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Quadratic-Mean Convergence and Mean-Square Stability for Discrete Linear Systems: A Hilbert-Space Approach

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Let H be a separable Hilbert space and let $\mathscr{H}$ be the Hilbert space of all second order H-valued random variables. This paper deals with limiting properties for random sequences in $\mathscr{H}$. Quadratic-mean convergence is investigated under the assumption of asymptotic weak uncorrelatedness. This leads to degenerate quadratic-mean limits. The mean-square stability problem for infinite-dimensional discrete linear systems driven by asymptotically uncorrelated input disturbances is analysed in detail. It is shown how mean-square stability acts on the quadratic-mean convergence of the state sequence.

1. Introduction

STOCHASTIC convergence for finite-dimensional random sequences, and asymptotic stability for finite-dimensional discrete linear systems driven by random disturbances, are presently rather settled and well-documented research topics (e.g. see [7] and [6], respectively). Although an infinite-dimensional treatment of these topics has already received some attention, several important questions still remain open. As far as stochastic convergence in infinite-dimensional linear topological spaces is concerned, see, for instance, the monograph [10]. Some partial results on stability properties for special classes of infinite-dimensional discrete stochastic linear systems can also be found in the recent literature (e.g. [3, 5, 13, 14]).

Here we shall be investigating quadratic-mean convergence and mean-square stability for infinite-dimensional discrete linear systems. The paper is organized as follows. Basic concepts that will be needed in the text are summarized in Section 2. The main results appear in Sections 3 to 5. In Section 3, we give a necessary and sufficient condition for quadratic-mean convergence of asymptotically weakly uncorrelated sequences, and we show that its quadratic-mean limit has to be a degenerate random variable. The mean-square stability problem deals with convergence preservation between input and state, for the expectation and correlation sequences. Such a problem is considered in Section 4, where the asymptotic behaviour of the state expectation and correlation sequences is analysed in detail. The results on quadratic-mean convergence and mean-square stability are combined together in Section 5. There, it is shown that state quadratic-mean convergence is too strong a requirement, whenever the input disturbance is asymptotically uncorrelated.

2. Preliminaries

In this section, we pose the notation and basic results that will be used throughout the text. Throughout this paper X and H will denote a Banach space and a separable nontrivial Hilbert space, with $\|\bullet\|$ and $\langle \bullet\,;\bullet \rangle$ standing for norm in X or H and inner product in H, respectively.

Deterministic Stability and Nuclear Operators

Let $\mathrm{B}[X]$ denote the Banach algebra of all bounded linear operators of X into itself, with $\|\bullet\|_0$ and $\mathrm{r}_\sigma(\bullet)$ standing for the uniform induced norm and spectral radius in $\mathrm{B}[X]$, respectively. Recall that

$$\mathrm{r}_\sigma(T)=\lim_{i\to\infty}\|T^i\|_0^{1/i}\leqslant\|T\|_0=\sup\{\|Tu\|:\|u\|=1\},$$

for any $T\in\mathrm{B}[X]$. By $\mathrm{B}_\infty[X]$ we shall denote the class of all compact operators from $\mathrm{B}[X]$. Now consider a sequence $(u_i\in X:i\geqslant 0)$ given by

$$u_{i+1}=Tu_i,\qquad u_0=u\in X.$$

The above free linear system (or equivalently, the operator $T\in\mathrm{B}[X]$) is: *uniformly asymptotically stable* if $\|T^i\|_0\to 0$ as $i\to\infty$, and *strongly asymptotically stable* if $\|T^iu\|\to 0$ as $i\to\infty$ for all $u\in X$.

Remark 1. Let us recall some well-known properties related to asymptotic stability that will suffice our needs in Section 4.

(a) The following assertions are equivalent (e.g. see [5]): (1) $\mathrm{r}_\sigma(T)<1$, (2) $\|T^i\|_0\to 0$ as $i\to\infty$, and (3) there exist real constants $\sigma\geqslant 1$ and $0<\alpha<1$ such that $\|T^i\|_0\leqslant\sigma\alpha^i$ for every $i\geqslant 0$.

(b) If the above holds (i.e. if $T\in\mathrm{B}[X]$ is uniformly asymptotically stable), then $(I-T)^{-1}=\sum_{j=0}^\infty T^j\in\mathrm{B}[X]$, where the convergence is in the uniform norm topology and I denotes the identity operator in $\mathrm{B}[X]$ (e.g. see [1, p. 567]).

(c) Obviously, uniform asymptotic stability implies strong asymptotic stability. The converse is true whenever $T\in\mathrm{B}_\infty[X]$, but it is not generally true for $T\in\mathrm{B}[X]$ (e.g. see [5]).

(d) Also recall that $|\alpha|<1$ if and only if

$$\lim_{i\to\infty}\sum_{j=0}^{i-1}\alpha^{i-j-1}\zeta_j=(1-\alpha)^{-1}\lim_{i\to\infty}\zeta_i$$

for every convergent scalar sequence $(\zeta_i:i\geqslant 0)$. This is a corollary of the Silverman–Toeplitz theorem (cf. [1, p. 75]).

(e) Consider an X-valued sequence $(u_i:i\geqslant 0)$ given by

$$u_{i+1}=Tu_i+v_i,\qquad u_0\in X,$$

where $T \in B[X]$ and $(v_i : i \geq 0)$ is a sequence in X. If T is uniformly asymptotically stable, then

$$\lim_{i\to\infty} u_i = (I - T)^{-1} \lim_{i\to\infty} v_i = \lim_{i\to\infty} \sum_{j=0}^{i-1} T^{i-j-1} v_j,$$

whenever $(v_i \in X : i \geq 0)$ converges in X. On the other hand, if $(u_i \in X : i \geq 0)$ converges in X for every initial condition $u_0 \in X$, to a limit which does not depend on u_0, then T is strongly asymptotically stable. This deterministic stability result can be derived easily by the above properties.

