WEAK ASYMPTOTIC STABILITY FOR DISCRETE LINEAR DISTRIBUTED SYSTEMS

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Abstract. Equivalent conditions for weak convergence of power operator sequences are established and, in particular, the special case of weak asymptotic stability for discrete time-invariant free linear systems in Hilbert space is considered. Necessary spectral conditions are provided, and the relationship between weak asymptotic stability and similarity to contraction is investigated.

Keywords. Linear systems; discrete-time systems; infinite-dimensional systems; stability; operator theory.

1. NOTATION AND TERMINOLOGY

Throughout this paper H will stand for a nontrivial complex separable Hilbert space, with inner product $<;>$ and norm $\|\ \|$. B[H] will denote the Banach algebra of all bounded linear operators of H into itself, with the upper star * denoting adjoint, as usual. We shall use the same symbol $\|\ \|$ to denote the uniform induced norm in B[H]. By an invertible operator in B[H] we mean an operator which is also bounded from below and maps H onto itself. G[H] will stand for the group of all invertible operators from B[H]. The orthogonal complement of any subset $M \subset H$ will be denoted by $M^{\perp}$, and the null space and range of an operator $T \in B[H]$ will be denoted by N(T) and R(T), respectively. For any $T \in B[H]$, $\rho(T)$ is its resolvent set, $\sigma(T)$ its spectrum, $\sigma_P(T)$ its point spectrum, $\sigma_R(T)$ its residual spectrum, and $\sigma_C(T)$ its continuous spectrum, as usual. Recall that the spectral radius $r_\sigma(T):=\sup_{\lambda\in\sigma(T)}|\lambda|$ and the numerical radius $\omega(T):=\sup_{\|x\|=1}|<Tx;x>|$ of any operator $T\in B[H]$ are related as follows: $r_\sigma(T)\leq \omega(T)\leq\|T\|\leq 2\omega(T)$, and $\omega(T)=\|T\| \Longrightarrow r_\sigma(T)=\|T\|$. T is said to be normaloid if $\omega(T)=\|T\|$, and spectraloid if $r_\sigma(T)=\omega(T)$. Clearly any normal operator is normaloid, any normaloid operator is spectraloid; and these inclusions of classes are both proper. By a contraction (a strict contraction) we mean an operator $T\in B[H]$ such that $\|T\|\leq 1$ ($\|T\|<1$). Thus, an operator $T\in B[H]$ is similar to a contraction (s.c.), or similar to a strict contraction (s.s.c.), iff there exists $Q\in G[H]$ such that $\|QTQ^{-1}\|\leq 1$, or $\|QTQ^{-1}\|<1$, respectively. We shall write $T_n \xrightarrow{u} T$, $T_n \xrightarrow{s} T$, or $T_n \xrightarrow{w} T$ if a given sequence of operators $\{T_n \in B[H];\ n\geq 1\}$ converges to $T\in B[H]$ uniformly (i.e. $\lim_{n\to\infty}\|T_n - T\| = 0$), strongly (i.e. $\lim_{n\to\infty}\|(T_n - T)x\| = 0\ \ \forall x\in H$), or weakly (i.e. $\lim_{n\to\infty}|<(T_n - T)\,x;y>| = 0\ \ \forall x,y\in H$, which is equivalent to $\lim_{n\to\infty}|<(T_n-T)x;x>|=0\ \ \forall x\in H$), respectively. The open unit disc and the unit circle (in the complex plane, centred at the origin) will be denoted by Δ and Γ, respectively. Finally, for any $\Lambda\subset\mathbb{C}$, set $\Lambda^* = \{\bar{\lambda}\in\mathbb{C}:\ \lambda\in\Lambda\}$, with the upper bar denoting complex conjugate.

