Automatica, Vol. 21, No. 2, pp. 117–128, 1985
Printed in Great Britain.

0005-1098/85 $3.00 + 0.00
Pergamon Press Ltd.

Survey Paper

Sensors and Controllers Location in Distributed Systems—A Survey*

C. S. KUBRUSLY†‡ and H. MALEBRANCHE†

An important problem regarding theory and practice of identification, state estimation, and control of distributed systems is the spatial location of sensors and controllers. A review of recent literature and a classification of methods indicate some directions for further research.

Key Words—Controllers location; distributed parameter systems; identification; optimal control; optimization; sensors location; state estimation.

Abstract—A survey of the field of optimal sensors and/or controllers location for dynamical distributed parameter systems modelled by partial differential equations is presented. The recent contributions in this field are grouped according to the main goal for which the location problem is developed, namely: system identification, state estimation, and optimal control. In order to pose the sensors and controllers location problem, the semigroup approach for modelling distributed linear systems is briefly reviewed together with its equivalent (infinite dimensional) and approximate (finite dimensional) Fourier expansion representations. After presenting a concise general review of the several methods considered in the current literature, a classification of methods is also proposed. The main classifying factor concerns the use of N-modal approximation schemes, and the different stages of the optimization procedure in which they are required.

1. *Introduction*

A FUNDAMENTAL problem towards identification, state estimation, and control of distributed systems is the sensors and controllers location (e.g. see Athans, 1970). This comprises the arrangement, in an optimal fashion, of a limited number of measurement transducers and control devices along the spatial domain. In this paper the several methods proposed for solving such a problem are reviewed and classified according to their main characteristics. To begin with it is advisable to give some abbreviations which will be of frequent usage throughout the text:

ODE: Ordinary Differential Equation(s).
PDE: Partial Differential Equation(s).
LPS: Lumped Parameter System(s).
DPS: Distributed Parameter System(s).
OCL: Optimal Controllers Location.
OSL: Optimal Sensors Location.

*Received 23 August 1983; revised 18 April 1984; revised 12 June 1984. The original version of this paper was presented at the 3rd IFAC Symposium on Control of Distributed Parameter Systems which was held in Toulouse, France during June, 1982. The Published Proceedings of this IFAC Meeting may be ordered from: Pergamon Press Limited, Headington Hill Hall, Oxford, OX3 0BW, England. This paper was recommended for publication in revised form by survey paper editors G. H. Saridis and K. J. Åström.

†Department of Research and Development, Scientific Computation Laboratory-LCC/CNPq, Rua Lauro Müller 455, Rio de Janeiro, 22290, Brazil.

‡Department of Electrical Engineering, Catholic University-PUC/RJ, Rua Marquês de São Vicente 209, Rio de Janeiro, 22453, Brazil.

DPS, as discussed here, will stand for dynamical systems governed by PDE (opposite to LPS which are described by ODE), as a class of dynamical systems modelled in an infinite dimensional state space (e.g. see Helton, 1976; Ray, 1978; Curtain and Pritchard, 1978; Pritchard, 1979; Fuhrmann, 1981; and Tzafestas and Stavroulakis, 1983). Therefore, it seems reasonable to say that the starting point for analysing DPS is the PDE literature. Presently the available literature in PDE is certainly huge. To mention just a few books, ranging from introductory to advanced texts, published over the past three decades, see for example Courant and Friedrichs, 1948; Courant and Hilbert, 1962; Garabedian, 1964; Weinberger, 1965; Friedman, 1969; Mikhlin, 1970; John, 1981; Ames, 1972; Treves, 1975; Schechter, 1977; Showalter, 1977; and Gustafson, 1980.