Let $T^* \in B[H]$ denote the adjoint of $T \in B[H]$, and recall that $\|T\|_0 = \|T^*\|_0$. The standard notation $T \geq 0$ will be used if a self-adjoint operator $T = T^* \in B[H]$ is nonnegative (i.e. $\langle Tu \,; u\rangle \geq 0$, $\forall u \in H$). We set $B[H]^+ = \{T \in B[H] : T \geq 0\}$, the closed convex cone of all nonnegative operators on H. Let $T^{\frac{1}{2}} \in B[H]^+$ be the (unique) square root of $T \in B[H]^+$, and set

$$|T| = (T^*T)^{\frac{1}{2}} \in B[H]^+$$

for any $T \in B[H]$, so that $B[H]^+ = \{T \in B[H] : T = |T|\}$. Since

$$\sum_n \langle |T| e_n \,; e_n\rangle = \sum_n \langle |T| f_n \,; f_n\rangle$$

for any orthonormal bases $(e_n : n \geq 1)$ and $(f_n : n \geq 1)$ for H, whenever the above series converge, set

$$B_1[H] = \left\{T \in B[H] : \sum_n \langle |T| e_n \,; e_n\rangle < \infty\right\},$$

i.e. the class of all nuclear (or trace-class) operators on H. Recall that $B_1[H] \subset B_\infty[H] \subset B[H]$. The subspace $B_1[H]$ is moreover a two-sided ideal of $B[H]$. For any $T \in B_1[H]$, define its trace as usual:

$$\operatorname{tr} T = \sum_n \langle Te_n \,; e_n\rangle,$$

which does not depend on the choice of the orthonormal basis $(e_n : n \geq 1)$ for H, and set

$$\|T\|_1 = \operatorname{tr} |T|.$$

Recall that $\operatorname{tr} : B_1[H] \to \mathbb{C}$ is a linear functional, and $\|\bullet\|_1$ is a norm in $B_1[H]$, the so-called trace norm. Actually, $(B_1[H], \|\bullet\|_1)$ is a Banach space. Notice that $B_1[H] = \{T \in B_\infty[H] : \|T\|_1 < \infty\}$. We set

$$B_1[H]^+ = B_1[H] \cap B[H]^+,$$

so that $\|T\|_1 = \operatorname{tr} T > 0$ whenever $T \neq 0 \in B_1[H]^+$. Given $u, v \in H$, define the outer product operator $(u \circ v) \in B_1[H]$ by

$$(u \circ v)s = \langle s \,; v\rangle u \quad \text{for all} \quad s \in H,$$

so that $(u \circ u) \in B_1[H]^+$. For a systematic presentation on nuclear operators, the reader is referred to [2, 8, 12].

Remark 2. The following further properties, which are readily verified, will also be needed in the sequel.

(a) $$\max\{|\operatorname{tr} T|, \|T\|_0\} \le \|T\|_1 = \|T^*\|_1 \quad \forall T \in B_1[H].$$

(b) $$\max\{\|TS\|_1, \|ST\|_1\} \le \|S\|_0 \|T\|_1 \quad \forall S \in B[H],\ \forall T \in B_1[H].$$

(c) $$\|u \circ v\|_0 = \|u \circ v\|_1 = \|u\|\,\|v\|, \quad \operatorname{tr}(u \circ v) = \langle u\,;v\rangle \quad \forall u,v \in H.$$

(d) $$(u \circ v)^* = v \circ u, \quad S(u \circ v)T^* = (Su)\circ(Tv) \quad \forall u,v \in H, \quad \forall S,T \in B[H].$$

(e) $$(u+v)\circ(r+s) = u\circ r + u \circ s + v \circ r + v \circ s \quad \forall u,v,r,s \in H.$$

(f) $$u \circ v - r \circ s = (u-r)\circ v + r \circ (v - s) \quad \forall u,v,r,s \in H.$$

Second-order H-valued Random Variables

Let $(\Omega, \mathscr{A}, \mu)$ be a probability space, where $\mathscr{A}$ is a σ-algebra of subsets of a nonempty basic set Ω and μ is a probability measure defined on $\mathscr{A}$. Let $\mathscr{H}$ be the set of equivalence classes of H-valued measurable maps x defined almost everywhere (a.e.) on Ω, such that the so-called *second-order* property

$$\|x\|_{\mathscr{H}}^2 := \mathcal{E}\,\|x\|^2 = \int_\Omega \|x(\omega)\|^2\, d\mu < \infty$$

holds true. Here $\mathcal{E}$ stands for the expectation of scalar-valued random variables. Recall that $x = y$ with probability one (w.p.1), which means that $x(\omega) = y(\omega)$ a.e. on Ω, if and only if $\|x - y\|_{\mathscr{H}} = 0$. Now set the following inner product in $\mathscr{H}$, which generates the above norm:

$$\langle x\,;y\rangle_{\mathscr{H}} := \mathcal{E}\langle x\,;y\rangle = \int_\Omega \langle x(\omega)\,;y(\omega)\rangle\, d\mu \quad \forall x,y \in \mathscr{H}.$$

Thus $\mathscr{H} = L_2(\Omega, \mu; H)$: *the Hilbert space of all second-order H-valued random variables*. Let $(e_n : n \ge 1)$ be any orthonormal basis for H, and let x and y be H-valued measurable maps defined a.e. on Ω. If $x,y \in \mathscr{H}$, then

$$\mathcal{E}\langle x;y\rangle = \mathcal{E}\sum_n \langle x;e_n\rangle\langle e_n;y\rangle = \sum_n \mathcal{E}(\langle x;e_n\rangle\langle e_n;y\rangle),$$

so that $\mathcal{E}\,\|x\|^2 = \mathcal{E}\sum_n |\langle x;e_n\rangle|^2 = \sum_n \mathcal{E}\,|\langle x;e_n\rangle|^2$. Conversely, if $\sum_n \mathcal{E}\,|\langle x;e_n\rangle|^2 < \infty$, then $x \in \mathscr{H}$. The above results can be derived easily by combining the Fourier-series theorem with the Lebesgue dominated-convergence theorem. They supply an equivalent definition for the inner product $\langle \bullet;\bullet\rangle_{\mathscr{H}}$ in $\mathscr{H}$, which will play a fundamental role for supporting the properties presented in Remark 3 below. Now let $x \in \mathscr{H}$, and consider the linear functional $\langle \bullet;x\rangle_{\mathscr{H}} : H \to \mathbb{C}$, given by

$$\langle u;x\rangle_{\mathscr{H}} = \mathcal{E}\langle u;x\rangle \quad \text{for all} \quad u \in H,$$

which is bounded. Then, by the Riesz representation theorem, there exists a unique element in H, say $\mathsf{E}x$, such that

$$\langle \mathsf{E}x;u\rangle = \langle x;u\rangle_{\mathscr{H}} \quad \forall u \in H.$$

$\mathsf{E}x$ is referred to as the *expectation* of $x \in \mathcal{H}$, and $\mathsf{E} : \mathcal{H} \to H$ is a bounded linear transformation. For any $x,y \in \mathcal{H}$, consider the sesquilinear functional

$$\langle (x \circ y)\bullet ; \bullet \rangle_{\mathcal{H}} : H \times H \to \mathbb{C},$$

given by

$$\langle (x \circ y)u ; v \rangle_{\mathcal{H}} = \mathcal{E}\langle (x \circ y)u ; v \rangle = \mathcal{E}(\langle u ; y \rangle \langle x ; v \rangle) \quad \text{for all} \quad u,v \in H,$$

which is bounded. Then (cf. [12: p. 120]) there exists a unique operator in $B[H]$, say $\mathbf{E}(x \circ y)$, such that

$$\langle \mathbf{E}(x \circ y)u ; v \rangle = \langle (x \circ y)u ; v \rangle_{\mathcal{H}} \quad \forall u,v \in H.$$