2. INTRODUCTION

Consider a discrete time-invariant distributed free linear (bounded) system, described by the following autonomous homogeneous difference equation in a separable Hilbert space H.

$$(1)\quad x_{n+1} = T\,x_n, \qquad x_0 = x \in H.$$

The model (1) (or equivalently the operator $T\in B[H]$) is strongly asymptotically stable if the state sequence $\{x_n = T^n x\in H;\ n\geq 0\}$ converges to zero for all initial conditions $x\in H$ (i.e. $T^n \xrightarrow{s} 0$). If the above convergence holds uniformly for all initial conditions $x\in H$, or equivalently if $\sup_{\|x\|\leq 1}\|T^n x\|\to 0$ as $n\to\infty$ (i.e. $\|T^n\|\to 0$ $n\to\infty$), then the model (1) (or the operator T) is said to be uniformly asymptotically stable (i.e. $T^n \xrightarrow{u} 0$). On the other hand, if the state sequence converges weakly to zero for all initial conditions $x\in H$ (i.e. if $<T^n x;y>\to 0$ as $n\to\infty$ $\forall x,y\in H$), then the model (1) (or the operator T) is said to be weakly asymptotically stable (i.e. $T^n \xrightarrow{w} 0$). Therefore, asymptotic stability of a discrete distributed linear system modelled as in (1) turns out to be equivalent to convergence to zero for the system operator power sequence $\{T^n \in B[H];\ n\geq 0\}$.

According to the topology for which the above sequence may converge to zero it is naturally associated the concepts of weak, strong and uniform asymptotic stability. These are trivially related as follows: $T^n \xrightarrow{u} 0 \Longrightarrow T^n \xrightarrow{s} 0 \Longrightarrow T^n \xrightarrow{w} 0$.

Uniform asymptotic stability for discrete infinite dimensional linear (bounded) systems has been investigated by several authors (e.g. see Zabczyk (1974, 1975), Kamen and Green (1980), Przyluski (1980, 1988), and Kubrusly (1985, 1989)). Strong convergence for power sequences of bounded linear operators on Hilbert space has received an unified treatment in the book of Nagy and Foias (1970), where its relationship with the invariant subspace problem (perhaps the most celebrated unsolved problem in operator theory) is deeply analysed. On the other hand, very little has been written on weak asymptotic stability for discrete infinite dimensional linear (bounded) systems, comparing with what has been done in the uniform topology.

There are at least two good reasons to attempt to an investigation of weak asymptotic stability. One of them is that state weak convergence to zero is enough to ensure (strong) output convergence to zero, when the system evolution is observed through a compact operator. Suppose $K \in B[H]$ is compact and let $\{z_n = Kx_n = KT^n x \in H;\ n \geq 0\}$ stand for the model output sequence, through which the system evolution is observed. Since a compact operator takes weakly convergent sequences into strongly convergent sequences, it follows that $\|z_n\| \to 0$ as $n \to \infty$ for all initial conditions $x \in H$ whenever $T^n \xrightarrow{w} 0$. In particular, if $\dim[R(K)] < \infty$, then we have an important special case which describes the situation where the infinite-dimensional linear model (1) can only be observed in a finite-dimensional subspace of H. A second motivation is that weak asymptotic stability is somewhat related with another important unsolved problem in operator theory, namely the characterization of similarity to contraction, as we point out in section 4 and discuss in section 5.

In this paper we shall be focusing on the convergence of power operator sequences in the weak topology and, in particular, on weak asymptotic stability. Our aim here is to establish equivalent conditions for weak convergence of power operator sequences, and to analyse the special case of weak convergence to zero. This will supply necessary spectral conditions for weak asymptotic stability. We shall also investigate in which extent (or for which class of operators) one can ensure an equivalence between weak asymptotic stability and similarity to contraction.

3. WEAKLY CONVERGENT POWER SEQUENCES

The purpose of this section is to present equivalent conditions for weak convergence of power operator sequences, as well as necessary spectral conditions. This will be achieved in Theorem 1 below. We begin by establishing two auxiliary results that will suffice our needs.

