of structural similarity. On the other hand, independence could not be relaxed to uncorrelatedness, as discussed in Remark 2.3. For a comparison of the independence conditions usually assumed in the stochastic bilinear systems literature the reader is referred to [4], where mean-square stability for a particular case within the class of models defined in Lemma 3.4 has been investigated.

The main results of the present paper appeared in Sections 3 and 4. The result of Section 3 was synthesized in Lemma 3.4, which was supported by Propositions 3.1 and 3.3. In Proposition 3.1 it was shown that the sequence of maps $\{\mathscr{A}_{w_i}(n) : \mathscr{I}_{w_i} \to \mathscr{H};\ n \geqslant 1\}$ converges, uniformly on $\mathscr{I}_{w_i} \subset \mathscr{H}$ and pointwise in $i$, for any structurally similar sequence $\{w_i \in \mathscr{H};\ i \geqslant 0\}$. Actually, the convergence also holds uniformly in $i$, whenever $\{w_i \in \mathscr{H};\ i \geqslant 0\}$ is also correlation dominated, as shown in Remark 3.2. The transition properties of the bounded linear-like limiting map $\mathscr{A}_{w_i} : \mathscr{I}_{w_i} \to \mathscr{H}$ where derived in Proposition 3.3.

The results of Section 4 are concerned with the reverse problem of approximating the state sequence generated by the infinite-dimensional model defined in Lemma 3.4, by state sequences generated by finite-dimensional models. As one would expect, uniform pointwise structure convergence naturally implies pointwise state convergence (cf. Corollary 4.1), so that pointwise state approximation is straightforward from Proposition 3.1. However, uniform state convergence does not generally follow from uniform structure equiconvergence, as summarized in Remark 4.2. A sufficient condition to ensure uniform state convergence out of uniform structure equiconvergence, so that uniform state approximation holds in that case, was given in Theorem 4.3.

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