$\mathbf{E}(x \circ y)$ is referred to as the *correlation,* and $\mathbf{E}[(x - \mathsf{E}x) \circ (y - \mathsf{E}y)] = \mathbf{E}(x \circ y) - \mathsf{E}x \circ \mathsf{E}y$ as the *covariance,* of $x \in \mathcal{H}$ and $y \in \mathcal{H}$. Actually, $\mathbf{E}(x \circ y) \in B_1[H]$ and $\mathbf{E}(x \circ x) \in B_1[H]^+$ for any $x,y \in \mathcal{H}$.

Remark 3. The following properties will be needed in the sequel. They are readily verified by using the equivalent forms for the inner product in $\mathcal{H}$ presented above. Let $x,y,w,z \in \mathcal{H}$ and $S,T \in B[H]$. Then:

(a) $\mathbf{E}(x \circ y) = \mathbf{E}(y \circ x)^*, \qquad \mathbf{E}(Sx \circ Ty) = S\mathbf{E}(x \circ y)T^*.$

(b) $\mathbf{E}[(x + y) \circ (w + z)] = \mathbf{E}(x \circ w) + \mathbf{E}(x \circ z) + \mathbf{E}(y \circ w) + \mathbf{E}(y \circ z).$

(c) $\mathbf{E}(x \circ x) - \mathbf{E}(y \circ y) = \mathbf{E}[(x - y) \circ (x - y)] + \mathbf{E}[(x - y) \circ y] + \mathbf{E}[y \circ (x - y)].$

(d) $\mathsf{E}(Tx) = T\mathsf{E}x, \qquad \|Tx\|_{\mathcal{H}} \leq \|T\|_0 \, \|x\|_{\mathcal{H}}.$

(e) $\|\mathsf{E}x\| \leq \|x\|_{\mathcal{H}}, \qquad \|\mathbf{E}(x \circ y)\|_0 \leq \|\mathbf{E}(x \circ y)\|_1 \leq \|x\|_{\mathcal{H}} \, \|y\|_{\mathcal{H}}.$

(f) $\operatorname{tr} \mathbf{E}(x \circ y) = \langle x ; y \rangle_{\mathcal{H}}, \qquad \|\mathbf{E}(x \circ x)\|_1 = \operatorname{tr} \mathbf{E}(x \circ x) = \|x\|^2_{\mathcal{H}}.$

Finally, consider the following standard concepts. The random variables $x,y \in \mathcal{H}$ are *uncorrelated* if $\mathbf{E}(x \circ y) = \mathsf{E}x \circ \mathsf{E}y$. They are *orthogonal* if $\langle x ; y \rangle_{\mathcal{H}} = 0$, or equivalently, if $\operatorname{tr} \mathbf{E}(x \circ y) = 0$ (cf. Remark 3(f)). A random sequence $(x_i \in \mathcal{H} : i \geq 0)$ is *stationary in expectation and correlation* if there exist $q \in H$ and $Q \in B_1[H]^+$ such that $\mathsf{E}x_i = q$ and $\mathbf{E}(x_i \circ x_i) = Q$ for every $i \geq 0$. It is *wide-sense stationary* if, in addition to the above conditions, there exists a sequence of operators $(Q_k \in B_1[H] \colon k \geq 1)$ such that $\mathbf{E}(x_{i+k} \circ x_i) = Q_k$ for every $i \geq 0$. We shall also need in the sequel a weaker version of uncorrelatedness and a stronger version of orthogonality, as follows. The random variables $x,y \in \mathcal{H}$ are *weakly uncorrelated* if $\langle x ; y \rangle_{\mathcal{H}} = \langle \mathsf{E}x ; \mathsf{E}y \rangle$, or equivalently, $\operatorname{tr} \mathbf{E}(x \circ y) = \operatorname{tr} (\mathsf{E}x \circ \mathsf{E}y)$ (cf. Remarks 2(c) and 3(f)). They are *hyperorthogonal* if $\mathbf{E}(x \circ y) = 0$, so that a random sequence $(x_i \in \mathcal{H} : i \geq 0)$ is a *white noise* if x_i and x_j are hyperorthogonal for every $i \neq j$.

3. Quadratic-mean convergence

The purpose of this section is to prove Lemma 1 below, which gives a necessary and sufficient condition for an asymptotically weakly uncorrelated sequence in $\mathscr{H}$ to be quadratic-mean convergent, and shows that its quadratic-mean limit has to be a degenerate random variable. We begin by posing the appropriate definitions.

DEFINITION 1 $x \in \mathscr{H}$ is said to be a *degenerate random variable* (or equivalently, the random variable $x \in \mathscr{H}$ *degenerates* to $q \in H$) if $x = q \in H$ w.p.1 (i.e. $\|x - q\|_{\mathscr{H}} = 0$). The sequence $(x_i \in \mathscr{H} : i \geq 0)$ is said to be a *degenerate random sequence* if $x_i \in \mathscr{H}$ degenerates for every $i \geq 0$. A random sequence $(x_i \in \mathscr{H} : i \geq 0)$ is *quadratic-mean convergent* (or equivalently, it *converges in the quadratic mean*) if $(x_i \in \mathscr{H} : i \geq 0)$ converges in $\mathscr{H}$. That is, if there exists $x \in \mathscr{H}$ such that $\|x_i - x\|_{\mathscr{H}} \to 0$ as $i \to \infty$, which is referred to as the *quadratic-mean limit of* $(x_i \in \mathscr{H} : i \geq 0)$. A random sequence $(x_i \in \mathscr{H} : i \geq 0)$ is *asymptotically weakly uncorrelated* or *asymptotically uncorrelated in the trace norm* if, respectively,

(a) $$|\langle x_i; x_j \rangle_{\mathscr{H}} - \langle \mathsf{E}x_i; \mathsf{E}x_j \rangle| \to 0 \quad \text{as} \quad i \to \infty \quad \forall j \geq 0,$$

(b) $$\|\mathbf{E}(x_i \circ x_j) - \mathsf{E}x_i \circ \mathsf{E}x_j\|_1 \to 0 \quad \text{as} \quad i \to \infty \quad \forall j \geq 0.$$

Remark 4. Note that (b)$\Rightarrow$(a) in the above Definition. Moreover, it is also easy to show that:

(a') $$|\langle x_i; x_j \rangle_{\mathscr{H}} - \langle q; \mathsf{E}x_j \rangle| \to 0 \quad \text{as} \quad i \to \infty \quad \forall j \geq 0,$$

whenever $(x_i \in \mathscr{H} : i \geq 0)$ is asymptotically weakly uncorrelated and $(\mathsf{E}x_i \in H : i \geq 0)$ converges weakly to $q \in H$;

(b') $$\|\mathbf{E}(x_i \circ x_j) - q \circ \mathsf{E}x_j\|_1 \to 0 \quad \text{as} \quad i \to \infty \quad \forall j \geq 0,$$

whenever $(x_i \in \mathscr{H} : i \geq 0)$ is asymptotically uncorrelated in the trace norm and $(\mathsf{E}x_i \in H : i \geq 0)$ converges (strongly) to $q \in H$. Furthermore, (b')$\Rightarrow$(a').

