IMA Journal of Mathematical Control & Information (1997) **14**, 385–399

Quadratic optimal control for discrete-time infinite-dimensional stochastic bilinear systems

O. L. V. COSTA

Department of Electronic Engineering, University of São Paulo, 05508 900 São Paulo SP, Brazil

C. S. KUBRUSLY

Catholic University PUC/RJ, and *National Laboratory for Scientific Computation (LNCC), 22290 160 Rio de Janeiro RJ, Brazil*

[Received 5 October 1995]

In this paper, we consider the class of infinite-dimensional discrete-time linear systems with multiplicative random disturbances (i.e. with the state multiplied by a random sequence), also known as stochastic bilinear systems. We formulate and solve the quadratic optimal-control problem for this class of systems subject to an arbitrary additive stochastic ℓ_2 input disturbance. Under assumptions that guarantee the existence of a solution to an algebraic Riccati-like operator equation (derived previously by the authors), we characterize a bounded linear operator that takes the additive stochastic ℓ_2 input disturbance and the initial condition into the optimal control law. Such a result generalizes, to the infinite-dimensional bilinear stochastic case, some known results for the deterministic linear case.

1. Introduction

Linear systems with multiplicative random disturbances (i.e. with the state multiplied by a random sequence), also known as stochastic bilinear systems, comprise an important subclass of stochastic systems which have lately received a great deal of attention. This is due, at least partly, to the various areas of application like, for instance, in population models, nuclear fission and heat transfer, immunology, etc. (e.g. see Refs 10 and 11; for further references see Ref. 7).

Quadratic optimal-control and H_∞-control problems, and their associated algebraic Riccati-like operator equations, for infinite-dimensional discrete bilinear systems operating in a stochastic environment have been recently considered [2, 3]. These generally mirror their linear counterparts (see e.g. Refs 4, 5, and 12; as a matter of fact, the results in Ref. 12 also reach bilinear models). Conditions for the existence and uniqueness of a solution to an algebraic Riccati-like operator equation were obtained in Ref. 2, with a view to solving the quadratic optimal-feedback-control problem for infinite-dimensional bilinear stochastic systems, for a class of independent zero-mean additive input disturbances. In the present paper, we consider the quadratic optimal-control problem (not necessarily in a feedback form) for an infinite-dimensional discrete-time stochastic bilinear system subject to an arbitrarily additive stochastic ℓ_2 input disturbance. Using the algebraic Riccati-like operator equation obtained in Ref. 2, we characterize a bounded linear operator that takes the additive stochastic ℓ_2 input disturbance and the initial condition into the optimal control law, which generalizes known results for the linear case (cf. Ref. 5 for the continuous-time case).

The present work is organized in the following way. In Section 2 we set out the notation that will be used throughout the paper, and Section 3 contains the basic assumptions upon which the model will be built. The model under consideration is described in Section 4. The main theorem is stated at the end of Section 5, where a bounded linear operator that takes the additive stochastic ℓ_2 input disturbance and initial condition into the optimal control law is established. Again this mirrors its linear counterpart [5]. An example for the particular case of independent zero-mean additive input disturbances is presented in Section 6, and the results compared with those obtained in Ref. 2.

2. Notation

Let $\mathcal{X}$ and $\mathcal{X}'$ be Banach spaces, and denote by $\mathrm{B}[\mathcal{X},\mathcal{X}']$ the Banach space of all bounded linear maps from $\mathcal{X}$ to $\mathcal{X}'$; for simplicity, we set $\mathrm{B}[\mathcal{X}] = \mathrm{B}[\mathcal{X},\mathcal{X}]$. We denote by $\mathrm{G}[\mathcal{X}]$ the group of all invertible operators in $\mathrm{B}[\mathcal{X}]$. The norms in $\mathcal{X}$ and $\mathcal{X}'$ and the induced uniform norm in $\mathrm{B}[\mathcal{X},\mathcal{X}']$ will all be denoted by $\|\cdot\|$, and $\mathrm{r}(\cdot)$ will stand for the spectral radius in the Banach algebra $\mathrm{B}[\mathcal{X}]$. For any nontrivial complex Hilbert space $\mathcal{H}$, we shall denote by $\langle\cdot\,;\cdot\rangle$ the inner product in $\mathcal{H}$ (we write $\langle\cdot\,;\cdot\rangle_{\mathcal{H}}$ with norm $\|\cdot\|_{\mathcal{H}}$ if $\mathcal{H}$ is a probabilistic space), and an asterisk will stand for adjoint as usual. Let $\mathrm{B}^+[\mathcal{H}]$ be the weakly closed convex cone of all self-adjoint nonnegative operators in $\mathrm{B}[\mathcal{H}]$, and set $\mathrm{G}^+[\mathcal{H}] = \mathrm{B}^+[\mathcal{H}] \cap \mathrm{G}[\mathcal{H}]$. Let $\mathrm{B}_1[\mathcal{H}]$ denote the class of all nuclear operators from $\mathrm{B}[\mathcal{H}]$ (see e.g. Refs 2, 3, 8, or 9) and set $\mathrm{B}_1^+[\mathcal{H}] = \mathrm{B}_1[\mathcal{H}] \cap \mathrm{B}^+[\mathcal{H}]$. Let $\ell_2(\mathcal{H}) \subset \oplus_{k=0}^{\infty}\mathcal{H}$ be the Hilbert space made up of all sequences $\{x_k \in \mathcal{H}; k \geqslant 0\}$ such that $\sum_{k=0}^{\infty}\|x_k\|^2 < \infty$.

Let (Ω,Σ,μ) be a probability space, where Σ is a sigmafield of subsets of a nonempty set Ω and μ a probability measure on Σ. Let $\mathscr{H} = \mathrm{L}_2(\Omega,\Sigma,\mu,\mathcal{H})$ denote the Hilbert space of all second-order $\mathcal{H}$-valued random variables with inner product given by $\langle x;y\rangle_{\mathscr{H}} = \mathrm{E}(\langle x;y\rangle)$ for all $x,y \in \mathscr{H}$, where E stands for the expectation of the underlying scalar-valued random variables. Accordingly, the norm of $x \in \mathscr{H}$ is given by $\|x\|_{\mathscr{H}} = \mathrm{E}(\|x\|^2)^{\frac{1}{2}}$. For any $x,y \in \mathscr{H}$, the expectation and correlation operators will be denoted by $\mathrm{E}'(x) \in \mathcal{H}$ and $\mathrm{E}'(x \circ y) \in \mathrm{B}_1[\mathcal{H}]$ respectively (e.g. see Ref. 8), with $\mathrm{E}'(x\circ x) \in \mathrm{B}_1^+[\mathcal{H}]$; they are uniquely defined by the formulae $\langle \mathrm{E}'(x);z\rangle = \mathrm{E}(\langle x;z\rangle)$ and $\langle \mathrm{E}'(x\circ y)w;z\rangle = \mathrm{E}(\langle w;y\rangle\langle x;z\rangle)$ for every w and z in $\mathscr{H}$. For any subsigmafield $\Sigma' \subseteq \Sigma$, the conditional expectation of $x \in \mathscr{H}$ will be denoted by $\mathrm{E}'(x\,|\,\Sigma') \in \mathscr{H}$, and the conditional expectation of the underlying scalar-valued random variable by $\mathrm{E}(\cdot\,|\,\Sigma')$. As usual, $\mathrm{E}'(x\,|\,\Sigma')$ is uniquely defined by the formula $\langle \mathrm{E}'(x\,|\,\Sigma');g\rangle = \mathrm{E}(\langle x;g\rangle\,|\,\Sigma')$ for all $g \in \mathcal{H}$. For any family $\{x_\iota \in \mathscr{H}; \iota \in \Phi \neq \varnothing\}$ set

$$\mathscr{I}_{\{x_\iota;\iota\in\Phi\}} = \{y \in \mathscr{H} : y \text{ is independent of } \{x_\iota \in \mathscr{H}; \iota \in \Phi\}\}.$$

Finally, for the product of operators $X_1,\ldots,X_n$, we use the operating-order convention: $\Pi_{k=1}^{n}X_k = X_n\Pi_{k=1}^{n-1}X_k$ $(n = 1,2,\ldots)$, with $\Pi_{k=i}^{j}X_k := I$ for $j < i$.