Three of the main problems in system theory (and in particular in DPS) are system identification, state estimation, and optimal control (cf. Ray and Lainiotis, 1978 and Stavroulakis, 1983). OCL and OSL will be regarded here as intermediate problems for considering the above mentioned 'final' problems. Although little literature has been written on DPS identification compared with what has been done for state estimation and optimal control, some survey papers have already appeared in this field. For instance, see Polis and Goodson, 1976; Kubrusly, 1977; Goodson and Polis, 1978; Ruberti, 1978; Burger and Chavent, 1979; Chavent, 1979 and Polis, 1982. On the other hand the current literature on state estimation is much richer. For some complete books and surveys on the state estimation problem in DPS, regarding both theory and applications, see for example Bensoussan, 1971; Phillipson, 1971; Curtain and Pritchard, 1978; Sawaragi, Soeda and Omatu, 1978; Curtain, 1975; Ray, 1975; Tzafestas, 1978 and Bencala and Seinfeld, 1979. The optimal control problem in DPS has also been reported in several books and surveys. For instance, see Wang, 1964; Lions, 1968, 1972, 1978, 1980; Butkovskiy, 1969; Balakrishnan, 1980; Aziz, Wingate and Balas, 1977; Curtain and Pritchard, 1978; Ahmed and Teo, 1981; Tzafestas, 1982; Robinson, 1974; Curtain, 1978 and Bensoussan, 1978.

The present survey is organized as follows. In Section 2 the semigroup approach for modelling DPS is briefly reviewed. After describing a linear model for DPS in a separable Hilbert space, and its equivalent (infinite dimensional) representation in terms of Fourier expansion, the so-called N-modal (finite dimensional) approximation is also presented. The OCL and OSL problems are motivated by the model dependence on the spatial location of controllers and sensors. Section 3 comprises a brief review of recent OCL and OSL literature, emphasizing the 'final' problem (i.e. system identification, state estimation, or optimal control) for which the OCL and OSL problems were developed. A classification of methods is proposed in Section 4, where the methods reviewed in Section 3 are compared according to their main structural characteristics. The paper ends with some comments and concluding remarks in Section 5.

2. *Modelling preliminaries*

This section contains a brief summary on the following topics: (1) linear models for DPS; (2) equivalent model description; (3) N-modal approximation; and (4) model dependence on the spatial location of controllers and sensors. The motivation for this will become clear in Section 3 when the several OSL and OCL methods will be reviewed. As it will be emphasized in Section 4, the major factor for classifying OSL and OCL methods will rely on when optimization techniques are applied, either before or after considering any model approximation; and the so-called N-modal is the most used approximation technique in the OSL and OCL literature. Illustrative examples of optimal location problems close the section.

Notation. The notation used in the paper is formally summarized as follows:

$\mathbb{R}^n$	n-dimensional Euclidean space
$\partial\Omega$	boundary of $\Omega \subset \mathbb{R}^n$.
$\langle ; \rangle_H$	inner product in a Hilbert space H.
$\| \|_H$	norm in a Hilbert space H
$D(L)$	domain of a transformation L
L^*	adjoint of a transformation L
$\mathrm{tr}[P]$	trace of a matrix P
$\mathrm{diag}(\mu_1, \ldots, \mu_k)$	diagonal matrix
$\dot{w}$	time derivative of w ($\dot{w} = \partial w/\partial t$).
E	The expectation operator, as usual
$M[\mathbb{R}^k, \mathbb{R}^l]$	linear space of all real matrices l by k ($M[\mathbb{R}^k] = M[\mathbb{R}^k, \mathbb{R}^k]$)
$Blt[X, Y]$	normed linear space of all bounded linear transformations of X into Y, X and Y being normed linear spaces ($Blt[X] = Blt[X, X]$).

The linear spaces l_2, $L_2(0, T)$, $L_2(\Omega)$, $L_2(0, T; H)$, $C(0, T)$, $C(\Omega)$, $C^2(\Omega)$, and $C(0, T; H)$ will have their standard meanings (e.g. see Curtain and Pritchard, 1977 and Leigh, 1980).