Proposition 1. Consider a complex double sequence (i.e. an infinite complex matrix) $\{\alpha_{nk} \in \mathbb{C};\ n,k \geq 1\}$, and let α be a positive constant. The following assertions are equivalent.

$$\text{(a)} \begin{cases} \text{(i)} & \lim_{n\to\infty} |\alpha_{nk}| = 0 \qquad \forall k \geq 1, \\ \text{(ii)} & |\alpha_{nk}| \leq \alpha \qquad \forall n,k \geq 1. \end{cases}$$

$$\text{(b)} \begin{cases} \text{(i)} & \lim_{n\to\infty} \sum_{k=1}^{\infty} |\alpha_{nk}||\xi_k| = 0 \\ & \forall x = (\xi_1, \xi_2, \dots) \in \ell_1, \\ \text{(ii)} & \sup_{n\geq 1} \sum_{k=1}^{\infty} |\alpha_{nk}||\xi_k| \leq \alpha \|x\|_1 \\ & \forall x = (\xi_1, \xi_2, \dots) \in \ell_1. \end{cases}$$

Sketch proof. Take an arbitrary $x = (\xi_1, \xi_2, \dots) \in \ell_1$. By the double limit theorem (e.g. see Hoffman (1975, p.180)) it is a simple matter to show that

$$\lim_{n\to\infty} \sup_{k\geq 1} |\alpha_{nk}||\xi_k| = 0$$

whenever (a) holds. Note that (a-ii) trivially implies that

$$\sum_{k=1}^{\infty} \sup_{n\geq 1} |\alpha_{nk}||\xi_k| < \infty \quad .$$

The dominated convergence theorem says that the above two conditions are enough to ensure that (b-i) holds (e.g. see Theorem 1 in Kubrusly (1986)), so that (a) $\Longrightarrow$ (b-i). Since (a-ii) $\Longrightarrow$ (b-ii) trivially, we get (a) $\Longrightarrow$ (b). On the other hand, take an arbitrary integer $\ell \geq 1$ and set $x_\ell = (0, \dots, 0, 1, 0, \dots) \in \ell_1$ with 1 at the ℓth position and zeros elsewhere, so that (b-i) $\Longrightarrow$ (a-i) and (b-ii) $\Longrightarrow$ (a-ii). Thus (b) $\Longrightarrow$ (a). □

Proposition 2. Let $\{e_k;\ k \geq 1\}$ be any orthonormal basis for H. The linear manifold $H_1 = \{x \in H:\ \sum_{k=1}^{\infty} |\langle x; e_k \rangle| < \infty\}$ is dense in H.

Sketch proof. Let $D \in B[H]$ be a diagonal operator with respect to a given orthogonal basis $\{e_k;\ k \geq 1\}$ for H. That is, set

$$Dx = \sum_{k=1}^{\infty} \lambda_k \langle x;e_k\rangle e_k \qquad \forall x\in H$$

for some bounded complex sequence $\{\lambda_k;\ k\ge 1\}$. Since D is normal, $\sigma_R(D)=\emptyset$. Now assume that $\lambda_k\ne 0$ for every $k\ge 1$, so that $0\notin\sigma_P(D)$. Hence $\overline{R(D)}=H$, because $0\in\sigma_C(D)\cup\rho(D)$. Assume further that $\sum_{k=1}^{\infty}|\lambda_k|^2<\infty$, so that

$$\sum_{k=1}^{\infty}|\langle Dx;e_k\rangle| \le$$
$$\left(\sum_{k=1}^{\infty}|\lambda_k|^2\right)^{1/2}\left(\sum_{k=1}^{\infty}|\langle x;e_k\rangle|^2\right)^{1/2}<\infty$$

for all $x\in H$. Thus $R(D)\subseteq H_1$. Then $\overline{H}_1=H$. □

Lemma 1. Take an arbitrary orthonormal basis $\{e_k;\ k\ge 1\}$ for H. Let $T\in B[H]$ and consider a sequence $\{T_n\in B[H];\ n\ge 1\}$. $T_n \xrightarrow{w} T$ if and only if $\sup_{n\ge 1}\|T_n\|<\infty$ and $\lim_{n\to\infty}\langle (T_n-T)e_k;e_\ell\rangle=0$ for every $k,\ell\ge 1$.