3. Assumptions

Throughout this paper, $\mathcal{H}, \mathcal{H}', \mathcal{H}''$, and $\mathcal{H}'''$ will stand for separable complex Hilbert spaces. Set $\mathscr{H} = \mathrm{L}_2(\Omega, \Sigma, \mu, \mathcal{H})$, $\mathscr{H}' = \mathrm{L}_2(\Omega, \Sigma, \mu, \mathcal{H}')$, $\mathscr{H}'' = \mathrm{L}_2(\Omega, \Sigma, \mu, \mathcal{H}'')$, and $\mathscr{H}''' = \mathrm{L}_2(\Omega, \Sigma, \mu, \mathcal{H}''')$, where (Ω, Σ, μ) is the underlying probability space. We assume that $\{w_i \in \mathscr{H}; i \geqslant 0\}$ is a stationary independent random sequence with expected value and correlation operator denoted by $s \in \mathcal{H}$ and $S \in \mathrm{B}_1^+[\mathcal{H}]$ respectively, and set $C = (S - s \circ s) \in \mathrm{B}_1^+[\mathcal{H}]$. On the probability space (Ω, Σ, μ), we consider a nondecreasing family of subsigmafields $\Sigma_n \subseteq \Sigma$ $(n = 0, 1, \ldots)$ such that the following properties are satisfied.

(P1) w_l is independent of Σ_n (that is, the sigmafield generated by w_l is independent of Σ_n) for all $l \geqslant n$,
(P2) w_n is Σ_{n+1}-measurable.

Set

$$\mathscr{X} = \{x = (x_0, x_1, \ldots) \in \ell_2(\mathscr{H}) : x_k \in \mathrm{L}_2(\Omega, \Sigma_k, \mu, \mathcal{H}) \ \forall k \geqslant 0\}.$$

It can be verified that $\mathscr{X}$ is a closed linear subspace of $\ell_2(\mathscr{H})$ and therefore a Hilbert space. In a similar way, we define the Hilbert spaces $\mathscr{V} \subset \ell_2(\mathscr{H}')$, $\mathscr{U} \subset \ell_2(\mathscr{H}'')$, and $\mathscr{Z} \subset \ell_2(\mathscr{H}''')$ by replacing $\mathcal{H}$ and $\mathscr{H}$ in the definition of $\mathscr{X}$ by $\mathcal{H}'$ and $\mathscr{H}'$, $\mathcal{H}''$ and $\mathscr{H}''$, and $\mathcal{H}'''$ and $\mathscr{H}'''$, respectively. It is easy to verify that

$$w_j \in \mathscr{I}_{\{x_0, \ldots, x_i; v_0, \ldots, v_i; u_0, \ldots, u_i; z_0, \ldots, z_i; w_0, \ldots, w_{i-1}\}} \quad \text{for all } j \geqslant i.$$

Notice that, if $v = (v_0, v_1, \ldots) \in \mathscr{V}$, then v_i may not be independent of past states x_k $(k \leqslant i)$. It has been shown in Ref. 3 how one can construct the spaces $\mathscr{X}$, $\mathscr{V}$, $\mathscr{U}$, and $\mathscr{Z}$, and decreasing family of subsigmafields Σ_n, out of a probability space (Ω, Σ, μ) which lead to the above properties.

4. Description of the problem

Consider a discrete-time bilinear system operating in a stochastic environment, whose model is given by the following infinite-dimensional difference equation:

$$x_{i+1} = \left(A_0 + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle\right) x_i + Bu_i + Dv_i, \qquad x_0 \in \mathscr{X}_0 := \mathrm{L}_2(\Omega, \Sigma_0, \mu, \mathcal{H}), \quad (1)$$

where $v = (v_0, v_1, \ldots) \in \mathscr{V}$ (the additive input disturbance), $u = (u_0, u_1, \ldots) \in \mathscr{U}$ (the additive control sequence), $\{w_i \in \mathscr{H}; i \geqslant 0\}$ is the multiplicative input sequence, $\{x_i \in \mathscr{H}; i \geqslant 0\}$ is the state sequence, $\{A_k \in \mathrm{B}[\mathcal{H}]; k \geqslant 0\}$ is a bounded sequence of operators, $D \in \mathrm{B}[\mathcal{H}', \mathcal{H}]$, $\mathrm{B} \in \mathrm{B}[\mathcal{H}'', \mathcal{H}]$, and $\{e_k; k \geqslant 1\}$ is an orthornormal basis for $\mathcal{H}$ made up of the eigenvectors of $S \in \mathrm{B}_1^+[\mathcal{H}]$. For simplicity, we write

$$\tilde{A}_i = A_0 + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle.$$

Since $x_0 \in \mathscr{X}_0$, $u \in \mathscr{U}$, and $v \in \mathscr{V}$, it follows from (1) and Property P2 above that x_i is Σ_i-measurable for each $i \geqslant 0$, and thus

$$w_0 \in \mathscr{I}_{\{x_0, u_0\}} \quad \text{and} \quad w_i \in \mathscr{I}_{\{x_0; u_0, \ldots, u_i; v_0, \ldots, v_i; w_0, \ldots, w_{i-1}\}}.$$

Set $R_i = \mathrm{E}'(v_i \circ v_i) \in \mathrm{B}_1^+[\mathcal{H}']$ and $Q_i = \mathrm{E}'(x_i \circ x_i) \in \mathrm{B}_1^+[\mathcal{H}]$ for every $i \geqslant 0$. If $u_i = -Kx_i$ for some $K \in \mathrm{B}[H, H'']$, then, by a straightforward modification of Lemma 2 in Ref. 8 and the fact that $w_i \in \mathscr{I}_{\{x_i, v_i\}}$, we can show that the state correlation sequence evolves as follows:

$$Q_{i+1} = \boldsymbol{F}_{BK}(Q_i) + \mathrm{E}'(F_{BK}x_i \circ Dv_i) + \mathrm{E}'(F_{BK}x_i \circ Dv_i)^* + DR_iD^*.$$

Here $\boldsymbol{F}_{BK}$ and F_{BK} are operators in $\mathrm{B}[\mathrm{B}[H]]$ and $\mathrm{B}[H]$, respectively, defined as

$$\boldsymbol{F}_{BK}(P) = F_{BK}PF_{BK}^* + \boldsymbol{T}(P) \quad \forall P \in \mathrm{B}[\mathcal{H}],$$

$$F_{BK} = (A_0 - BK) + \sum_{k=1}^{\infty} \langle s; e_k \rangle A_k \in \mathrm{B}[\mathcal{H}],$$

with $\boldsymbol{T} \in \mathrm{B}[\mathrm{B}[H]]$ given by

$$\boldsymbol{T}(P) = \sum_{k,l=1}^{\infty} \langle Ce_k; e_l \rangle A_k P A_l^* \quad \forall P \in \mathrm{B}[\mathcal{H}],$$

Associated with $\boldsymbol{T}$ and $\boldsymbol{F}_{BK}$, we set $\boldsymbol{T}^{\#} \in \mathrm{B}[\mathrm{B}[\mathcal{H}]]$ and $\boldsymbol{F}_{BK}^{\#} \in \mathrm{B}[\mathrm{B}[\mathcal{H}]]$ as follows: for all $P \in \mathrm{B}[\mathcal{H}]$,

$$\boldsymbol{T}^{\#}(P) = \sum_{k,l=1}^{\infty} \langle Ce_k; e_l \rangle A_l^* P A_k, \qquad \boldsymbol{F}_{BK}^{\#}(P) = F_{BK}^* P F_{BK} + \boldsymbol{T}^{\#}(P).$$

Moreover, set $\boldsymbol{F}^{\#} = \boldsymbol{F}_0^{\#}$, $\boldsymbol{F} = \boldsymbol{F}_0$, and $F = F_0$, and define

$$\Gamma_B = \{K \in \mathrm{B}[\mathcal{H}, \mathcal{H}'']: \ \mathrm{r}(\boldsymbol{F}_{BK}^{\#}) < 1\}.$$