A linear model for DPS in $L_2(\Omega)$. Technical details are omitted throughout this section and the reader is here, once and for all, referred to the available literature. As far as Hilbert space methods are concerned see, for instance, Kato, 1980; Akhiezer and Glazman, 1981; Naylor and Sell, 1982 or Weidmann, 1980, among others. Classical references for the semigroup theory are Hille and Phillips, 1957; Dunford and Schwartz, 1958 and Yosida, 1980. For an introduction to semigroups towards control theory see, for example, Balakrishnan, 1980 and Curtain and Pritchard, 1978.

Let U (the input or control space), H (the state space), and Z (the observation or output space) be separable Hilbert spaces, and consider a linear dynamical system modelled by an autonomous inhomogeneous abstract differential equation as

$$\dot{y} = Ay + Bu; \quad y(0) = y_0 \in H, \tag{1}$$

where $u \in L_2(0, T; U)$, $B \in Blt[U, H]$, and the (closed linear but possibly unbounded) operator $A: D(A) \to H$ is the infinitesimal generator of a strongly continuous semigroup $\{T_t \in Blt[H]; t \geq 0\}$, whose domain $D(A)$ is dense in H. The mild solution of (1) is given by

$$y(t) = T_t y_0 + \int_0^t T_{t-\tau} Bu(\tau)\, d\tau \tag{2}$$

with $y \in C(0, T; H)$. Furthermore, let $v, z \in L_2(0, T; Z)$ and $C \in Blt[H, Z]$, and suppose the state y is observed according to the following measurement equation

$$z = Cy + v. \tag{3}$$

Now set $H = L_2(\Omega)$, Ω being a simply connected open set in $\mathbb{R}^n$, and consider a linear time-invariant DPS governed by a parabolic PDE as in (1). For example suppose a special case where the system operator A is a second order elliptic self-adjoint one of the form

$$A = \sum_{v} \alpha_v D^v, \tag{4}$$

with

$$v = (v_1, \ldots, v_n) \in I^n, \quad v_q \in I = \{0, 1, 2\}; \quad \forall q = 1, 2, \ldots, n$$

$$|v| = \sum_{q=1}^{n} v_q, \text{ such that } 0 \leq |v| \leq 2, \qquad \alpha_v \in C^2(\Omega)$$

$$D^v = \frac{\partial^{|v|}}{\partial x_1^{v_1} \ldots \partial x_n^{v_n}}: \quad H^2(\Omega) \to L_2(\Omega)$$

$$H^2(\Omega) = \{w \in L_2(\Omega): \quad D^v w \in L_2(\Omega); \quad |v| = 1, 2\}$$

$$D(A) = \{w \in H^2(\Omega): \quad Lw = 0 \text{ on } \partial\Omega\}$$

where L denotes a linear operator defined on $\partial\Omega$ (standing for the boundary conditions). Moreover, assume that there exists an infinite divergent real sequence $\{\lambda_i; i = 1, 2, \ldots\}$ of eigenvalues of A, which is bounded above and non-increasingly ordered. That is

$$A\phi_i = \lambda_i \phi_i$$

$$\lambda_{i+1} \leq \lambda_i < \zeta < \infty$$

$$|\lambda_i| \to \infty \text{ as } i \to \infty$$

where $\{\phi_i \in D(A); i = 1, 2, \ldots\}$ is an orthonormal basis for $L_2(\Omega)$ of eigenvectors of A. Then the solution in (2) has an unique Fourier series expansion

$$y(t) = \sum_{i=1}^{\infty} a^i(t)\phi_i \tag{5}$$

with coefficients $a^i \in C(0, T)$

$$a^i(t) = \langle y(t); \phi_i \rangle_{L_2(\Omega)}.$$

Hence, for $y(t) \in D(A)$

$$Ay(t) = \sum_{i=1}^{\infty} \lambda_i a^i(t)\phi_i \tag{6}$$

and the semigroup $\{T_t \in Blt[L_2(\Omega)]; t \geq 0\}$ generated by A is given by

