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MODELLING OF INFINITE-DIMENSIONAL DISCRETE BILINEAR SYSTEMS IN A STOCHASTIC ENVIRONMENT*

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Abstract: Let $\mathcal{H}$ be the Hilbert space of all second order H-valued random variables, where H is a nontrivial separable Hilbert space. In this paper it is shown that infinite-dimensional discrete-time bilinear models driven by H-valued random sequences can be rigorously defined as the uniform limit in $\mathcal{H}$ of finite-dimensional bilinear models. Existence and uniqueness of solutions for such infinite-dimensional models can be established by assuming only independence and structural similarity for the stochastic environment under consideration.

1. INTRODUCTION

One has noticed lately a remarkable research effort towards an important subclass of nonlinear dynamical systems, namely bilinear systems (e.g. see the surveys in [1],[6] and [7]). The major part of the available literature is related to determistic, continuous-time, and finite-dimensional models; although some important contributions outside the above category have already been published (e.g. see the references in [3] and [5]). Here, we shall be considering discrete-time bilinear system operating in a stochastic environment, whose model is formally given by the following difference equation.

$$x_{i+1} = [A_0 + \sum_{k\geq 1} A_k<w_i;e_k>]x_i + u_i \quad ,$$

where $\{A_k;\ k\geq 0\}$ is a sequence of bounded linear operators on some separable Hilbert space H, $\{e_k;\ k\geq 1\}$ is an orthonormal basis for H, and $\{u_i;\ i\geq 0\}$, $\{w_i;\ i\geq 0\}$ and $\{x_i;\ i\geq 0\}$ are H-valued random sequences. If H is finite-dimensional, then such a model characterizes precisely a finite-dimensional stochastic discrete bilinear system, since the above series is finite. The purpose of the present paper is to give a rigorous definition of the above stochastic discrete bilinear model, when H is infinite-dimensional. It will be shown that the infinite-dimensional model under consideration can be properly defined as the uniform limit of finite-dimensional models on the Hilbert space $\mathcal{H}$ of all second order H-valued random variables. The existence and uniqueness of solutions for such a limiting model will also be established.

2. NOTATIONAL AND CONCEPTUAL PRELIMINARIES

We shall assume throughout this paper that H is a separable nontrivial Hilbert space. $\|\ \|$ and $<;>$ will stand for norm and inner product in H, respectively. Let B[H] denote the Banach algebra of all bounded linear transformations of H into itself. We shall use the same symbol $\|\ \|$ to denote the uniform induced norm in B[H]. Let $T^*\in B[H]$ be the adjoint of $T\in B[H]$, and set $B[H]^+=\{T\in B[H]:\ 0\leq T=T^*\}$, the closed convex cone of all self-adjoint nonnegative (i.e. $0\leq <Th;h>\ \forall h\in H$) operators on H. For $T\in B[H]^+$ we define the trace of T as usual.

$$tr(T) \overset{\text{def.}}{=} \sum_{k=1}^{\infty} <Te_k;e_k> = \sum_{k=1}^{\infty} \lambda_k \quad ,$$

where $\{e_k;\ k\geq 1\}$ is any orthonormal basis for H, and $\{\lambda_k\geq 0;\ k\geq 1\}$ is the set of all eigenvalues of T, each of them counted according to its multiplicity. Let $B_1[H]^+=\{T\in B[H]^+: tr(T)<\infty\}$ denote the class of all nonnegative nuclear operators on H. For a brief presentation on nuclear (or trace-class) operators, the reader is referred to [10].

Let (Ω,Σ,μ) be a probability space, where Σ is a σ-algebra of subsets of a nonempty basic set Ω, and μ is a probability measure on Σ. Let $\mathcal{H}$ be the set of equivalence classes of H-valued measurable maps x defined almost everywhere (a.e.) on Ω, such that

$$\|x\|_{\mathcal{H}}^2 \overset{\text{def.}}{=} \varepsilon\{\|x(\omega)\|^2\} = \int_{\Omega} \|x(\omega)\|^2\, d\mu < \infty \quad ,$$

where ε stands for the expectation operator for scalar-valued random variables. The above is the so-called second order property. Now set the following inner product in $\mathcal{H}$

$$<x;y>_{\mathcal{H}} \overset{\text{def.}}{=} \varepsilon\{<x(\omega);y(\omega)>\} = \int_{\Omega} <x(\omega);y(\omega)>d\mu$$