Let $M \in \mathrm{B}^+[\mathcal{H}]$ and $N \in \mathrm{G}^+[\mathcal{H}]$, and take $\boldsymbol{v} = (v_0, v_1, \ldots) \in \mathscr{V}$ arbitrarily fixed. Set $\hat{\mathscr{U}} = \{\boldsymbol{u} = (u_0, u_1, \ldots) \in \mathscr{U} : \boldsymbol{x} = (x_0, x_1, \ldots) \in \mathscr{X}$, where $\boldsymbol{x}$ is given by (1)$\}$ (so that $\|\boldsymbol{x}\|_{\mathscr{X}} < \infty$ whenever $\boldsymbol{u} \in \hat{\mathscr{U}}$). For any $\boldsymbol{u} = (u_0, u_1, \ldots) \in \hat{\mathscr{U}}$ and $x_0 \in \mathscr{X}_0$, set

$$J(x_0, \boldsymbol{u}) = \sum_{i=0}^{\infty} (\|M^{\frac{1}{2}}x_i\|_{\mathscr{X}}^2 + \|N^{\frac{1}{2}}u_i\|_{\mathscr{X}''}^2) = \sum_{i=0}^{\infty} \mathrm{E}(\langle Mx_i; x_i \rangle + \langle Nu_i; u_i \rangle). \tag{2}$$

The quadratic stochastic optimal-control problem associated with the above discrete model that we shall be addressing in this paper is that of finding the control in $\hat{\mathscr{U}}$ that minimizes (2); that is, find $\hat{\boldsymbol{u}} \in \hat{\mathscr{U}}$ such that

$$J(x_0, \hat{\boldsymbol{u}}) = \inf_{\boldsymbol{u} \in \hat{\mathscr{U}}} J(x_0, \boldsymbol{u}). \tag{3}$$

5. Main results

Consider the setup of the previous sections. The following proposition and theorem will be required in the rest of the paper; they were proved in Refs 3 and 2 respectively. The definition below, which was introduced in Ref. 2 is necessary for the theorem statement.

Proposition 1 Consider model (1) with $u_i = -Kx_i$ for some $K \in \mathrm{B}[\mathcal{H}, \mathcal{H}'']$. If $\mathrm{r}(\boldsymbol{F}_{BK}^{\#}) < 1$, then $\boldsymbol{x} = (x_0, x_1, \ldots) \in \mathscr{X}$ for every $\boldsymbol{v} = (v_0, v_1, \ldots) \in \mathscr{V}$ and $x_0 \in \mathscr{X}_0$. Moreover $\|\boldsymbol{x}\|_{\mathscr{X}} \leqslant d(\|x_0\|_{\mathscr{X}_0} + \|y\|_{\mathscr{V}})$ for some nonnegative constant d.

Definition. Take a separable Hilbert space $\mathcal{H}_0$ and $L \in \mathrm{B}[\mathcal{H}, \mathcal{H}_0]$ arbitrary. The pair $(L, \boldsymbol{F})$ is detectable if there exists $\hat{B} \in \mathrm{B}[\mathcal{H}_0, \mathcal{H}]$ such that $\mathrm{r}(\boldsymbol{F}^{\sharp}_{\hat{B}L}) < 1$.

THEOREM 1 If $(M, \boldsymbol{F})$ is detectable and Γ_B is not empty, then there exits a unique $P \in \mathrm{B}^{+}[\mathcal{H}]$ such that

$$M = P - \boldsymbol{F}^{\sharp}(P) + F^{*}PB(N + B^{*}PB)^{-1}B^{*}PF = P - \boldsymbol{F}^{\sharp}_{BK_P}(P) - K_P^{*}K_P, \tag{4}$$

with

$$K_P := (N + B^{*}PB)^{-1}B^{*}PF. \tag{5}$$

Moreover $K_P \in \Gamma_B$. Furthermore, the optimal stabilizing feedback solution to (3) (i.e. the optimal solution to (3) of the form $\{u_i = -Kx_i; i \geqslant 0\}$ over all $K \in \Gamma_B$) is obtained for $K = K_P$, whenever $\boldsymbol{v} \in \mathscr{V}_{w}$. Here $\mathscr{V}_{w} \subseteq \mathscr{V}$ is the class of all zero-mean independent random sequences from $\ell_2(\mathscr{H}')$ that are independent of $\{w_i \in \mathscr{H}; i \geqslant 0\}$ and x_0.

The purpose of the present paper is to extend the final part of Theorem 1 to allow a larger class of additive input disturbances $\boldsymbol{v}$. The apriori constraint of feedback control is dismissed, and the additive disturbance $\boldsymbol{v}$ will be allowed to lie in $\mathscr{V}$ rather than in $\mathscr{V}_{w}$ (see Theorem 2 at the end of this section).

The next five propositions will be required for proving Lemmas 1 and 2 below, from which we shall conclude the final result in Theorem 2. For fixed and arbitrary $\boldsymbol{v} \in \mathscr{V}$, and for P and K_P as in (4)–(5), set

$$r_k = \sum_{j=0}^{\infty} \mathrm{E}'\left(\left[\prod_{l=0}^{j-1}(\tilde{A}_{k+j-l} - BK_P)^{*}\right] PDv_{k+j} \,\middle|\, \Sigma_{k+1}\right) \quad \forall k \geqslant 0. \tag{6}$$

PROPOSITION 2 Suppose that $(M, \boldsymbol{F})$ is detectable and Γ_B is not empty. Then

$$\sum_{k=0}^{\infty} \| r_k \|^2_{\mathscr{H}} \leqslant c \| \boldsymbol{v} \|^2_{\mathscr{V}} \quad \text{for some } c \geqslant 0.$$

Proof. Set

$$\zeta_{k,j} = \mathrm{E}'\left(\left[\prod_{l=0}^{j-1}(\tilde{A}_{k+j-l} - BK_P)^{*}\right] PDv_{k+j} \,\middle|\, \Sigma_{k+1}\right).$$

First we show that

$$\| \zeta_{k,j} \|^2 \leqslant \| \boldsymbol{F}^{\sharp j}_{BK_P} \| \, \mathrm{E}(\| PDv_{k+j} \|^2 \,|\, \Sigma_{k+1}). \tag{7}$$

Indeed, from the fact that $\zeta_{k,j}$ is Σ_{k+1}-measurable, we have from Hölder's inequality that

$$\| \zeta_{k,j} \|^2 = \left\langle \mathrm{E}'\left(\left[\prod_{l=0}^{j-1}(\tilde{A}_{k+j-l} - BK_P)^{*}\right] PDv_{k+j} \,\middle|\, \Sigma_{k+1}\right); \zeta_{k,j} \right\rangle$$

$$= \mathrm{E}\left(\left\langle \left[\prod_{l=0}^{j-1}(\tilde{A}_{k+j-l} - BK_P)^{*}\right] PDv_{k+j} \right\rangle \,\middle|\, \Sigma_{k+1}\right)$$

$$= \mathrm{E}\left(\left\langle PDv_{k+j}; \left[\prod_{l=1}^{j}(\tilde{A}_{k+l} - BK_P)\right]\zeta_{k,j}\right\rangle \middle| \Sigma_{k+1}\right)$$

$$\leqslant \mathrm{E}\left(\|PDv_{k+j}\| \left\| \left[\prod_{l=1}^{j}(\tilde{A}_{k+l} - BK_P)\right]\zeta_{k,j}\right\| \middle| \Sigma_{k+1}\right)$$

$$\leqslant \mathrm{E}(\|PDv_{k+j}\|^2 \mid \Sigma_{k+1})^{\frac{1}{2}} \mathrm{E}\left(\left\| \left[\prod_{l=1}^{j}(\tilde{A}_{k+l} - BK_P)\right]\zeta_{k,j}\right\|^2 \middle| \Sigma_{k+1}\right)^{\frac{1}{2}}.$$

But, recalling that w_l is independent of Σ_{k+1} for every $l \geqslant k+1$ (by Property P1), it follows (cf. the proof of Proposition 2 in Ref. 3), that

$$\mathrm{E}\left(\left\| \left[\prod_{l=1}^{j}(\tilde{A}_{k+l} - BK_P)\right]\zeta_{k,j}\right\|^2 \middle| \Sigma_{k+1}\right) \leqslant \|\boldsymbol{F}_{BK_P}^{*j}\| \|\zeta_{k,j}\|^2.$$