$$T_t y(\tau) = \sum_{i=1}^{\infty} e^{\lambda_i t} a^i(\tau)\phi_i.$$

Now set $U = \mathbb{R}^p$, and assume that the input transformation $B \in Blt[\mathbb{R}^p, L_2(\Omega)]$ is such that

$$Bu(t) = \sum_{j=1}^{p} \beta_j u_j(t)$$

with $\beta_j \in L_2(\Omega)$ and $u_i \in L_2(0, T)$ for each $j = 1, 2, \ldots, p$. By the Fourier series expansion of β_j one gets

$$Bu(t) = \sum_{i=1}^{\infty} \langle u(t); b_i \rangle_{\mathbb{R}^p} \phi_i \tag{7}$$

where

$$u = (u_1, \ldots, u_p) \in L_2(0, T; \mathbb{R}^p)$$

$$b_i = (\langle \beta_1; \phi_i \rangle_{L_2(\Omega)}, \ldots, \langle \beta_p; \phi_i \rangle_{L_2(\Omega)}).$$

Finally set $Z = \mathbb{R}^m$, and let the output transformation $C \in Blt[L_2(\Omega), \mathbb{R}^m]$ be given by

$$Cy(t) = (\langle y(t); \gamma_1 \rangle_{L_2(\Omega)}, \ldots, \langle y(t); \gamma_m \rangle_{L_2(\Omega)}) \tag{8}$$

where $\gamma_k \in L_2(\Omega)$ for each $k = 1, 2, \ldots, m$, such that

$$z_k(t) = \langle y(t); \gamma_k \rangle_{L_2(\Omega)} + v_k(t)$$

with $z = (z_1, \ldots, z_m)$, $v = (v_1, \ldots, v_m) \in L_2(0, T; \mathbb{R}^m)$.

An equivalent model in l_2. Equations (5)–(8) supply an equivalent representation in l_2 for the system model (1)–(4) in $L_2(\Omega)$, as follows:

$$\dot{a} = \mathrm{A}a + \mathrm{B}u; \quad a(0) \in l_2 \tag{9}$$

where $\mathrm{B} \in \mathrm{B}lt[\mathbb{R}^p, l_2]$ is such that

$$\mathrm{B}u(t) = (\langle Bu(t);\phi_1\rangle_{\mathrm{L}_2(\Omega)}, \langle Bu(t);\phi_2\rangle_{\mathrm{L}_2(\Omega)}, \ldots)$$
$$= (\langle u(t);b_1\rangle_{\mathbb{R}^p}, \langle u(t);b_2\rangle_{\mathbb{R}^p}, \ldots)$$

and $\mathrm{A}:\mathrm{D(A)} \to l_2$ is a closed densely defined linear operator,

$$\mathrm{A}a(t) = (\langle Ay(t);\phi_1\rangle_{\mathrm{L}_2(\Omega)}, \langle Ay(t);\phi_2\rangle_{\mathrm{L}_2(\Omega)}, \ldots)$$
$$= (\lambda_1 a^1(t), \lambda_2 a^2(t), \ldots)$$
$$\mathrm{D(A)} = \{\omega = (\omega_1, \omega_2, \ldots) \in l_2: \sum_{i=1}^{\infty} |\lambda_i \omega_i|^2 < \infty\}$$

generating a strongly continuous semigroup $\{\mathrm{T}_t \in \mathrm{B}lt[l_2]; t \geq 0\}$

$$\mathrm{T}_t a(\tau) = (\langle T_t y(\tau);\phi_1\rangle_{\mathrm{L}_2(\Omega)}, \langle T_t y(\tau);\phi_2\rangle_{\mathrm{L}_2(\Omega)}, \ldots)$$
$$= (e^{\lambda_1 t} a^1(\tau), e^{\lambda_2 t} a^2(\tau), \ldots).$$