Proof. The "only if" part follows immediately by the Banach-Steinhaus theorem. On the other hand, suppose $\sup_{n\ge 1}\|T_n\|<\infty$ (so that $\sup_{n\ge 1}|\langle (T_n-T)x;y\rangle|\le (\sup_{n\ge 1}\|T_n\|+\|T\|)\,\|x\|\,\|y\|\ \ \forall x,y\in H$), and also that $\lim_{n\to\infty}\langle (T_n-T)e_k;e_\ell\rangle=0$ for every $k,\ell\ge 1$, for some orthonormal basis $\{e_k;\ k\ge 1\}$ for H. Thus, by the Fourier series theorem,

$$\limsup_{n\to\infty}|\langle (T_n-T)e_k;y\rangle| \le$$
$$\limsup_{n\to\infty}\sum_{\ell=1}^{\infty}|\langle (T_n-T)e_k;e_\ell\rangle|\,|\langle e_\ell;y\rangle| = 0$$

for each $k\ge 1$ and every $y\in H_1$, since (a) ⟹ (b-i) in Proposition 1, with $H_1\subset H$ defined as in Proposition 2. Hence

$$\limsup_{n\to\infty}|\langle x;(T_n-T)^*y\rangle| \le$$
$$\limsup_{n\to\infty}\sum_{k=1}^{\infty}|\langle (T_n-T)e_k;y\rangle|\,|\langle x;e_k\rangle| = 0$$

for every $x,y\in H_1$, since (a) ⟹ (b-i) in Proposition 1. Therefore, $\lim_{n\to\infty}\langle (T_n-T)x;y\rangle=0$ for every $x,y\in H_1$. However, the limit actually holds for every $x,y\in H$ (i.e. $T_n\xrightarrow{w}T$), since $\sup_{n\ge 1}\|T_n\|<\infty$ and $\overline{H}_1=H$ according to Proposition 2 (e.g. see Weidmann (1980, p.81)). □

Theorem 1. Let $T,P\in B[H]$, and let $\{e_k;\ k\ge 1\}$ be an arbitrary orthonormal basis for H. The following assertions are equivalent.

(a) $T^n \xrightarrow{w} P$.

(b) $PT = TP = P^2 = P$ and $(T-P)^n \xrightarrow{w} 0$.

(c) $\sup_{n\ge 1}\|T^n\| < \infty$ and
$\lim_{n\to\infty}\langle T^n e_k;e_\ell\rangle = \langle Pe_k;e_\ell\rangle \qquad \forall k,\ell\ge 1.$

Moreover, if the above holds, then

(d) $r_\sigma(T) \le 1$,

(e) $\sigma_R(T) \subseteq \Delta$,

(f) $\sigma_P(T) \subseteq \Delta \cup \{1\}$,

(g) $P = 0 \iff \sigma_P(T)\subseteq\Delta \iff 1\notin\sigma_P(T)$.

Proof. First note that the equivalence (a) ⟺ (c) is a particular case of Lemma 1. Recall that $r_\sigma(T)^n=r_\sigma(T^n)\le\|T^n\|$ for every $n\ge 1$, so that $r_\sigma(T)\le 1$ whenever $\sup_{n\ge 1}\|T^n\|<\infty$. Thus (c) ⟹ (d). It is readily verified by induction that $(T-P)^n=T^n-P$ for every $n\ge 1$ whenever $PT=TP=P^2=P$, so that (b) ⟹ (a). From now on, suppose (a) holds. Then $T^{n+1}\xrightarrow{w}P$, $T^{n+1}=T^nT\xrightarrow{w}PT$, and $T^{n+1}=TT^n\xrightarrow{w}TP$. Hence, by uniqueness of the weak limit, $PT=TP=P$; which implies by induction that $PT^n=P$ for every $n\ge 1$, so that $P=PT^n\xrightarrow{w}P^2$. Thus $PT=TP=P^2=P$, which implies by induction that $(T-P)^n=T^n-P$ for every $n\ge 1$, so that $(T-P)^n\xrightarrow{w}0$ (i.e. (a) ⟹ (b)). Now recall that $\sigma_P(T)\cup\sigma_R(T)=\sigma_P(T)\cup\sigma_P(T^*)^*$ for any $T\in B[H]$. If $\lambda\in\sigma_P(T)\cup\sigma_R(T)$, then there exists $x_o\in H$, with $\|x_o\|=1$, such that $Tx_o=\lambda x_o$ or $T^*x_o=\overline{\lambda}x_o$. Hence, in both cases,

$$\lambda^n = \lambda^n\langle x_o;x_o\rangle =$$
$$\langle T^n x_o;x_o\rangle \to \langle Px_o;x_o\rangle\in\mathbb{C} \qquad \text{as} \qquad n\to\infty.$$