Therefore

$$\|\zeta_{k,j}\| \leqslant \mathrm{E}(\|PDv_{k+j}\|^2 \mid \Sigma_{k+1})^{\frac{1}{2}} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}},$$

which ensures our first claim. From Theorem 1 it follows that $r(\boldsymbol{F}_{BK_P}^{*}) < 1$ (i.e. $K_P \in \Gamma_B$), and hence

$$\sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^p < \infty$$

for every $p > 0$ (cf. Ref. 6). Thus, from (6), (7), and the triangle inequality in $\mathscr{H}$,

$$\|r_k\|_{\mathscr{H}} \leqslant \sum_{j=0}^{\infty} \left\| \mathrm{E}'\left(\left[\prod_{l=0}^{j-1}(\tilde{A}_{k+j-l} - BK_P)^*\right] PDv_{k+j} \middle| \Sigma_{k+1}\right)\right\|_{\mathscr{H}} = \sum_{j=0}^{\infty} \mathrm{E}(\|\zeta_{k,j}\|^2)^{\frac{1}{2}}$$

$$\leqslant \sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}} \mathrm{E}(\mathrm{E}(\|PDv_{k+j}\|^2 \mid \Sigma_{k+1}))^{\frac{1}{2}} = \sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}} \mathrm{E}(\|PDv_{k+j}\|^2)^{\frac{1}{2}}$$

$$= \|PD\| \sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{4}} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{4}} \|v_{k+j}\|_{\mathscr{H}'}$$

$$\leqslant \|PD\| \left(\sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}}\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}} \|v_{k+j}\|_{\mathscr{H}'}^2\right)^{\frac{1}{2}}.$$

However, for arbitrary integer $n \geqslant 0$,

$$\sum_{k=0}^{n}\sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}} \|v_{k+j}\|_{\mathscr{H}'}^2 = \sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}} \sum_{k=0}^{n} \|v_{k+j}\|_{\mathscr{H}'}^2$$

$$\leqslant \sum_{j=0}^{\infty} \|\boldsymbol{F}_{BK_P}^{*j}\|^{\frac{1}{2}} \sum_{m=0}^{\infty} \|v_m\|_{\mathscr{H}'}^2.$$

Therefore,

$$\sum_{k=0}^{\infty}\|r_k\|_{\mathcal{X}}^2 \leqslant \|PD\|^2\left(\sum_{j=0}^{\infty}\|F_{BK_P}^{*j}\|^{\frac{1}{2}}\right)^2\|v\|_{\mathcal{V}}^2. \quad \square$$

Take $v \in \mathcal{V}$ and set $\hat{r}_k = \mathrm{E}'(r_k \mid \Sigma_k)$, with r_k given in (6), for each $k \geqslant 0$. As we shall see in the next proposition, $\hat{r} := (\hat{r}_0, \hat{r}_1, \ldots) \in \mathfrak{X}$. Now consider the transformation $R : \mathcal{V} \to \mathfrak{X}$ such that $Rv = \hat{r}$ for all $v \in \mathcal{V}$.

Proposition 3 Suppose that (M, F) is detectable, and that Γ_B is not empty. Then $R \in \mathrm{B}[\mathcal{V}, \mathfrak{X}]$ and $\|\hat{r}\|_{\mathfrak{X}}^2 \leqslant c\|v\|_{\mathcal{V}}^2$ for c as in Proposition 2 above.

Proof. From linearity of the conditional expectation operator and (6), it is immediate to verify that R is linear. Clearly $\hat{r}_k$ is Σ_k-measurable. Moreover,

$$\begin{aligned}\|\hat{r}_k\|_{\mathcal{X}}^2 &= \mathrm{E}(\langle \mathrm{E}'(r_k \mid \Sigma_k); \hat{r}_k\rangle) = \mathrm{E}(\mathrm{E}\langle r_k; \hat{r}_k\rangle \mid \Sigma_k)) = \mathrm{E}(\langle r_k; \hat{r}_k\rangle)\\ &= \langle r_k; \hat{r}_k\rangle_{\mathcal{X}} \leqslant \|r_k\|_{\mathcal{X}}\|\hat{r}_k\|_{\mathcal{X}'},\end{aligned}$$

and hence $\|\hat{r}_k\|_{\mathcal{X}} \leqslant \|r_k\|_{\mathcal{X}}$. Thus, from Proposition 2, $\|\hat{r}\|_{\mathfrak{X}}^2 \leqslant c\|v\|_{\mathcal{V}}^2$. $\square$

Proposition 4 Suppose that (M, F) is detectable and Γ_B is not empty. Then, for each $k = 0, 1, \ldots$,

$$\sum_{j=0}^{\infty}\mathrm{E}'\left(\left[\prod_{l=0}^{j}(\tilde{A}_{k+j-l} - BK_P)^*\right]PDv_{k+j}\,\middle|\,\Sigma_k\right) = \mathrm{E}'((\tilde{A}_k - BK_P)^* r_k \mid \Sigma_k).$$

Proof. Set

$$\tilde{\zeta}_{k,j} = (\tilde{A}_k - BK_P)^*\zeta_{k,j}$$

with $\zeta_{k,j}$ as in Proposition 2. Then, for any $g \in \mathcal{H}$,

$$\begin{aligned}\sum_{j=0}^{\infty}\mathrm{E}\big(|\langle\tilde{\zeta}_{k,j}; g\rangle|\,|\,\Sigma_k\big) &\leqslant \sum_{j=0}^{\infty}\mathrm{E}(\|\tilde{\zeta}_{k,j}\| \mid \Sigma_k)\|g\|\\ &\leqslant \sum_{j=0}^{\infty}\mathrm{E}(\|\tilde{A}_k - BK_P\|\|\zeta_{k,j}\| \mid \Sigma_k)\|g\|\\ &\leqslant \sum_{j=0}^{\infty}\mathrm{E}(\|\tilde{A}_k - BK_P\|^2 \mid \Sigma_k)^{\frac{1}{2}}\mathrm{E}(\|\zeta_{k,j}\|^2 \mid \Sigma_k)^{\frac{1}{2}}\|g\|\\ &= \mathrm{E}(\|\tilde{A}_k - BK_P\|^2)^{\frac{1}{2}}\|g\|\sum_{j=0}^{\infty}\mathrm{E}(\|\zeta_{k,j}\|^2 \mid \Sigma_k)^{\frac{1}{2}},\end{aligned}$$

where, in the last equality, we have used the fact that w_k is independent of Σ_k (by Property P1), so that $\mathrm{E}(\|\tilde{A}_k - BK_P\|^2 \mid \Sigma_k) = \mathrm{E}(\|\tilde{A}_k - BK_P\|^2)$. Moreover, from the triangle inequality and the proof of Proposition 2,

$$\mathrm{E}\left(\left[\sum_{j=0}^{\infty}\mathrm{E}(\|\zeta_{k,j}\|^2\,|\,\Sigma_k)^{\frac{1}{2}}\right]^2\right)^{\frac{1}{2}} \leqslant \sum_{j=0}^{\infty}\mathrm{E}(\mathrm{E}(\|\zeta_{k,j}\|^2\,|\,\Sigma_k))^{\frac{1}{2}} = \sum_{j=0}^{\infty}\mathrm{E}(\|\zeta_{k,j}\|^2)^{\frac{1}{2}} < \infty.$$

Hence, with probability 1, $\sum_{j=0}^{\infty}\mathrm{E}(\|\zeta_{k,j}\|^2\,|\,\Sigma_k)^{\frac{1}{2}} < \infty$ and thus, with probability 1,

$$\sum_{j=0}^{\infty}\mathrm{E}(|\langle\tilde{\zeta}_{k,j};g\rangle|\,|\,\Sigma_k) < \infty.$$

The above result and the bounded-convergence theorem [1] implies that

$$\sum_{j=0}^{\infty}\mathrm{E}(\langle\tilde{\zeta}_{k,j};g\rangle\,|\,\Sigma_k) = \mathrm{E}\left(\sum_{j=0}^{\infty}\langle\tilde{\zeta}_{k,j};g\rangle\,\middle|\,\Sigma_k\right).$$

The above equality, linearity of the operators $\tilde{A}_k - BK_P$ for each realization of the random variable w_k, and linearity of the inner product in the first argument, lead to

$$\begin{aligned}\sum_{j=0}^{\infty}\mathrm{E}(\langle\tilde{\zeta}_{k,j};g\rangle\,|\,\Sigma_k) &= \mathrm{E}\left(\sum_{j=0}^{\infty}\langle\tilde{\zeta}_{k,j};g\rangle\,\middle|\,\Sigma_k\right) = \mathrm{E}\left(\left\langle\sum_{j=0}^{\infty}\tilde{\zeta}_{k,j};g\right\rangle\,\middle|\,\Sigma_k\right)\\ &= \mathrm{E}(\langle(\tilde{A}_k - BK_P)^*r_k;g\rangle\,|\,\Sigma_k) = \langle\mathrm{E}'((\tilde{A}_k - BK_P)^*r_k\,|\,\Sigma_k);g\rangle.\end{aligned}$$