PROPOSITION 1 *A random variable $x \in \mathscr{H}$ degenerates to $q \in H$ if and only if $\mathsf{E}x = q$ and $\mathbf{E}(x \circ x) = q \circ q$.*

Proof. This is clear by Remarks 2(c) and 3(c, e, f). □

PROPOSITION 2 *The sequence $(x_i \in \mathscr{H} : i \geq 0)$ converges in the quadratic mean to $x \in \mathscr{H}$ if and only if $\|\mathbf{E}[(x_i - x) \circ (x_i - x)]\|_1 \to 0$ as $i \to \infty$, which implies that*

$$\|\mathsf{E}x_i - \mathsf{E}x\| \to 0 \quad \textit{and} \quad \|\mathbf{E}(x_i \circ x_i) - \mathbf{E}(x \circ x)\|_1 \to 0 \quad \textit{as} \quad i \to \infty.$$

Proof. The equivalent form of stating quadratic-mean convergence is straightforward from Remark 3(f). The remaining results follow from Remarks 3(c, e). □

PROPOSITION 3 *If a given sequence $(x_i \in \mathscr{H} : i \geq 0)$ is such that*

$$\|\mathsf{E}x_i - q\| \to 0 \quad \textit{and} \quad \|\mathbf{E}(x_i \circ x_i)\|_1 \to \|q \circ q\|_1 \quad \textit{as} \quad i \to \infty,$$

for some $q \in H$, then it converges in the quadratic mean to the degenerate quadratic-mean limit $q \in H$.

Proof. Actually, by the definition of $\mathsf{E}x_i$,

$$\|x_i - q\|^2_{\mathscr{H}} = \|x_i\|^2_{\mathscr{H}} + \|q\|^2 - 2\,\mathrm{Re}\,\langle q; \mathsf{E}x_i\rangle \to 0 \quad \text{as} \quad i \to \infty,$$

since $\|x_i\|_{\mathscr{H}} \to \|q\|$ as $i \to \infty$, by Remarks 2(c) and 3(f). □

PROPOSITION 4 *If $(x_i \in \mathscr{H} : i \geqslant 0)$ is asymptotically weakly uncorrelated and converges in the quadratic mean, then it has a degenerate quadratic-mean limit $q = \lim_{i\to\infty} \mathsf{E}x_i \in H$.*

Proof. Recall that, for every $i,k \geqslant 0$,

$$\|x_{i+k} - x_i\|^2_{\mathscr{H}} = \|x_{i+k}\|^2_{\mathscr{H}} + \|x_i\|^2_{\mathscr{H}} - 2\,\mathrm{Re}\,\langle x_{i+k}; x_i\rangle_{\mathscr{H}}.$$

Since $\lim_{i\to\infty} \|x_i - x\|_{\mathscr{H}} = 0$ for some $x \in \mathscr{H}$, it follows that

$$\limsup_{i\to\infty}\,\sup_{k\geqslant 0} \|x_{i+k} - x_i\|_{\mathscr{H}} = 0$$

(i.e. $(x_i : i \geqslant 0)$ is a Cauchy sequence in $\mathscr{H}$) and $\lim_{i\to\infty} \|x_i\|_{\mathscr{H}} = \|x\|_{\mathscr{H}}$. Moreover, by Proposition 2, $\lim_{i\to\infty} \|\mathsf{E}x_i - \mathsf{E}x\| = 0$. Therefore, by Remarks 2(c), 3(f), and 4,

$$\begin{aligned} 0 &= \limsup_{i\to\infty}\,\sup_{k\geqslant 0} \|x_{i+k} - x_i\|^2_{\mathscr{H}} \geqslant \lim_{i\to\infty}\lim_{k\to\infty} \|x_{i+k} - x_i\|^2_{\mathscr{H}} \\ &= \lim_{i\to\infty} (\|x\|^2_{\mathscr{H}} + \|x_i\|^2_{H} - 2\,\mathrm{Re}\,\langle \mathsf{E}x; \mathsf{E}x_i\rangle) \\ &= 2(\|x\|^2_{\mathscr{H}} - \|\mathsf{E}x\|^2) = 2\,\mathrm{tr}\,[\mathbf{E}(x \circ x) - \mathsf{E}x \circ \mathsf{E}x]. \end{aligned}$$

But $\mathbf{E}(x \circ x) - \mathsf{E}x \circ \mathsf{E}x \in \mathrm{B}_1[H]^+$, so that $\mathbf{E}(x \circ x) = \mathsf{E}x \circ \mathsf{E}x$. Thus $x = \mathsf{E}x \in H$ w.p.1, according to Proposition 1. □

LEMMA 1 *If $(x_i \in \mathscr{H} : i \geqslant 0)$ is asymptotically weakly uncorrelated, then it is quadratic-mean convergent if and only if*

$$\|\mathbf{E}(x_i \circ x_i) - q \circ q\|_1 \to 0 \quad \textit{as} \quad i \to \infty$$

with $q = \lim_{i\to\infty} \mathsf{E}x_i$. Moreover, if the above holds, then the quadratic-mean limit of $(x_i \in \mathscr{H} : i \geqslant 0)$ degenerates to $q \in H$.

Proof. Combine the results in Propositions 1 to 4. □

Remark 5. The following conclusions can be drawn from Lemma 1.

(a) An asymptotically weakly uncorrelated random sequence $(x_i \in \mathscr{H} : i \geqslant 0)$ will not converge in the quadratic mean whenever its correlation sequence

$$(\mathbf{E}(x_i \circ x_i) \in \mathrm{B}_1[H]^+ : i \geqslant 0)$$

converges (in any topology) to a limit $Q \neq q \circ q$, where $q = \lim_{i\to\infty} \mathsf{E}x_i \in H$.