Thus, either $|\lambda|<1$ or $\lambda=1$. Therefore

(h) $\sigma_P(T)\cup\sigma_R(T)\subseteq\Delta\cup\{1\}$,

which implies (f). Moreover, if $P=0$, then $\lambda^n\to 0$ as $n\to\infty$, so that $|\lambda|<1$. Hence

(i) $P = 0 \implies \sigma_P(T)\cup\sigma_R(T)\subseteq\Delta$.

We have already verified that $(I-T)P=0$, so that $R(P)\subseteq N(I-T)$. Therefore

(j) $P\ne 0 \iff R(P)\ne\{0\} \implies N(I-T)\ne\{0\} \iff 1\in\sigma_P(T)$.

By (i) and (j) we finally get: $1\notin\sigma_R(T)$, so that (h) implies (e); and $\sigma_P(T)\subseteq\Delta \implies 1\notin\sigma_P(T) \implies P=0 \implies \sigma_P(T)\subseteq\Delta$, so that (g) holds. □

4. WEAK ASYMPTOTIC STABILITY

Take an arbitrary $T\in B[H]$, and recall that $r_\sigma(T)=\inf_{Q\in G[H]}\|QTQ^{-1}\|$, so that T is similar to a strict contraction if and only if $r_\sigma(T)<1$ (cf. Rota (1960)). According to the Gelfand formula ($r_\sigma(T)=\lim_{n\to\infty}\|T^n\|^{1/n}$), and recalling again that $r_\sigma(T)^n=r_\sigma(T^n)\le\|T^n\|$ for every $n\ge 1$, it follows that $r_\sigma(T)<1$ if and only if $\|T^n\|\to 0$ as $n\to\infty$. Summing up leads to the following well-known equivalence.

(2) $r_\sigma(T)<1 \iff T^n\xrightarrow{u}0 \iff$ T is s.s.c.

In this section we shall investigate what can be said, in the light of the above equivalence, when uniform asymptotic stability is relaxed to weak asymptotic stability. More precisely, we shall be looking for a relationship between weak asymptotic stability and similarity to contraction, as well as for the role played by the parts of the spectrum in the characterization of such a relationship. From Theorem 1 we immediately get the following result.

$$(3)\quad T^n \xrightarrow{w} 0 \Longrightarrow \begin{cases} \sup_{n\geq 1}\|T^n\|<\infty \;(\Longrightarrow r_\sigma(T)\leq 1), \\ \sigma_P(T) \cup \sigma_R(T) \subseteq \Delta. \end{cases}$$

Thus, according to (2), if the continuous spectrum does not intersect the unit circle (trivial example: compact operators), then all the three concepts of asymptotic stability coincide. In other words

$$(4)\quad T^n \xrightarrow{u} 0 \Longleftrightarrow \begin{cases} T^n \xrightarrow{w} 0, \\ \sigma_C(T) \cap \Gamma = \emptyset. \end{cases}$$

Now notice that weak asymptotic stability and similarity to contraction share some common properties. For instance, if T is similar to a contraction, than it is power bounded (i.e. $\sup_{n\geq 1}\|T^n\|<\infty$). This is trivially verified (recall that $T^n=Q^{-1}C^nQ$ for every $n\geq 1$ whenever $QTQ^{-1}=C$). Moreover, if T is similar to a contraction, then its residual spectrum is contained in the open unit disc. This is a consequence of the following fact: $\lambda\in\sigma_R(C) \Longrightarrow |\lambda|<\|C\|$ (recall that $\sigma_R(C)=\sigma_P(C^*)^*\backslash\sigma_P(C)$ for any $C\in B[H]$, and take an arbitrary $\lambda\in\sigma_R(C)$, so that $0<\|Cx-\lambda x\|^2 = \|Cx\|^2 + |\lambda|^2\|x\|^2 - 2\mathrm{Re}\langle\bar{\lambda}x;C^*x\rangle = \|Cx\|^2-|\lambda|^2\|x\|^2$ for some $0\neq x\in H$, which implies that $|\lambda|<\|C\|$). Since similarity preserves the spectrum and its parts (e.g. see Halmos (1982, p.42)), $\sigma_R(T)=\sigma_R(C)$ whenever T is similar to C. If C is a contraction, then $\sigma_R(T)=\sigma_R(C)\subseteq\Delta$. Summing up we get