On the other hand, from linearity of the inner product in the first argument again, and the fact that w_k is Σ_{k+1}-measurable (cf. Property P2), we get,

$$\begin{aligned}\sum_{j=0}^{\infty}\mathrm{E}(\langle\tilde{\zeta}_{k,j};g\rangle\,|\,\Sigma_k) &= \sum_{j=0}^{\infty}\langle\mathrm{E}'(\tilde{\zeta}_{k,j}\,|\,\Sigma_k);g\rangle = \left\langle\sum_{j=0}^{\infty}\mathrm{E}'(\tilde{\zeta}_{k,j}\,|\,\Sigma_k);g\right\rangle\\ &= \left\langle\sum_{j=0}^{\infty}\mathrm{E}'\left(\left[\prod_{l=0}^{j}(\tilde{A}_{k+j-l} - BK_P)^*\right]PDv_{k+j}\,\middle|\,\Sigma_k\right);g\right\rangle.\end{aligned}$$

Since the above identities hold for all $g \in \mathcal{H}$, the desired result follows. □

Next consider the transformation U defined on $\mathscr{X}_0 \oplus \mathscr{V}$ such that $U(x_0, v) = \hat{u} := (\hat{u}_0, \hat{u}_1, \ldots)$ for all $x_0 \in \mathscr{X}_0$ and $v \in \mathscr{V}$, where $(\hat{u}_0, \hat{u}_1, \ldots)$ is recursively defined as follows:

(i) $\hat{x}_0 = x_0$;
(ii) for $k = 0, 1, \ldots$:

$$\eta_{k+1} = P\tilde{A}_k\hat{x}_k + r_k - PB(N + B^*PB)^{-1}B^*(PF\hat{x}_k + \hat{r}_k), \tag{8}$$

$$\hat{u}_k = -N^{-1}B^*\mathrm{E}'(\eta_{k+1}\,|\,\Sigma_k), \tag{9}$$

$$\hat{x}_{k+1} = \tilde{A}_k\hat{x}_k + B\hat{u}_k + Dv_k. \tag{10}$$

LEMMA 1 Suppose that (M, F) is detectable and Γ_B is not empty. Then $U \in \mathrm{B}[\mathscr{X}_0 \oplus \mathscr{V}, \mathscr{U}]$ with range$(U) \subseteq \hat{\mathscr{U}}$. Moreover, for each $k = 0, 1, \ldots$,

$$\hat{u}_k = -[K_P\hat{x}_k + (N + B^*PB)^{-1}B^*\hat{r}_k]. \tag{11}$$

Proof. To verify the identity in (11), proceed as follows. According to the independence of w_k and Σ_k (by Property P1), and recalling that $\hat{x}_k$ is Σ_k-measurable, we obtain, in a similar way as in the proof of Propositions 2 and 3 of Ref. 8, that

$$\mathrm{E}'(\tilde{A}_k\hat{x}_k \mid \Sigma_k) = F\hat{x}_k. \tag{12}$$

Thus, from (5), (8), (9) and (12), and recalling that $\hat{x}_k$ and $\hat{r}_k$ are Σ_k-measurable, we get

$$\begin{aligned}
\hat{u}_k &= -N^{-1}B^*\mathrm{E}'(\eta_{k+1} \mid \Sigma_k)\\
&= -N^{-1}B^*\mathrm{E}'\big(P\tilde{A}_k\hat{x}_k + r_k - PB(N+B^*PB)^{-1}B^*(PF\hat{x}_k+\hat{r}_k) \mid \Sigma_k\big)\\
&= -N^{-1}B^*[P\mathrm{E}'(\tilde{A}_k\hat{x}_k \mid \Sigma_k) + \mathrm{E}(r_k \mid \Sigma_k) - PB(N+B^*PB)^{-1}B^*(PF\hat{x}_k+\hat{r}_k)]\\
&= -N^{-1}[B^*PF\hat{x}_k + B^*\hat{r}_k - B^*PB(N+B^*PB)^{-1}B^*(PF\hat{x}_k+\hat{r}_k)]\\
&= -N^{-1}[I - B^*PB(N+B^*PB)^{-1}]B^*(PF\hat{x}_k+\hat{r}_k)\\
&= -(N+B^*PB)^{-1}B^*(PF\hat{x}_k+\hat{r}_k)\\
&= -[K_P\hat{x}_k + (N+B^*PB)^{-1}B^*\hat{r}_k].
\end{aligned}$$

Now let $V : \mathscr{V} \to \mathscr{X}$ be a transformation such that $V(v) := \hat{v} = (\hat{v}_0, \hat{v}_1, \ldots)$ where

$$\hat{v}_k = -B(N+B^*PB)^{-1}B^*\hat{r}_k + Dv_k.$$

From Proposition 3 it follows that V is linear and $\|\hat{v}\|_{\mathscr{X}} \leqslant d_0\|v\|_{\mathscr{V}}$ for some $d_0 \geqslant 0$, so that $V \in \mathrm{B}[\mathscr{V}, \mathscr{X}]$. From (10) and (11),

$$\hat{x}_{k+1} = (\tilde{A}_k - BK_P)\hat{x}_k + \hat{v}_k; \tag{13}$$

also, from Theorem 1, $\mathrm{r}(\boldsymbol{F}^{\#}_{BK_P}) < 1$ (i.e. $K_P \in \Gamma$). These results and Proposition 1 yield

$$\|\hat{x}\|_{\mathscr{X}} \leqslant d_1(\|x_0\|_{\mathscr{X}_0} + \|v\|_{\mathscr{V}}) \tag{14}$$

for some $d_1 \geqslant 0$. From (11), (14), and Proposition 3, we obtain

$$\|U(x_0, v)\|_{\mathscr{U}} = \left(\sum_{k=0}^{\infty}\mathrm{E}(\|\hat{u}_k\|^2)\right)^{\frac{1}{2}} \leqslant d(\|x_0\|_{\mathscr{X}_0} + \|v\|_{\mathscr{V}})$$

for some $d \geqslant 0$. Linearity of the operator U is immediate from expressions (11) and (13), by linearity of the operators R and V. Moreover $U(x_0, v) \in \hat{\mathscr{U}}$ according to (14), completing the proof of Lemma 1. □

PROPOSITION 5 Suppose that $(M, \boldsymbol{F})$ is detectable and Γ_B is not empty. Then

$$\mathrm{E}'(\tilde{A}_k^*\eta_{k+1} \mid \Sigma_k) = \eta_k - M\hat{x}_k \quad \text{for each } k = 1, 2, \ldots.$$

Proof. From (8) it follows that

$$\begin{aligned}
\mathrm{E}'(\tilde{A}_k^*\eta_{k+1} \mid \Sigma_k) &= \mathrm{E}'\big(\tilde{A}_k[P\tilde{A}_k\hat{x}_k + r_k - PB(N+B^*PB)^{-1}B^*(PF\hat{x}_k+\hat{r}_k)] \mid \Sigma_k\big)\\
&= \mathrm{E}'(\tilde{A}_k^*P\tilde{A}_k\hat{x}_k \mid \Sigma_k) + \mathrm{E}(\tilde{A}_k^*r_k \mid \Sigma_k)\\
&\quad - \mathrm{E}'\big(\tilde{A}_k^*PB(N+B^*PB)^{-1}B^*(PF\hat{x}_k+\hat{r}_k) \mid \Sigma_k\big).
\end{aligned}$$

From the independence of w_k and Σ_k (by Property P1), and recalling that $\hat{x}_k$ and $\hat{r}_k$ are Σ_k-measurable, we get, using the same arguments as in (12) and in Ref. 8, that

$$\mathrm{E}'(\tilde{A}_k^* PB(N + B^*PB)^{-1}B^*(PF\hat{x}_k + \hat{r}_k) \mid \Sigma_k) = F^*PB(N + B^*PB)^{-1}B^*(PF\hat{x}_k + \hat{r}_k)$$

and

$$\langle \mathrm{E}'(\tilde{A}_k^* PA_k\hat{x}_k \mid \Sigma_k); g\rangle = \mathrm{E}(\langle \tilde{A}_k^* PA_k\hat{x}_k; g\rangle \mid \Sigma_k) = \langle \boldsymbol{F}^{\#}(P)x_k; g\rangle$$

for all $g \in \mathcal{H}$, so that

$$\mathrm{E}'(\tilde{A}_k^* P\tilde{A}_k\hat{x}_k \mid \Sigma_k) = \boldsymbol{F}^{\#}(P)\hat{x}_k.$$