(b) Let $(x_i \in \mathscr{H} : i \geqslant 0)$ be an asymptotically weakly uncorrelated random sequence. If it is stationary in expectation and correlation, then it will converge in the quadratic mean only if it is a constant degenerate random sequence, i.e. such that $x_i = \mathsf{E}x_i = q \in H$ w.p.1 for every $i \geqslant 0$.

(c) In particular, if $(x_i \in \mathscr{H} : i \geq 0)$ is a wide-sense stationary zero-mean white noise, then it will converge in the quadratic mean only if it is a null degenerate random sequence, i.e. such that $x_i = 0 \in H$ w.p.1 for every $i \geq 0$.

We recall finally that stochastic convergence (and, in particular, quadratic-mean convergence) of H-valued random sequences to a degenerate random variable has been investigated by several authors. For instance, see the results on stochastic approximation algorithms in Hilbert space in [4, 9, 11]. Indeed, from the point of view of parametric estimation, convergence to a degenerate random variable plays an important role, even though this may be regarded as a hard constraint from the point of view of stochastic stability. We shall return to this point in Section 5.

4. Mean-square stability

Consider a discrete time-invariant linear dynamical system in H, driven by a second-order H-valued random sequence, as follows.

$$(1) \qquad x_{i+1} = Ax_i + w_i, \qquad x_0 \in \mathscr{H},$$

where $A \in \mathrm{B}[H]$ is the system operator and $(w_i \in \mathscr{H} : i \geq 0)$ denotes the input disturbance. On iterating (1) from $i = k$ onwards we get, by induction,

$$(2) \qquad x_{i+k} = A^i x_k + \sum_{j=0}^{i-1} A^{i-j-1} w_{j+k},$$

for every $k \geq 0$ and $i \geq 1$. Hence, setting $k = 0$, it is clear that $\|x_i\|_{\mathscr{H}} < \infty$ for every $i \geq 0$, according to Remark 3(d), so that $x_i \in \mathscr{H}$ for every $i \geq 0$. Now set, for every $i \geq 0$,

$$(3) \qquad V_i = A\mathbf{E}(x_i \circ w_i) + \mathbf{E}(w_i \circ x_i)A^* + \mathbf{E}(w_i \circ w_i).$$

Actually, $V_i \in \mathrm{B}_1(H)$, since the above correlations are always in $\mathrm{B}_1[H]$, which is a two-sided ideal of $\mathrm{B}[H]$.

Remark 6. Throughout this section, $\boldsymbol{L}$ will stand for an operator in $\mathrm{B}[\mathrm{B}[H]]$ associated with $A \in \mathrm{B}[H]$, which is given by

$$\boldsymbol{L}P = APA^* \quad \forall P \in \mathrm{B}[H].$$

It is a simple matter to verify that $\boldsymbol{L}$ is $\mathrm{B}_1[H]^+$-invariant. Then set

$$\boldsymbol{L}_1 = \boldsymbol{L}\restriction_{\mathrm{B}_1[H]} \in \mathrm{B}[\mathrm{B}_1[H]],$$

the restriction of $\boldsymbol{L}$ in the Banach space $\mathrm{B}_1[H]$. It is readily verified that $\boldsymbol{L} \in \mathrm{B}[\mathrm{B}[H]]$ is uniformly asymptotically stable if and only if $A \in \mathrm{B}[H]$ is uniformly asymptotically stable, since

$$\|\boldsymbol{L}^i\|_{\mathrm{B}[\mathrm{B}[H]]} = \sup_{\|P\|_0 = 1} \|A^i P A^{*i}\|_0 = \|A^i\|_0^2 \quad \forall i \geq 0.$$

Moreover, $L_1 \in B[B_1[H]]$ is uniformly asymptotically stable whenever $A \in B[H]$ is uniformly asymptotically stable, since by Remark 2(b),

$$\|L_1^i\|_{B[B_1[H]]} = \sup_{\|P\|_1=1} \|A^i P A^{*i}\|_1 \leq \sup_{\|P\|_1=1} \|A^i\|_0^2 \|P\|_1 = \|A^i\|_0^2 \quad \forall i \geq 0.$$

PROPOSITION 5 *The state sequence* ($x_i \in \mathscr{H} : i \geq 0$) *generated by* (1) *has the following properties* (*the empty sum denoting zero*):

(a) $$\mathsf{E}x_{i+1} = A\mathsf{E}x_i + \mathsf{E}w_i \quad \forall i \geq 0.$$

(b) $$\mathsf{E}x_{i+k} = A^i\mathsf{E}x_k + \sum_{j=0}^{i-1} A^{i-j-1}\mathsf{E}w_{j+k} \quad \forall i,k \geq 0.$$

(c) $$\mathsf{E}(x_{i+1} \circ x_{i+1}) = L\mathsf{E}(x_i \circ x_i) + V_i \quad \forall i \geq 0.$$

(d) $$\mathsf{E}(x_{i+k} \circ x_{i+k}) = L^i\mathsf{E}(x_k \circ x_k) + \sum_{j=0}^{i-1} L^{i-j-1}V_{j+k} \quad \forall i,k \geq 0.$$

(e) $$\mathsf{E}(x_{i+k+1} \circ x_i) = A\mathsf{E}(x_{i+k} \circ x_i) + \mathsf{E}(w_{i+k} \circ x_i) \quad \forall i,k \geq 0.$$

(f) $$\mathsf{E}(x_{i+k} \circ x_i) = A^k\mathsf{E}(x_i \circ x_i) + \sum_{j=0}^{k-1} A^{k-j-1}\mathsf{E}(w_{j+i} \circ x_i) \quad \forall i,k \geq 0.$$

Proof. The results in (a) and (b) are straightforward from (1) and (2), respectively. It is a simple matter to show the result in (c) by (1), (3), and Remarks 3(a, b). On iterating (c) from $i = k$ onwards we get the result in (d) by induction. The result in (e) is readily verified by (1) and Remarks 3(a, b). On iterating (e) from $k = 0$ onwards we get the result in (f) by induction. □

Now consider the following assumptions regarding the asymptotic behaviour of the input disturbance.