$$(5)\quad T \text{ is s.c.} \Longrightarrow \begin{cases} \sup_{n\geq 1}\|T^n\|<\infty \;(\Longrightarrow r_\sigma(T)\leq 1), \\ \sigma_R(T) \subseteq \Delta. \end{cases}$$

For compact operators, similarity to contraction is equivalent to power boundedness (cf. Nagy (1958)). For spectraloid operators this is also true, since T is similar to a contraction whenever $\omega(T)\leq 1$ (cf. Nagy and Foias (1970, p.95)). An example of a power bounded operator which is not similar to a contraction was given by Foguel (1964). However, such an operator, say F, is not weakly asymptotically stable. Indeed, by the very construction of F (cf. Foguel (1964) and Halmos (1964)), it is readily verified that $\sigma_P(F)=\Delta$, $\sigma_R(F)=\emptyset$, $\sigma_C(F)=\Gamma$, and $F^n \not\xrightarrow{w} 0$. Thus, not only the converse of (5) fails, but it also fails the converse of (3). Couldn't one find another power bounded operator, which is also not similar to a contraction, but weakly asymptotically stable? Putting it in another way. Is every weakly asymptotically stable operator similar to a contraction? Since the inclusion of the point spectrum in the open unit disc is a necessary condition for weak asymptotic stability, and since this is clearly not necessary for similarity to contraction (e.g. take the identity operator), the converse of the above question should read as follows. Does similarity to contraction imply weak asymptotic stability for an operator with point spectrum contained in the open unit disc? Let us synthesize the above questions.

<u>Question 1</u>. $T^n \xrightarrow{w} 0 \Longrightarrow T$ is s.c.?

<u>Question 2</u>. T is s.c. and $\sigma_P(T)\subseteq\Delta \Longrightarrow T^n \xrightarrow{w} 0$?

5. CONCLUDING REMARKS

There are some partial evidences that the answer to Question 1 may be negative, since a positive answer would lead to a universal model for strong asymptotic stability. Actually, a positive answer to Question 1 would trivially ensure that " $T^n \xrightarrow{s} 0 \Longrightarrow T$ is s.c. ". However, if the above assertion holds, then (e.g. see Kubrusly (1988)) strong asymptotic stability turns out to be equivalent to similarity to part of the adjoint of a shift operator. On the other hand, Question 1 has clearly a positive answer for those classes of operators for which power boundedness implies similarity to contraction (e.g. compact or spectraloid operators).

Finally, we show that Question 2 is equivalent to the following one.

<u>Question 2'</u>. U is unitary and $\sigma_P(U)=\emptyset \Longrightarrow U^n \xrightarrow{w} 0$?

Recall that $U\in B[H]$ is unitary iff $U\in G[H]$ and $U^{-1}=U^*$, so that $\sigma(U)\subseteq\Gamma$ and $r_\sigma(U)=\|U\|=1$. Thus, a positive answer to Question 2 trivially implies a positive answer to Question 2' (since U is a contraction). To verify the converse, suppose T is similar to a contraction C and set $Z(C)=\{x\in H: \langle C^nx;x\rangle\to 0 \text{ as } n\to\infty\}$. Note that $Z(C)=H \Longleftrightarrow C^n \xrightarrow{w} 0 \Longleftrightarrow T^n \xrightarrow{w} 0$ (since $QT^n=C^nQ$ for every $n\geq 1$, for some $Q\in G[H]$). If $Z(C)\neq H$, then $\|C\|=1$ (reason: $Z(C)\neq H \Longrightarrow T^n \not\xrightarrow{w} 0 \Longrightarrow T^n \not\xrightarrow{u} 0 \Longrightarrow T$