for all $x,y\in\mathcal{H}$, which induces the above norm in $\mathcal{H}$. Thus $\mathcal{H}=L_2(\Omega,\mu;H)$: *the Hilbert space of all second order H-valued random variables*. For any $x\in\mathcal{H}$ consider the sesquilinear functional $\varepsilon\{<\cdot;x(\omega)>\ <x(\omega);\cdot>\}:H^2\to\mathbb{C}$, which is bounded. Then (cf. [10, p.120]) there exists a unique operator in $B[H]^+$, say $E\{x\circ x\}$, referred to as the *correlation* of $x\in\mathcal{H}$, such that

$$<E\{x\circ x\}f;g> = \varepsilon\{<f;x(\omega)>\ <x(\omega);g>\} \qquad \forall f,g\in H \quad .$$

Remark 1: It is a simple matter to show that $tr(E\{x\circ x\})=\|x\|_{\mathcal{H}}^2$ for any $x\in\mathcal{H}$, so that $E\{x\circ x\}\in B_1[H]^+$ for every $x\in\mathcal{H}$. Moreover, it is also easy to show that $\|Tx\|_{\mathcal{H}}\leq\|T\|\ \|x\|_{\mathcal{H}}$ for every $x\in\mathcal{H}$ and $T\in B[H]$.

Now consider a family $\{x_\xi\in\mathcal{H};\ \xi\in\Xi\neq\emptyset\}$ of random variables. For each $\xi\in\Xi$ let $\{e_{\xi,k};\ k\geq 1\}$ be an orthonormal basis for H made up of all eigenvectors of $E\{x_\xi\circ x_\xi\}\in B_1[H]^+$, whose existence is ensured by the Spectral Theorem (e.g. see [8, p.460]). Such a family is said to be *structurally similar* if there exists an orthonormal basis for H, say $\{e_k;\ k\geq 1\}$, such that $\{e_{\xi,k};\ k\geq 1\}=\{e_k;\ k\geq 1\}$ for every $\xi\in\Xi$. $\{e_k;\ k\geq 1\}$ is referred to as the *common orthonormal basis* for H of $\{x_\xi\in\mathcal{H};\ \xi\in\Xi\}$. For any family $\{x_\xi\in\mathcal{H};\ \xi\in\Xi\neq\emptyset\}$ we set

$$\mathcal{I}_{\{x_\xi;\ \xi\in\Xi\}} = \{y\in\mathcal{H}:\ y \text{ is independent of } \{x_\xi\in\mathcal{H};\ \xi\in\Xi\}\} \ .$$

In particular, for any $x\in\mathcal{H}$, $\mathcal{I}_x=\{y\in\mathcal{H}: y \text{ is independent } x\in\mathcal{H}\}$.

Remark 2: Note that $y\in\mathcal{I}_x \iff x\in\mathcal{I}_y$. The following well-known independence properties (e.g. see [9]) will be needed in the sequel.

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(a) If $x\in I_y$ then, for every measurable functionals $\phi,\psi:H\to\mathbb{C}$,

$$\varepsilon\{\phi[x(\omega)]\ \psi[y(\omega)]\} = \varepsilon\{\phi[x(\omega)]\}\ \varepsilon\{\psi[y(\omega)]\}\ .$$

(b) If $\{y_\upsilon\in H;\ \upsilon\in T\neq\emptyset\}$ is independent of $\{x_\xi\in H;\ \xi\in\Xi\neq\emptyset\}$ then, for any finite subset $\{x_{\xi_k};\ 1\le k\le n\}$ of $\{x_\xi;\ \xi\in\Xi\}$,

$$N(x_{\xi_1},\ldots,x_{\xi_n}) \in I_{\{y_\upsilon;\ \upsilon\in T\}}$$

for every measurable map $N:H^n\to H$.

3. INFINITE-DIMENSIONAL STOCHASTIC BILINEAR MODEL

Consider the infinite-dimensional stochastic discrete bilinear model that has been formally introduced in section 1. The purpose of this section is to give a rigorous definition for such a model. This will be achieved in Lemma 1 below. We begin by establishing two auxiliary results that will suffice our needs.