Therefore, from (4) and (5), and recalling that $\hat{r}_k = \mathrm{E}(r_k \mid \Sigma_k)$, we get

$$\begin{aligned}
\mathrm{E}'(\tilde{A}_k^*\eta_{k+1} \mid \Sigma_k) &= \boldsymbol{F}^{\#}(P)\hat{x}_k + \mathrm{E}'(\tilde{A}_k^* r_k \mid \Sigma_k) - F^*PB(N + B^*PB)^{-1}B^*(PF\hat{x}_k + \hat{r}_k) \\
&= [\boldsymbol{F}^{\#}(P) - F^*PB(N + B^*PB)^{-1}B^*PF]\hat{x}_k + \mathrm{E}'((\tilde{A}_k - BK_P)^* r_k \mid \Sigma_k) \\
&= (P - M)\hat{x}_k + \mathrm{E}'((\tilde{A}_k - BK_P)^* r_k \mid \Sigma_k) \quad (k \geqslant 0). \qquad (15)
\end{aligned}$$

From (10) and (11) it follows that

$$\hat{x}_k - Dv_{k-1} = \tilde{A}_{k-1}\hat{x}_{k-1} + B\hat{u}_{k-1} = \tilde{A}_{k-1}\hat{x}_{k-1} - B(N + B^*PB)^{-1}B^*(PF\hat{x}_k + \hat{r}_k).$$

Recalling that v_{k-1} is Σ_k-measurable, we obtain from (6) and Proposition 4 that

$$\begin{aligned}
r_{k-1} &= PDv_{k-1} + \sum_{j=1}^{\infty} \mathrm{E}\left(\left[\prod_{l=0}^{j-1}(\tilde{A}_{k-1+j-l} - BK_P)^*\right] PDv_{k-1+j} \,\middle|\, \Sigma_k\right) \\
&= PDv_{k-1} + \sum_{j=0}^{\infty} \mathrm{E}\left(\left[\prod_{l=0}^{j}(\tilde{A}_{k+j-l} - BK_P)^* PDv_{k+j}\right. \,\middle|\, \Sigma_k\right) \\
&= PDv_{k-1} + \mathrm{E}'((\tilde{A}_k - BK_P)^* r_k \mid \Sigma_k).
\end{aligned}$$

Putting these results together, we get from (5), (8), (10), and (11) that

$$\begin{aligned}
\mathrm{E}'(\tilde{A}_k^*\eta_{k+1} \mid \Sigma_k) &= (P - M)\hat{x}_k + r_{k-1} - PDv_{k-1} = P(\hat{x}_k - Dv_{k-1}) + r_{k-1} - M\hat{x}_k \\
&= P[\tilde{A}_{k-1}\hat{x}_{k-1} - B(N + B^*PB)^{-1}B^*(PF\hat{x}_k + \hat{r}_k)] + r_{k-1} - M\hat{x}_k \\
&= P\tilde{A}_{k-1}\hat{x}_{k-1} + r_{k-1} - PB(N + B^*PB)^{-1}B^*(PF\hat{x}_k + \hat{r}_k) - M\hat{x}_k \\
&= \eta_k - M\hat{x}_k. \quad \square
\end{aligned}$$

Set $\eta_0 = \tilde{A}_0^*\eta_1 + Mx_0$. For $\boldsymbol{v}$ and x_0 fixed, take $\boldsymbol{u} = (u_0, u_1, \ldots) \in \hat{\mathcal{U}}$ and consider $\boldsymbol{x} = (x_0, x_1, \ldots) \in \mathcal{X}$ given by (1) as a function of such a $\boldsymbol{u}$.

Proposition 6 Suppose that $(M, \boldsymbol{F})$ is detectable and Γ_B is not empty. Then

$$\sum_{k=0}^{\infty} \mathrm{E}(\langle x_k; M\hat{x}_k\rangle - \langle (Bu_k + Dv_k); \eta_{k+1}\rangle) = \mathrm{E}(\langle x_0; \eta_0\rangle).$$

Proof. From Proposition 5 and (1), recalling that η_k and x_k are Σ_k-measurable, we get

$$
\begin{aligned}
&\sum_{k=0}^{n} \mathrm{E}\big(\langle x_k; M\hat{x}_k\rangle - \langle (Bu_k + Dv_k); \eta_{k+1}\rangle\big) \\
&= \sum_{k=0}^{n} \mathrm{E}\big(\langle x_k; -\mathrm{E}'(\tilde{A}_k^*\eta_{k+1} \mid \Sigma_k) + \eta_k\rangle - \langle (x_{k+1} - \tilde{A}_k x_k); \eta_{k+1}\rangle\big) \\
&= \sum_{k=0}^{n} \mathrm{E}\Big(\mathrm{E}\big(\langle x_k; (-\tilde{A}_k^*\eta_{k+1} + \eta_k)\rangle \mid \Sigma_k\big) - \langle (x_{k+1} - \tilde{A}_k x_k); \eta_{k+1}\rangle\Big) \\
&= \sum_{k=0}^{n} \mathrm{E}(-\langle x_k; \tilde{A}_k^*\eta_{k+1}\rangle + \langle x_k; \eta_k\rangle - \langle x_{k+1}; \eta_{k+1}\rangle + \langle \tilde{A}_k x_k; \eta_{k+1}\rangle) \\
&= \sum_{k=0}^{n} \mathrm{E}(-\langle \tilde{A}_k^* x_k; \eta_{k+1}\rangle + \langle x_k; \eta_k\rangle - \langle x_{k+1}; \eta_{k+1}\rangle + \langle \tilde{A}_k x_k; \eta_{k+1}\rangle) \\
&= \sum_{k=0}^{n} \mathrm{E}(-\langle x_{k+1}; \eta_{k+1}\rangle + \langle x_k; \eta_k\rangle) = \mathrm{E}(-\langle x_{n+1}; \eta_{n+1}\rangle + \langle x_0; \eta_0\rangle).
\end{aligned}
$$

From (8) and arguments as in the proof of Proposition 4, it follows that $\|\eta_{n+1}\|_{\mathscr{X}} \to 0$ as $n \to \infty$ (in fact, $\sum_{k=0}^{\infty} \|\eta_k\|_{\mathscr{X}}^2 < \infty$). Since $\boldsymbol{u} \in \hat{\mathscr{U}}$, it follows that $\boldsymbol{x} \in \mathscr{X}$ and hence $\|x_{n+1}\|_{\mathscr{X}} \to 0$ as $n \to \infty$. Therefore

$$
\mathrm{E}(|\langle x_{n+1}; \eta_{n+1}\rangle|) \leqslant \|x_{n+1}\|_{\mathscr{X}} \|\eta_{n+1}\|_{\mathscr{X}} \to 0 \quad \text{as } n \to \infty.
$$

Thus

$$
\sum_{k=0}^{\infty} \mathrm{E}\big(\langle x_k; M\hat{x}_k\rangle - \langle (Bu_k + Dv_k); \eta_{k+1}\rangle\big) = \mathrm{E}(\langle x_0; \eta_0\rangle). \quad \square
$$

LEMMA 2 Suppose that $(M, \boldsymbol{F})$ is detectable and Γ_B is not empty. Then, for $x_0 \in \mathscr{X}_0$ and $\boldsymbol{v} \in \mathscr{V}$ fixed and any $\boldsymbol{u} \in \hat{\mathscr{U}}$,

$$
\begin{aligned}
J(x_0, \boldsymbol{u}) + J(x_0, \hat{\boldsymbol{u}}) &= 2\,\mathrm{Re}\,\mathrm{E}(\langle x_0; \eta_0\rangle) \\
&+ \sum_{k=0}^{\infty} [\|M^{\frac{1}{2}}(x_k - \hat{x}_k)\|_{\mathscr{X}}^2 + \|N^{\frac{1}{2}}(u_k - \hat{u}_k)\|_{\mathscr{X}^*}^2 + 2\,\mathrm{Re}\,\mathrm{E}(\langle \eta_{k+1}; Dv_k\rangle)]
\end{aligned}
$$