ASSUMPTIONS 1.ν ($\nu = 0, 1$)

(a) $$\|\mathsf{E}(x_0 \circ w_i) - \mathsf{E}x_0 \circ \mathsf{E}w_i\|_\nu \to 0 \quad \text{as} \quad i \to \infty,$$

(b) $$\sup_{j \neq i} \|\mathsf{E}(w_j \circ w_i) - \mathsf{E}w_j \circ \mathsf{E}w_i\|_\nu \to 0 \quad \text{as} \quad i \to \infty.$$

ASSUMPTION 2 There exists $r \in H$ such that

$$\|\mathsf{E}w_i - r\| \to 0 \quad \text{as} \quad i \to \infty.$$

ASSUMPTIONS 3.ν ($\nu = 0, 1$) There exists $R \in B[H]^+$ such that

$$\|\mathsf{E}(w_i \circ w_i) - R\|_\nu \to 0 \quad \text{as} \quad i \to \infty.$$

Remark 7. Clearly, Assumptions 1.1 and 3.1 imply Assumptions 1.0 and 3.0, respectively. Note that the convergence in Assumption 3 necessarily implies that $R \in B[H]^+$, since $B[H]^+$ is closed in $B[H]$, and the trace-norm convergence in Assumption 3.1 also implies that $R \in B_1[H]$. On the other hand, just the uniform convergence in Assumption 3.0 does not necessarily imply that R is a nuclear operator. However, if we expect that a random sequence ($w_i \in \mathscr{H} : i \geq 0$) has a correlation sequence converging uniformly to a correlation operator, such that $R = \mathbf{E}(w \circ w)$ for some $w \in \mathscr{H}$, we have to assume further that R is nuclear in Assumption 3.0, since $\mathbf{E}(w \circ w) \in B_1[H]^+$.

The purpose of this section is to consider the following stability questions. Under which conditions is it the case that: (i) the state sequence $(x_i \in \mathscr{H} : i \geq 0)$ has a convergent expectation sequence and a correlation sequence converging to a correlation operator whenever the input sequence $(w_i \in \mathscr{H} : i \geq 0)$ has the same property; (ii) the state expectation and correlation limits remain unchanged if the initial condition $x_0 \in \mathscr{H}$ is perturbed? Such questions will be investigated in Lemma 2 below, but first let us pose our stability problem properly.

Definition 2 Set either $v = 0$ or $v = 1$, and consider the following statement. For any initial condition $x_0 \in \mathscr{H}$ and input disturbance $(w_i \in \mathscr{H} : i \geq 0)$ satisfying Assumptions 1.v, 2, and 3.v with $R \in B_1[H]^+$, there exist $q \in H$ and $Q \in B_1[H]^+$ independent of $x_0 \in \mathscr{H}$ such that

$$\|\mathbf{E}x_i - q\| \to 0 \quad \text{as} \quad i \to \infty, \qquad \|\mathbf{E}(x_i \circ x_i) - Q\|_v \to 0 \quad \text{as} \quad i \to \infty.$$

The linear system in (1) is *mean square stable* if the above statement holds for $v = 0$, and it is *mean square stable in the trace norm* if the above statement holds for $v = 1$.

Lemma 2 *Consider the linear system in* (1) *and the following assertions.*

(a) $A \in B[H]$ *is uniformly asymptotically stable.*

($a_1.v$) $$\|\mathbf{E}(x_i \circ w_{i+k}) - \mathbf{E}x_i \circ \mathbf{E}w_{i+k}\|_v \to 0$$ *as* $i \to \infty$ $\forall k \geq 0$, *and as* $k \to \infty$ $\forall i \geq 0$.

($a_2.v$) $$\|\mathbf{E}(x_i \circ x_j) - \mathbf{E}x_i \circ \mathbf{E}x_j\|_v \to 0 \quad \text{as} \quad i \to \infty \quad \forall j \geq 0.$$

(a_3) $(x_i \in \mathscr{H} : i \geq 0)$ *is asymptotically weakly uncorrelated.*

(b) *There exists* $q \in H$ *such that*:

$$\|\mathbf{E}x_i - q\| \to 0 \quad \text{as} \quad i \to \infty.$$

(b_1) $$q = (I - A)^{-1} r \in H.$$

(b_2) $$\left\|\sum_{j=0}^{i} A^j r - q\right\| \to 0 \quad \text{as} \quad i \to \infty.$$

($b_3.v$) $$\|\mathbf{E}(x_i \circ w_{i+k}) - q \circ r\|_v \to 0 \quad \text{as} \quad i \to \infty \quad \forall k \geq 0.$$

(c) *There exists* $Q \in B[H]^+$ *such that*

$$\|\mathbf{E}(x_i \circ x_i) - Q\|_0 \to 0 \quad \text{as} \quad i \to \infty.$$

(c_1) $Q = (\boldsymbol{I} - \boldsymbol{L})^{-1} V \in B[H]^+$, *where*
$$V = A(q \circ r) + (r \circ q)A^* + R \in B[H],$$

which is the only solution of $Q = AQA^* + V$.

(c_2) $$\left\|\sum_{j=0}^{i} A^j V A^{*j} - Q\right\|_0 \to 0 \quad \text{as} \quad i \to \infty.$$

(c_3) $$Q \in B_1[H]^+ \Leftrightarrow R \in B_1[H]^+.$$

(c_4) $$Q = q \circ q \Leftrightarrow R = r \circ r.$$

(d) *There exists* $Q \in B_1[H]^+$ *such that*

$$\|\mathbf{E}(x_i \circ x_i) - Q\|_1 \to 0 \quad as \quad i \to \infty.$$

(d_1) $$\left\|\sum_{j=0}^{i} A^j V A^{*j} - Q\right\|_1 \to 0 \quad as \quad i \to \infty.$$

We claim that the implications among the above assertions indicated below hold true.

$$(\mathrm{a}) \Rightarrow \begin{cases} (a_1.v) & \text{under Assumption } 1.v \quad (v = 0, 1), \\ (b, b_1) & \text{under Assumption } 2, \\ (c, c_1) & \text{under Assumptions } 1.v, 2, \text{ and } 3.v \quad (v = 0, 1), \\ (d, d_1) & \text{under Assumptions } 1.1, 2, \text{ and } 3.1. \end{cases}$$

$$(\mathrm{a}, \mathrm{a}_1.v) \Rightarrow (\mathrm{a}_2.v) \quad (v = 0, 1), \qquad (\mathrm{a}_2.1) \Rightarrow (\mathrm{a}_3), \qquad (\mathrm{a}, \mathrm{b}_1) \Rightarrow (\mathrm{b}_2),$$

$$(\mathrm{a}_1.v, \mathrm{b}) \Rightarrow (\mathrm{b}_3.v) \quad \text{under Assumption } 2 \quad (v = 0, 1),$$

$$(\mathrm{a}, \mathrm{c}_1) \Rightarrow (\mathrm{c}_2, \mathrm{c}_3), \qquad (\mathrm{b}_1, \mathrm{c}_1) \Rightarrow (\mathrm{c}_4), \qquad (\mathrm{d}) \Rightarrow (\mathrm{c}).$$

Moreover, consider the following further assertions.