Proposition 1: Let $\{w_\xi\in H;\ \xi\in\Xi\neq\emptyset\}$ be a structurally similar family with a common orthonormal basis $\{e_k;\ k\ge1\}$ for H. For each $\xi\in\Xi$ and for every $n\ge1$, set

$$A_{w_\xi}(n) = A_0 + \sum_{k=1}^{n} A_k\langle w_\xi;e_k\rangle : I_{w_\xi} \to H\ ,$$

where $\{A_k\in B[H];\ k\ge0\}$ is an uniformly bounded sequence of operators. We claim that, for each $\xi\in\Xi$, the sequence of maps $\{A_{w_\xi}(n):I_{w_\xi}\to H;\ n\ge1\}$ converges uniformly, or equivalently, for each $\xi\in\Xi$ there exists a map $A_{w_\xi}:I_{w_\xi}\to H$ such that

$$\sup_{0\neq v\in I_{w_\xi}} \frac{\| A_{w_\xi}(n)v - A_{w_\xi}v\|_H}{\| v\|_H} \to 0 \quad\text{as}\quad n\to\infty\ .$$

Such a map has the following properties.

$$\| A_{w_\xi}v\|_H \le (\| A_0\| + \sup_{k\ge1}\| A_k\|\ \ \| w_\xi\|_H)\| v\|_H\ ,$$

$$A_{w_\xi}\alpha v = \alpha A_{w_\xi}v\ ,$$

for every $\alpha\in\mathbb{C}$ and $v\in I_{w_\xi}$, and for each $i\ge1$

$$A_{w_\xi}(\textstyle\sum_{j=0}^{i} v_j) = \sum_{j=0}^{i} A_{w_\xi}v_j$$

whenever $v_j\in I_{w_\xi}$ for every $j=0,1,\ldots,i$ and $(\sum_{j=0}^{k}v_j)\in I_{w_\xi}$ for every $k=1,\ldots,i$, which happens whenever $w_\xi\in I_{\{v_0,v_1,\ldots,v_i\}}$ according to Remark 2(b).

Proof: For each $\xi\in\Xi$ we get from Remark 2(a), and according to the definition of the correlation operator, that

$$\langle A_k\langle w_\xi;e_k\rangle v\ ;\ A_\ell\langle w_\xi;e_\ell\rangle v\rangle_H = \langle E\{w_\xi o w_\xi\}e_\ell;e_k\rangle\ \langle A_k v;A_\ell v\rangle_H$$

for every $k,\ell\ge1$, whenever $v\in I_{w_\xi}$. Moreover, since $\{w_\xi\in H;\ \xi\in\Xi\}$ is structurally similar,

$$E\{w_\xi o w_\xi\}e_k = \lambda_{\xi,k}e_k,\ \sum_{k=1}^{\infty}\lambda_{\xi,k} = tr(E\{w_\xi o w_\xi\}) = \| w_\xi\|_H^2 < \infty$$

for each $\xi\in\Xi$ and every $k\ge1$, where $\lambda_{\xi,k}\ge0$ is the eigenvalue of $E\{w_\xi o w_\xi\}\in B_1[H]^+$ associated with the common eigenvector e_k for each $k\ge1$. Hence, for each $\xi\in\Xi$,

$$\| \sum_{k=m}^{p} A_k\langle w_\xi;e_k\rangle v\|_H^2 = \sum_{k,\ell=m}^{p}\langle A_k\langle w_\xi;e_k\rangle v\ ;\ A_\ell\langle w_\xi;e_\ell\rangle v\rangle_H$$

$$=\sum_{k,\ell=m}^{p}\lambda_{\xi,\ell}\langle e_\ell;e_k\rangle\ \langle A_k v;A_\ell v\rangle_H \le \sup_{m\le k\le p}\| A_k\|^2\| v\|_H^2 \sum_{k=m}^{p}\lambda_{\xi,k}$$

for all $v\in I_{w_\xi}$ and for any $1\le m\le p$, according to Remark 1. Therefore,

$$\sup_{\upsilon\ge1}\ \sup_{0\neq v\in I_{w_\xi}} \frac{\| A_{w_\xi}(n+\upsilon)v - A_{w_\xi}(n)v\|_H}{\| v\|_H} \to 0 \quad\text{as}\quad n\to\infty\ ,$$

so that uniform convergence follows for each $\xi\in\Xi$. The remaining bounded linear-like properties of the map A_{w_ξ} (which just fails to be a bounded linear one because its domain I_{w_ξ} is not a linear subspace of H) are easily verified. Boundedness and homogeneity are trivial. The additivity property can be established by induction (cf. [4]).