Proof. First notice that

$$
\begin{aligned}
\sum_{k=0}^{\infty} \mathrm{E}(|\langle \eta_{k+1}; Dv_k\rangle|) &\leqslant \sum_{k=0}^{\infty} \mathrm{E}(\|\eta_{k+1}\| \|Dv_k\|) \leqslant \sum_{k=0}^{\infty} \|\eta_{k+1}\|_{\mathscr{X}} \|Dv_k\|_{\mathscr{X}} \\
&\leqslant \left(\sum_{k=0}^{\infty} \|\eta_{k+1}\|_{\mathscr{X}}^2\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} \|Dv_k\|_{\mathscr{X}}^2\right)^{\frac{1}{2}} < \infty,
\end{aligned}
$$

so that the last term in the above sum is well defined. According to (9), and recalling that u_k is Σ_k-measurable and $N \in \mathrm{G}^+[\mathcal{H}]$, we get

$$\begin{aligned}
\mathrm{E}(\langle Bu_k;\eta_{k+1}\rangle) &= \mathrm{E}\big(\mathrm{E}(\langle Bu_k;\eta_{k+1}\rangle \,|\, \Sigma_k)\big) = \mathrm{E}\big(\langle Bu_k;\mathrm{E}'(\eta_{k+1}\,|\,\Sigma_k)\rangle\big)\\
&= \mathrm{E}\big(\langle u_k;B^*\mathrm{E}'(\eta_{k+1}\,|\,\Sigma_k)\rangle\big) = \mathrm{E}\big(\langle Nu_k;N^{-1}B^*\mathrm{E}'(\eta_{k+1}\,|\,\Sigma_k)\rangle\big)\\
&= -\mathrm{E}(\langle Nu_k;\hat{u}_k\rangle).
\end{aligned}$$

Thus, from (2) and Proposition 6,

$$\begin{aligned}
&J(x_0,\boldsymbol{u}) - 2\,\mathrm{Re}\,\mathrm{E}(\langle x_0;\eta_0\rangle)\\
&= \sum_{k=0}^{\infty}\{\,\|M^{\frac12}x_k\|^2_{\mathcal{X}} + \|N^{\frac12}u_k\|^2_{\mathcal{X}''} - 2\,\mathrm{Re}\,[\mathrm{E}(\langle x_k;M\hat{x}_k\rangle - \langle (Bu_k+Dv_k);\eta_{k+1}\rangle)]\\
&\quad + \|M^{\frac12}\hat{x}_k\|^2_{\mathcal{X}} + \|N^{\frac12}\hat{u}_k\|^2_{\mathcal{X}''} - \|M^{\frac12}\hat{x}_k\|^2_{\mathcal{X}} - \|N^{\frac12}\hat{u}_k\|^2_{\mathcal{X}''}\}\\
&= \sum_{k=0}^{\infty}[\,\|M^{\frac12}x_k\|^2_{\mathcal{X}} + \|M^{\frac12}\hat{x}_k\|^2_{\mathcal{X}} - 2\,\mathrm{Re}\,\mathrm{E}(\langle M^{\frac12}x_k;M^{\frac12}\hat{x}_k\rangle + \|N^{\frac12}u_k\|^2_{\mathcal{X}}\\
&\quad + \|N^{\frac12}\hat{u}_k\|^2_{\mathcal{X}''} - 2\,\mathrm{Re}\,\mathrm{E}(\langle N^{\frac12}u_k;N^{\frac12}\hat{u}_k\rangle) + 2\,\mathrm{Re}\,\mathrm{E}(\langle \eta_{k+1};Dv_k\rangle)] - J(x_0;\hat{\boldsymbol{u}}).\\
&= \sum_{k=0}^{\infty}[\,\|M^{\frac12}(x_k-\hat{x}_k)\|^2_{\mathcal{X}} + \|N^{\frac12}(u_k-\hat{u}_k)\|^2_{\mathcal{X}''} + 2\,\mathrm{Re}\,\mathrm{E}(\langle \eta_{k+1};Dv_k\rangle)] - J(x_0;\hat{\boldsymbol{u}}).
\end{aligned}$$

□

We can now present the main result of the paper.

THEOREM 2 If $(M,\boldsymbol{F})$ is detectable and Γ_B is not empty, then U, as defined by (8), (9), and (10), belongs to $\mathrm{B}[\mathscr{X}_0 \oplus \mathscr{V}, \mathscr{U}]$. Moreover, for any $x_0 \in \mathscr{X}_0$ and $\boldsymbol{v} \in \mathscr{V}$, the sequence $\hat{\boldsymbol{u}} = U(x_0,\boldsymbol{v})$ belongs to $\hat{\mathscr{U}}$ and it is the unique optimal solution to (3). Furthermore,

$$\inf_{\boldsymbol{u}\in\hat{\mathscr{U}}} J(x_0,\boldsymbol{u}) = J(x_0,\hat{\boldsymbol{u}}) = \mathrm{Re}\,\mathrm{E}(\langle \eta_0;x_0\rangle) + \sum_{k=0}^{\infty}\mathrm{Re}\,\mathrm{E}(\langle \eta_{k+1};Dv_k\rangle).$$

Proof. From Lemma 1 it follows that $U \in \mathrm{B}[\mathscr{X}_0 \oplus \mathscr{V}, \mathscr{U}]$ with $U(x_0,\boldsymbol{v}) \in \hat{\mathscr{U}}$ for every $x_0 \in \mathscr{X}_0$ and $\boldsymbol{v} \in \mathscr{V}$. From Lemma 2 it is immediate to conclude that the infimum of $J(x_0,\boldsymbol{u})$ is achieved if and only if $\boldsymbol{u} = \hat{\boldsymbol{u}}$. □

6. Concluding remarks

In this final section we shall consider the particular assumption made in Ref. 2 and compare the present result with the one obtained there. Thus suppose that $\boldsymbol{v} = (v_0, v_1, \ldots) \in \mathscr{V}_{\boldsymbol{w}} \subset \mathscr{V}$ (see definition of $\mathscr{V}_{\boldsymbol{w}}$ in Theorem 1). In this particular case there is no loss of generality in assuming that the construction of Σ_n is such that v_l is independent of Σ_n (by Property P1) whenever $l > n$. From the definition of r_k (see (6)), and using the fact that v_l is zero-mean, independent of w_i and Σ_{k+1} for $i \geqslant 0$ and $l > k+1$, and Σ_{k+1}-measurable for $l = k$, $k+1$, it follows that

$$r_k = PDv_k + (F - BK_P)^* PDv_{k+1}. \tag{16}$$

Moreover, repeating the above arguments, we obtain $\hat{r}_k = \mathrm{E}(r_k \mid \Sigma_k) = PDv_k$. From this and (13) we get

$$\hat{x}_{k+1} = (\tilde{A}_k - BK_P)\hat{x}_k + \hat{v}_k, \qquad \hat{v}_k = [I - B(N + B^*PB)^{-1}B^*P]Dv_k. \tag{17}$$

In a similar fashion, we get from (15) and (16) that

$$\mathrm{E}'(\tilde{A}_k^* \eta_{k+1} \mid \Sigma_k) + M\hat{x}_k = P\hat{x}_k + \mathrm{E}'\big((\tilde{A}_k - BK_P)^* r_k \mid \Sigma_k\big) = P\hat{x}_k + (F - BK_P)^* PDv_k;$$

hence, from Proposition 5,

$$\eta_{k+1} = \mathrm{E}'(\tilde{A}_{k+1}^* \eta_{k+2} \mid \Sigma_{k+1}) + M\hat{x}_{k+1} = P\hat{x}_{k+1} + (F - BK_P)^* PDv_{k+1} \tag{18}$$

for every $k \geqslant 0$. Let us show now that

$$\mathrm{E}(\langle \eta_{k+1}; Dv_k \rangle) = \mathrm{E}(\langle P\hat{x}_{k+1}; Dv_k \rangle) = \mathrm{E}\big(\langle [P - PB(N + B^*PB)^{-1}B^*P]Dv_k; Dv_k \rangle\big). \tag{19}$$

From (17) it follows that, for $k \geqslant 0$,

$$\hat{x}_{k+1} = \left[\prod_{l=0}^{k} (\tilde{A}_l - BK_P)\right] x_0 + \sum_{j=1}^{k} \left[\prod_{l=j}^{k} (\tilde{A}_l - BK_P)\right] \hat{v}_{j-1} + \hat{v}_k.$$