(b′) *Assertion* (b) *holds true for every* $x_0 \in \mathcal{H}$ *satisfying Assumption* 1.0(a), *and the limit* $q \in H$ *does not depend on* x_0.

(c′) *Assertion* (c) *holds true for every* $x_0 \in \mathcal{H}$ *satisfying Assumption* 1.0(a), *and the limit* $Q \in B[H]^+$ *does not depend on* x_0.

(d′) *Assertion* (d) *holds true for every* $x_0 \in \mathcal{H}$ *satisfying Assumption* 1.1(a), *and the limit* $Q \in B_1[H]^+$ *does not depend on* x_0.

We also claim that each of (b′), (c′), *and* (d′) *imply* (a) *whenever* $A \in B_\infty[H]$.

Proof. By the system solution in (2) with $k = 0$, Remarks 3(a, b) and 2(b, d, e), and Proposition 5(b) with $k = 0$ we get, for every $i \geq 0$ and $k \geq 0$,

$$\|\mathbf{E}(x_i \circ w_{i+k}) - \mathbf{E}x_i \circ \mathbf{E}w_{i+k}\|_v \leq \|A^i\|_0 \|\mathbf{E}(x_0 \circ w_{i+k}) - \mathbf{E}x_0 \circ \mathbf{E}w_{i+k}\|_v + \sup_{0 \leq j \leq i-1} \|\mathbf{E}(w_j \circ w_{i+k}) - \mathbf{E}w_j \circ \mathbf{E}w_{i+k}\|_v \sum_{j=0}^{i-1} \|A^j\|_0,$$

which goes to 0 as $k \to \infty$, for every $i \geq 0$, according to Assumption $1.v$; also it goes to zero as $i \to \infty$, for every $k \geq 0$, according to Remark 1(a), whenever assertion (a) and Assumption $1.v$ hold true. (Here, we read the empty sum, and empty supremum in $\mathbb{R}_{\geq 0}$, as zero.) Thus (a)$\Rightarrow$($\mathrm{a}_1.v$). By Proposition 5(b, f) and Remarks 2(a, b, d, e) and 3(a) we have

$$\|\mathbf{E}(x_{i+l} \circ x_i) - \mathbf{E}x_{i+l} \circ \mathbf{E}x_i\|_v \leq \|A^l\|_0 \|\mathbf{E}(x_i \circ x_i) - \mathbf{E}x_i \circ \mathbf{E}x_i\|_v + \sum_{k=0}^{l-1} \|A^{l-k-1}\|_0 \|\mathbf{E}(x_i \circ w_{k+i}) - \mathbf{E}x_i \circ \mathbf{E}w_{k+i}\|_v,$$

for every $i \geq 0$ and $l \geq 0$. Thus, $(a, a_1.v) \Rightarrow (a_2.v)$, according to Remarks 1(a, d). Moreover, $(a_2.1) \Rightarrow (a_3)$, according to Remark 4. By Assumption 2, Proposition 5(a), and Remark 1(e) with $X = H$, it follows that $(a) \Rightarrow (b, b_1)$. Moreover, $(a, b_1) \Rightarrow (b_2)$, since

$$\left\|\sum_{j=0}^{i} A^j r - q\right\| = \left\|\left(\sum_{j=0}^{i} A^j - (I-A)^{-1}\right) r\right\| \leq \left\|\sum_{j=0}^{i} A^j - (I-A)^{-1}\right\|_0 \|r\| \to 0 \quad \text{as} \quad i \to \infty,$$

according to Remark 1(b). By using Remarks 2(c, f) we get

$$\|\mathbf{E}(x_i \circ w_{i+k}) - q \circ r\|_v \leq \|\mathbf{E}(x_i \circ w_{i+k}) - \mathbf{E}x_i \circ \mathbf{E}w_{i+k}\|_v + \|\mathbf{E}x_i - q\| \, \|\mathbf{E}w_{i+k}\| + \|q\| \, \|\mathbf{E}w_{i+k} - r\|,$$

for every $i, k \geq 0$. Hence $(a_1.v, b) \Rightarrow (b_3.v)$ under Assumption 2, by taking the limit as $i \to \infty$. Now, under Assumptions 1.0, 2, and 3.0, set

$$V = A(q \circ r) + (r \circ q)A^* + R \in B[H]$$

whenever (b) holds true. By the definition of V_i in (3), and Remarks 2(b, d) and 3(a), we get, for every $i \geq 0$,

$$\|V_i - V\|_0 \leq 2 \, \|A\|_0 \, \|\mathbf{E}(x_i \circ w_i) - q \circ r\|_0 + \|\mathbf{E}(w_i \circ w_i) - R\|_0 \to 0 \quad \text{as} \quad i \to \infty,$$

whenever (a) holds true, since $(a) \Rightarrow (a_1.0, b) \Rightarrow (b_3.0)$. Then, by Proposition 5(c) and Remark 1(e) with $X = B[H]$, the assertion (a) implies the convergence in (c), with $Q = (\boldsymbol{I} - \boldsymbol{L})^{-1} V \in B[H]$, which is the only solution of $V = (\boldsymbol{I} - \boldsymbol{L})Q = Q - AQA^*$. Actually, $Q \in B[H]^+$, since $\mathbf{E}(x_i \circ x_i) \in B[H]^+$ $(i \geq 0)$ and $B[H]^+$ is closed in $B[H]$. Thus $(a) \Rightarrow (c, c_1)$. Moreover, $(a, c_1) \Rightarrow (c_2)$, since

$$\left\|\sum_{j=0}^{i} A^j V A^{*j} - Q\right\|_0 = \left\|\left(\sum_{j=0}^{i} \boldsymbol{L}^j - (\boldsymbol{I} - \boldsymbol{L})^{-1}\right) V\right\|_0 \leq \left\|\sum_{j=0}^{i} \boldsymbol{L}^j - (\boldsymbol{I} - \boldsymbol{L})^{-1}\right\|_{B[B[H]]} \|V\|_0 \to 0 \quad \text{as} \quad i \to \infty$$

according to Remark 1(b). Now we show that $(a, c_1) \Rightarrow (c_3)$. Actually, by (c_1),

$$Q \in B_1[H] \quad \Rightarrow \quad R = Q - AQA^* - A(q \circ r) - (r \circ q)A^* \in B_1[H].$$