Proposition 2: Consider a sequence $\{v_i\in H;\ i\ge0\}$. Let $\{w_i\in H;\ i\ge0\}$ be a structurally similar sequence such that $w_0\in I_{v_0}$ and

$$w_j \in I_{\{w_0,\ldots,w_{j-1},v_0,v_1,\ldots,v_j\}}$$

for every $j\ge1$. Then

$$v_i \in I_{w_i}\ ,\quad A_{w_{i-1}}\cdots A_{w_j}v_j \in I_{w_i} \qquad \forall j=0,\ldots,i-1\ ,$$

$$\sum_{j=0}^{k}A_{w_{i-1}}\cdots A_{w_j}v_j \in I_{w_i} \qquad \forall k=0,\ldots,i-1\ ,$$

$$\sum_{j=0}^{i-1}A_{w_{i-1}}\cdots A_{w_j}v_j + v_i \in I_{w_i}\ ,$$

$$A_{w_i}[\textstyle\sum_{j=0}^{i-1}A_{w_{i-1}}\cdots A_{w_j}v_j + v_i] = \sum_{j=0}^{i}A_{w_i}\cdots A_{w_j}v_j\ ,$$

for every $i\ge1$, with $A_{w_j}:I_{w_j}\to H$ defined as in Proposition 1.

Proof: By Remark 2(b) and Proposition 1 it is a simple matter to show that the desired result holds for $i=1$. Now suppose it holds for some $i\ge1$. Hence, according to Remark 2(b),

$$w_{i+1} \in I_{\{w_0,\ldots,w_i,v_0,v_1,\ldots,v_{i+1}\}} \Longrightarrow$$

$$\Longrightarrow \begin{cases} v_{i+1} \in I_{w_{i+1}}\ ,\ A_{w_i}\cdots A_{w_j}v_j \in I_{w_{i+1}} & \forall j=0,\ldots,i\ ,\\ \sum_{j=0}^{k}A_{w_i}\cdots A_{w_j}v_j \in I_{w_{i+1}} & \forall k=0,\ldots,i\ ,\\ \sum_{j=0}^{i}A_{w_i}\cdots A_{w_j}v_j + v_{i+1} \in I_{w_{i+1}}\ . \end{cases}$$

Then, by the additivity property of $A_{w_{i+1}}$ in Proposition 1,

$$A_{w_{i+1}}[\textstyle\sum_{j=0}^{i}A_{w_i}\cdots A_{w_j}v_j + v_{i+1}] = \sum_{j=0}^{i+1}A_{w_{i+1}}\cdots A_{w_j}v_j\ .$$

Thus the result holds for $i+1$, which concludes the proof by induction.

Lemma 1: Let $\{w_i\in H;\ i\ge0\}$ be a structurally similar sequence with a common orthonormal basis $\{e_k;\ k\ge1\}$ for H. Set

$$A_{w_i} = A_0 + \sum_{k=1}^{\infty}A_k\langle w_i;e_k\rangle : I_{w_i} \to H$$

for every $i\ge0$, as defined in Proposition 1, where $\{A_k\in B[H];\ k\ge0\}$ is uniformly bounded. Given $x_0\in H$ and $\{u_i\in H;\ i\ge0\}$, assume further that $w_0\in I_{x_0}$ and

$$w_j \in I_{\{x_0,u_0,\ldots,u_{j-1},w_0,\ldots,w_{j-1}\}}$$

for every $j\geq 1$. Then the difference equation in H

$$x_{i+1} = [A_0 + \sum_{k=1}^{\infty} A_k\langle w_i;e_k\rangle]x_i + u_i$$

has a unique solution, which lies in I_{w_i} for every $i\geq 0$, given by $x_1=A_{w_0}x_0+u_0$ and, for every $i\geq 2$,

$$x_i = A_{w_{i-1}}\cdots A_{w_0}x_0 + \sum_{j=1}^{i-1} A_{w_{i-1}}\cdots A_{w_j}u_{j-1} + u_{i-1} \quad .$$