Let $\{f_m; m \geqslant 1\}$ be any orthornormal basis for the Hilbert space $\mathcal{H}$. From the independence of v_k and x_0, v_l, and w_i for $l < k$ and $i \geqslant 0$, and since $\mathrm{E}'(v_k) = 0$, we have

$$\begin{aligned}
\mathrm{E}(\langle P\hat{x}_{k+1}; Dv_k \rangle) &= \sum_{m=1}^{\infty} \mathrm{E}(\langle P\hat{x}_{k+1}; f_m \rangle \langle f_m; Dv_k \rangle) \\
&= \sum_{m=1}^{\infty} \mathrm{E}\left(\left\langle P \left[\prod_{l=0}^{k} (\tilde{A}_l - BK_P)\right] x_0; f_m \right\rangle\right) \mathrm{E}(\langle f_m; Dv_k \rangle) \\
&\quad + \sum_{m=1}^{\infty} \sum_{j=1}^{k} \mathrm{E}\left(\left\langle P \left[\prod_{l=j}^{k} (\tilde{A}_l - BK_P)\right] \hat{v}_{j-1}; f_m \right\rangle\right) \mathrm{E}(\langle f_m; Dv_k \rangle) + \mathrm{E}(\langle P\hat{v}_k; Dv_k \rangle) \\
&= \mathrm{E}(\langle P\hat{v}_k; Dv_k \rangle) = \mathrm{E}\big(\langle P[I - B(N + B^*PB)^{-1}B^*P]Dv_k; Dv_k \rangle\big).
\end{aligned}$$

Similarly we can show that $\mathrm{E}\big(\langle (F - BK_P)^* PDv_{k+1}; Dv_k \rangle\big) = 0$. Therefore, from (18), it follows that

$$\begin{aligned}
\mathrm{E}(\langle \eta_{k+1}; Dv_k \rangle) &= \mathrm{E}\big(\langle P\hat{x}_{k+1} + (F - BK_P)^* PDv_{k+1}; Dv_k \rangle\big) \\
&= \mathrm{E}(\langle P\hat{x}_{k+1}; Dv_k \rangle) + \mathrm{E}\big(\langle (F - BK_P)^* PDv_{k+1}; Dv_k \rangle\big) \\
&= \mathrm{E}(\langle P\hat{x}_{k+1}; Dv_k \rangle) = \mathrm{E}\big(\langle P[I - B(N + B^*PB)^{-1}B^*P]Dv_k; Dv_k \rangle\big),
\end{aligned}$$

proving (19). Recalling that (see (18))

$$\mathrm{E}'(\eta_0 \mid \Sigma_0) = \mathrm{E}'(\tilde{A}_0^* \eta_1 \mid \Sigma_0) + Mx_0 = Px_0 + (F - BK_P)^* PDv_0,$$

and repeating the same arguments used above, we obtain

$$\begin{aligned}\mathrm{E}(\langle x_0;\eta_0\rangle) &= \mathrm{E}\big(\mathrm{E}(\langle x_0;\eta_0\rangle \mid \Sigma_0)\big)\\ &= \mathrm{E}\big(\langle x_0;\mathrm{E}'(\eta_0\mid\Sigma_0)\rangle\big) = \mathrm{E}\big(\langle x_0; Px_0 + (F-BK_P)^*PDv_0\rangle\big)\\ &= \mathrm{E}(\langle Px_0;x_0\rangle) + \mathrm{E}\big(\langle x_0;(F-BK_P)^*PDv_0\rangle\big) = \mathrm{E}(\langle Px_0;x_0\rangle). \qquad (20)\end{aligned}$$

From (19), (20), and Theorem 2 it follows that the optimal solution for this case is given by

$$\hat{u}_k = -K_P\hat{x}_k + (N+B^*PB)^{-1}B^*PDv_k \qquad (21)$$

with associated cost

$$J(x_0,\hat{\boldsymbol{u}}) = \mathrm{E}(\langle Px_0;x_0\rangle + \sum_{k=0}^{\infty}\mathrm{E}(\langle [P-PB(N+B^*PB)^{-1}B^*P]Dv_k;Dv_k\rangle).$$

Notice that

$$0 \leqslant P^{\frac{1}{2}}(1+P^{\frac{1}{2}}BN^{-1}B^*P^{\frac{1}{2}})^{-1}P^{\frac{1}{2}} = P - PB(N+B^*PB)^{-1}B^*P \leqslant P,$$

and therefore

$$0 \leqslant J(x_0;\hat{\boldsymbol{u}}) \leqslant \mathrm{E}(\langle Px_0;x_0\rangle) + \sum_{k=0}^{\infty}\mathrm{E}(\langle PDv_k;Dv_k\rangle). \qquad (22)$$

The term on the right-hand side of (22) represents the cost associated with the optimal solution derived in Ref. 2 (i.e. $\hat{u}_k = -K_Px_k$), which is larger than the one obtained here. This is so because feedback solutions were imposed in Ref. 2 (see Theorem 1) but not here. Note that $\hat{u}_k$ obtained in (21) has a linear (feedback) term in $\hat{x}_k$, as well as a linear term in v_k. If $v_k = 0$ for $k \geqslant 0$, then the solution in (21) and that in Ref. 2 coincide. In other words, for the case with no additive input disturbance, the best of all solutions actually is in a feedback form, with the optimal feedback loop characterized by the linear operator K_P given in (5).

In summary, by allowing any $\boldsymbol{v} \in \mathscr{V}$, the sequence $\hat{\boldsymbol{u}} = (\hat{u}_0,\hat{u}_1,\ldots) = U(x_0,\boldsymbol{v})$, with $\hat{u}_k$ as in (11), minimizes (3) (by Theorem 2). By restricting $\boldsymbol{v}$ to $\mathscr{V}_w \subset \mathscr{V}$ and imposing $u_k = Kx_k$ for some $K \in \Gamma_B$, it follows that $u_k = -K_Px_k$ minimizes (3) (by Theorem 1). At the extreme case of $\boldsymbol{v} = 0$, the same feedback solution $u_k = -K_Px_k$ also is the minimizing solution to (3), even though a feedback restriction is not imposed apriori.

Acknowledgements

This research was supported in part by CNPq (Brazilian National Research Council) and FAPESP (Research Council of the State of São Paulo).

References

1. Billingsley, P., 1985. *Probability and measure* (2nd edn). Wiley, New York.
2. Costa, O. L. V., & Kubrusly, C. S., 1993. Riccati equation for infinite-dimensional discrete bilinear systems. *IMA J. Mathematical Control & Information* 10, 273–91.
3. Costa, O. L. V., & Kubrusly, C. S., 1996. State feedback H_∞-optimal control for

discrete-time infinite-dimensional stochastic bilinear systems. To appear in *J. Mathematical Systems, Estimation & Control.*
4. HAGER, W. W., & HOROWITZ, L. L., 1976. Convergence and properties of the discrete Riccati operator equation and the associated optimal control and filtering problems. *SIAM J. Control & Optimization* **14**, 295–312.
5. KEULEN, B. V., PETERS, M., & CURTAIN, R., 1993. H_∞-control with state feedback: the infinite-dimensional case. *J. Mathematical Systems, Estimation, & Control* **3**, 1–39.
6. KUBRUSLY, C. S., 1985. Mean square stability for discrete bounded linear systems in Hilbert space. *SIAM J. Control & Optimization* **23**, 19–29.
7. KUBRUSLY, C. S., 1986. On discrete stochastic bilinear systems stability. *J. Mathematical Analysis and its Applications* **113**, 36–58.
8. KUBRUSLY, C. S., 1987. On the existence, evolution and stability of infinite-dimensional stochastic discrete bilinear models. *Control Theory Advances & Techniques* **3**, 271–87.
9. KUBRUSLY, C. S., & COSTA, O. L. V., 1992. Mean square stability for discrete bilinear systems in Hilbert space. *Systems & Control Letters* **19**, 205–11.
10. MOHLER, R. R., & RUBERTI, A., (Eds) 1975. *Recent development in variable structure systems economics and biology.* Lectures Notes in Economics and Mathematical Systems, Vol 111. Springer, Berlin.
11. MOHLER, R. R., & KOLODZIEJ, W. J., 1980. An overview of stochastic bilinear control processes. *IEEE Transactions on Systems, Man, & Cybernetics* **10**, 913–19.
12. ZABCZYK, J., 1975. On optimal stochastic control of discrete-time systems in Hilbert space. *SIAM J. Control* **13**, 1217–34.