Also note that $V \in B_1[H]$ whenever $R \in B_1[H]$. Thus, by Remark 2(b) and according to assertion (a) and Remark 1(a),

$$R \in B_1[H] \quad \Rightarrow \quad \sup_{i \geq 0} \left\|\sum_{j=0}^{i} A^j V A^{*j}\right\|_1 \leq \|V\|_1 \sum_{j=0}^{\infty} \|A^j\|_0^2 < \infty,$$

which, together with the uniform convergence in (c_2), implies that $Q \in B_1[H]$ (cf. [12 : p. 179]). To verify that $(b_1, c_1) \Rightarrow (c_4)$, notice that $Aq = q - r$ whenever (b_1) holds true. Hence, by Remarks 2(d, e),

$$A(q \circ q)A^* = q \circ q + r \circ r - q \circ r - r \circ q, \qquad V = q \circ r + r \circ q + R - 2r \circ r.$$

Therefore $q \circ q = A(q \circ q)A^* + V - (R - r \circ r)$. Then $(b_1, c_1) \Rightarrow (c_4)$ since, according to (c_1), $Q \in B[H]^+$ is the only solution of $Q = AQA^* + V$. Now retain Assumption 2 and replace Assumptions 1.0 and 3.0 by the stronger Assumptions 1.1 and 3.1. By Remarks 2(c, b, d) and 3(a), it follows that $V \in B_1[H]$ and that $\|V_i - V\|_1 \leq 2\,\|A\|_0\,\|\mathsf{E}(x_i \circ w_i) - q \circ r\|_1 + \|\mathsf{E}(w_i \circ w_i) - R\|_1 \to 0$ as $i \to \infty$, whenever (a) holds true, since (a)$\Rightarrow(a_1.1, b)\Rightarrow(b_3.1)$. Then, by Proposition 5(c) and Remarks 1(b, e) with $X = B_1[H]$, we have (a)$\Rightarrow$(d) with $Q = (\boldsymbol{I} - \boldsymbol{L}_1)^{-1}V = (\boldsymbol{I} - \boldsymbol{L})^{-1}V \in B_1[H]^+$, since (a)$\Rightarrow(c, c_1)$. Therefore,

$$\left\|\sum_{j=0}^{i} A^j V A^{*j} - Q\right\|_1 = \left\|\left(\sum_{j=0}^{i} \boldsymbol{L}_1^j - (\boldsymbol{I} - \boldsymbol{L}_1)^{-1}\right)V\right\|_1 \leq \left\|\sum_{j=0}^{i} \boldsymbol{L}_1^j - (\boldsymbol{I} - \boldsymbol{L}_1)^{-1}\right\|_{B[B_1[H]]} \|V\|_1$$

which tends to 0 as $i \to \infty$, by Remark 1(b). Thus (a)$\Rightarrow(d, d_1)$. The derivation of (d)$\Rightarrow$(c) is trivial (cf. Remark 2(a)). Finally, let

$$H_0 = \{p \in H : p = \mathsf{E}x_0 \quad \text{for some} \quad x_0 \in \mathcal{H} \text{ satisfying Assumption 1.0(a)}\},$$

$$B_1[H]_v^+ = \{P \in B_1[H]^+ : P = \mathsf{E}(x_0 \circ x_0) \quad \text{for some} \quad x_{(0)} \in \mathcal{H} \text{ satisfying Assumption } 1.v(a)\}.$$

It is a simple matter to verify that $H_0 = H$ and $B_1[H]_v^+ = B_1[H]^+$ for $v = 0, 1$. Then, by Proposition 5(a, c) and Remark 1(e) with $X = H$, $X = B[H]$, or $X = B_1[H]$, it follows that each of (b′), (c′), or (d′) imply (a), respectively, whenever $A \in B_\infty[H]$, according to Remark 1(c). □

5. Concluding remarks

In this paper, we have considered asymptotic properties for Hilbert-space-valued random sequences, with applications to the stability analysis of infinite-dimensional discrete linear systems.

In Lemma 1 was given a necessary and sufficient condition for an asymptotically weakly uncorrelated random sequence to be quadratic-mean convergent, and it was shown that its quadratic-mean limit necessarily degenerates to its expectation limit. Therefore, the possibility of an asymptotically weakly uncorrelated (or, in particular, an uncorrelated) sequence to converge in the quadratic mean to a nondegenerate random variable has been dismissed. Further conclusions have been drawn from Lemma 1 in Remark 5.

A fairly complete set of results, on the asymptotic behaviour of discrete linear systems in Hilbert space driven by random disturbances, was derived in Lemma 2. In particular, it was established that uniform asymptotic stability for the free system is sufficient to ensure mean-square stability (and also mean-square stability in the trace norm) for the randomly driven system in (1) and, on the other hand, that such a condition is also necessary whenever the system operator is compact. The limiting state moments were given in terms of the limiting input moments, and it was shown which properties are preserved between input and state limiting correlations.

According to Definition 2, mean-square stability means essentially that convergence in H of the expectation sequence, and convergence in $B[H]$ of the correlation sequence to an operator in $B_1[H]^+$ (such that its uniform limit is a correlation operator for some random variable in $\mathcal{H}$), are preserved through the linear system in (1). If, instead of investigating mean-square stability, we turn our attention to analysing whether convergence in $\mathcal{H}$, rather than in H and $B[H]$, is preserved through the linear system in (1) (i.e. whether quadratic-mean convergence is preserved between input and state, which essentially means *quadratic-mean stability*), then we come across with some noteworthy conclusions. First, by combining Lemmas 1 and 2, it can be verified that the following remark holds true.

Remark 8. Take the linear system in (1) under Assumption 1.1, and consider the following assertions.

(a) $(x_i \in \mathcal{H} : i \geqslant 0)$ is quadratic mean convergent.

(b) $(w_i \in \mathcal{H} : i \geqslant 0)$ is quadratic mean convergent.

If $A \in B[H]$ is uniformly asymptotically stable, then (a) and (b) are equivalent. Moreover, if they hold true, then the quadratic-mean limits in (a) and (b) degenerate, respectively, to

$$q = \lim_{i \to \infty} \mathsf{E}x_i \in H, \qquad r = \lim_{i \to \infty} \mathsf{E}w_i \in H.$$

Now, by Remarks 5 and 8, we conclude finally that quadratic-mean convergence for the state sequence is too stringent a result to be sought when the input disturbance is an uncorrelated random sequence (or, more generally, when it satisfies Assumption 1.1). Actually, if the free system is uniformly asymptotically stable, so that mean-square stability is ensured by Lemma 2, then quadratic-mean convergence between uncorrelated input and state is preserved. However, in such a case, quadratic-mean convergence will happen only if the quadratic-mean limits are degenerate random variables. Therefore, state quadratic-mean convergence will never happen if the uncorrelated input is a nondegenerate random sequence that is stationary in expectation and correlation. In particular, quadratic-mean convergence for the state sequence will never happen if the input disturbance is a non-trivial zero-mean wide-sense stationary white noise.

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