Proof: Set $v_0=x_0$ and $v_{j+1}=u_j$ for every $j\geq 0$, such that $w_0\in I_{v_0}$ and

$$w_j \in I_{\{w_0,\ldots,w_{j-1},v_0,v_1,\ldots,v_j\}}$$

for every $j\geq 1$. Then, by Proposition 2,

$$\text{(a)} \quad \begin{cases} \sum_{j=0}^{i-1} A_{w_{i-1}}\cdots A_{w_j}v_j + v_i \in I_{w_i} \\ A_{w_i}\left[\sum_{j=0}^{i-1} A_{w_{i-1}}\cdots A_{w_j}v_j + v_i\right] = \sum_{j=0}^{i} A_{w_i}\cdots A_{w_j}v_j \end{cases}$$

for every $i\geq 1$. Therefore, by setting

$$\text{(b)} \quad x_i = \sum_{j=0}^{i-1} A_{w_{i-1}}\cdots A_{w_j}v_j + v_i \in I_{w_i}$$

for every $i\geq 1$, we get

$$\text{(c)} \quad x_{i+1} = A_{w_i}x_i + v_{i+1} \quad , \quad x_0=v_0 \quad ,$$

for every $i\geq 0$. On the other hand, if $\{x_i\in H;\ i\geq 0\}$ solves the difference equation (c) then, from (a) it is readily verified by induction that (b) holds true for every $i\geq 1$.

4. CONCLUDING REMARKS

This paper dealt with modelling of infinite-dimensional discrete bilinear systems driven by H-valued second order random sequences. The stochastic environment, under which the system is supposed to operate, was characterized by independence and structural similarity only. No assumption on stationarity was required, and the probability distributions involved were allowed to be arbitrary and unknown.

The main results were presented in Lemma 1, which were supported by Propositions 1 and 2. In Proposition 1 it was established the existence of the bounded linear-like map $A_{w_i}:I_{w_i}\to H$ by uniform convergence arguments. Its transition properties were derived in Proposition 2. Such a map plays a fundamental role in bilinear modelling, since it characterizes the multiplicative action of the input over the state. Existence and uniqueness of solutions for infinite-dimensional stochastic discrete bilinear models were established in Lemma 1.

Finally, it is worth remarking on the independence conditions assumed so far. Let $x_0\in H$, $\{u_i\in H;\ i\geq 0\}$ and $\{w_i\in H;\ i\geq 0\}$ be the random disturbances involved in Lemma 1, and consider the following conditions:

(1) $x_0\in I_{\{(u_i,w_i);\,i\geq 0\}}$, and $\{(u_i,w_i);\ i\geq 0\}$ is an independent sequence in H^2.

(2) $u_i(\omega)=w_i(\omega)$ for all $\omega\in\Omega$ and every $i\geq 0$, and $\{x_0,w_i;\ i\geq 0\}$ is an independent sequence.

(3) $x_0\in I_{\{u_i,w_i;\ i\geq 0\}}$, and $\{u_i;\ i\geq 0\}$ and $\{w_i;\ i\geq 0\}$ are independent sequences, which are independent of each other.

(4) $x_0\in I_{\{u_i,w_i;\ i\geq 0\}}$, $\{u_i;\ i\geq 0\}$ and $\{w_i,\ i\geq 0\}$ are independent sequences, and $\{u_j,w_j\}$ is independent of $\{u_i,w_i;\ j\neq i\geq 0\}$ for every $j\geq 0$.

(5) $x_0\in I_{\{u_0,w_0\}}$, and $\{u_j,w_j\}$ is independent of $\{x_0,u_0,\ldots,u_{j-1},w_0,\ldots,w_{j-1}\}$ for every $j\geq 0$.

It is a simple matter to show that

$$\left.\begin{matrix}(1)\\(2)\\(3)\end{matrix}\right\} \Longrightarrow (4) \Longrightarrow (5) \quad .$$

Note that condition (5) is certainly stronger than what is actually needed to ensure the results in Lemma 1. However, condition (5) may look somewhat artificial, so that the stronger condition in (4) is sometimes assumed in the related literature (e.g. see [3]). Indeed the even stronger conditions (1), (2) and (3), which may eventually be more appealing, are very often assumed for modelling stochastic bilinear systems (e.g. see [5],[11] and [2], respectively). It is also worth remarking that conditions (2) and (3) represent rather different situations, which turn out to suffice our modelling purposes.

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