is not s.s.c. $\Longrightarrow \|C\|=1$). Since $\|C\|=1$, $Z(C)$ is a subspace (i.e. a closed linear manifold) of H which reduces C, and $U:=C|_{Z(C)^\perp}$ in $B[Z(C)^\perp]$ is unitary (cf. Foguel (1963)). Note that $Z(C)^\perp$ is nontrivial (because $\overline{Z(C)}=Z(C)\neq H$). Since $Z(C)$ reduces C, $C^n=V^n \oplus U^n$ for every $n\geq 1$, with $V:=C|_{Z(C)}$ in $B[Z(C)]$, and $V^n \xrightarrow{w} 0$, according to the definition of the reducing subspace $Z(C)$. Since U is unitary and $Z(C)^\perp$ is nontrivial, $\sigma_p(U) \subseteq (\sigma_p(C) \cap \Gamma)$. Now suppose $\sigma_p(T) \subseteq \Delta$, and recall that $\sigma_p(C)=\sigma_p(T)$. Thus $\sigma_p(U)=\emptyset$. Therefore, if Question 2' has a positive answer, then $U^n \xrightarrow{w} 0$, so that $C^n \xrightarrow{w} 0$. Hence $T^n \xrightarrow{w} 0$ (i.e. $Z(C)=H$). Conclusion: a positive answer to Question 2' implies a positive answer to Question 2.

REFERENCES

Foguel, S.R. (1963). Powers to a contraction in Hilbert spaces. *Pacific J. Math.*, 13, 551-562.

Foguel, S.R. (1964). A counterexample to a problem of Sz.-Nagy. *Proc. Amer. Math. Soc.*, 15, 788-790.

Halmos, P.R. (1964). On Foguel's answer to Nagy's question. *Proc. Amer. Math. Soc.*, 15, 791-793.

Halmos, P.R. (1982). *A Hilbert Space Problem Book*. 2nd edn. Springer-Verlag, New York.

Hoffman, K. (1975). *Analysis in Euclidean Spaces*. Prentice-Hall, Englewood Cliffs.

Kamen, E.W. and W.L. Green (1980). Asymptotic stability of linear difference equations defined over a commutative Banach algebra. *J. Math. Anal. Appl.*, 75, 584-601.

Kubrusly, C.S. (1985). Mean square stability for discrete bounded linear systems in Hilbert space. *SIAM J. Control Optimiz.*, 25, 19-29.

Kubrusly, C.S. (1986). On convergence of nuclear and correlation operators in Hilbert space. *Mat. Aplic. Comp.*, 5, 265-282.

Kubrusly, C.S. (1988). Questões sobre similaridade a contração. *Anais Sem. Bras. Anal.*, 28, 381-388.

Kubrusly, C.S. (1989). A note on the Lyapunov equation for discrete linear systems in Hilbert space. *Appl. Math. Lett.*, to appear.

Sz.-Nagy, B. (1958). Completely continuous operators with uniformly bounded iterates. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 3, 89-93.

Sz.-Nagy, B. and C. Foias (1970). *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam.

Przyluski, K.M. (1980). The Lyapunov equation and the problem of stability for linear bounded discrete-time systems in Hilbert space. *Appl. Math. Optim.*, 6, 97-112.

Przyluski, K.M. (1988). Stability of linear infinite-dimensional systems revisited. *Int. J. Control*, 48, 513-523.

Rota, G.-C. (1960). On models for linear operators. *Comm. Pure Appl. Math.*, 13, 469-472.

Weidmann, J. (1980). *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York.

Zabczyk, J. (1974). Remarks on the control of discrete-time distributed parameter systems. *SIAM J. Control*, 12, 721-735.

Zabczyk, J. (1975). On optimal sothcastic control of discrete-time systems in Hilbert space. *SIAM J. Control*, 13, 1217-1234.