The mild solution of (9) is then given by

$$a(t) = \mathrm{T}_t a(0) + \int_0^t \mathrm{T}_{t-\tau} \mathrm{B}u(\tau)\,\mathrm{d}\tau$$

with $a = (a^1, a^2, \ldots) \in \mathrm{C}(0, \mathrm{T}; l_2)$. By the Fourier expansion of γ_k in (8) one gets the following equivalent representation in l_2 for the measurement equation (3) in $\mathrm{L}_2(\Omega)$

$$z = \mathrm{C}a + v \tag{10}$$

where $\mathrm{C} \in \mathrm{B}lt[l_2, \mathbb{R}^m]$ is given by

$$\mathrm{C}a(t) = (\langle a(t);c_1\rangle_{l_2}, \ldots, \langle a(t);c_m\rangle_{l_2})$$

with

$$c_k = (\langle \gamma_k;\phi_1\rangle_{\mathrm{L}_2(\Omega)}, \langle \gamma_k;\phi_2\rangle_{\mathrm{L}_2(\Omega)}, \ldots)$$

for each $k = 1, 2, \ldots, m$.

An approximate model in $\mathbb{R}^N$. The so-called N-modal approximation consists in truncating the Fourier series expansions involved in their first N terms, yielding a Galerkin-like approximation for the state y in (5),

$$y_N(t) = \sum_{i=1}^{N} a^i(t)\phi_i$$

$y_N \in \mathrm{C}(0, \mathrm{T}; \mathrm{H}_N)$, where H_N is the N-dimensional linear subspace of $\mathrm{L}_2(\Omega)$ spanned by $\{\phi_i; i = 1, 2, \ldots, N\}$. This supplies an approximate representation in $\mathbb{R}^N$ for the equivalent system model (9) in l_2, given by

$$\dot{a}_N = \mathrm{A}_N a_N + \mathrm{B}_N u; \quad a_N(0) \in \mathbb{R}^N \tag{11}$$

where

$$\mathrm{A}_N = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathrm{M}[\mathbb{R}^N]$$
$$\mathrm{B}_N = [b_1, \ldots, b_N]^* \in \mathrm{M}[\mathbb{R}^p, \mathbb{R}^N]$$

whose solution is

$$a_N(t) = \mathrm{T}_t^N a_N(0) + \int_0^t \mathrm{T}_{t-\tau}^N \mathrm{B}_N u(\tau)\,\mathrm{d}\tau$$

with $a_N = (a^1, \ldots, a^N) \in \mathrm{C}(0, \mathrm{T}; \mathbb{R}^N)$, and

$$\mathrm{T}_t^N = e^{\mathrm{A}_N t} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_N t}) \in \mathrm{M}[\mathbb{R}^N]; \quad t \geq 0.$$

The observation $z_N = (z_1^N, \ldots, z_m^N) \in L_2(0, \mathrm{T}; \mathbb{R}^m)$ is given by the following approximate version of (10)

$$z_N = \mathrm{C}_N a_N + v \tag{12}$$

with

$$\mathrm{C}_N = [c_1^N \ldots c_m^N]^* \in \mathrm{M}[\mathbb{R}^N, \mathbb{R}^m]$$
$$c_k^N = (\langle \gamma_k;\phi_1\rangle_{\mathrm{L}_2(\Omega)}, \ldots, \langle \gamma_k;\phi_N\rangle_{\mathrm{L}_2(\Omega)})$$

for each $k = 1, \ldots, m$.

Model dependence on the spatial location of controllers and sensors. Suppose the input transformation B in (7) depends on a vector $\mathbf{x}^c = (x_1^c, \ldots, x_p^c) \in \mathbb{R}^{np}$ as follows. Let the input (or control) coefficients depend on $\mathbf{x}^c$ in the following way:

$$\beta_j = \beta_{x_j^c}; \quad x_j^c \in \Omega \subset \mathbb{R}^n,$$

where $\mathbf{x}^c$ describes the controllers spatial location, such that $B = B(\mathbf{x}^c) \in \mathrm{B}lt[\mathbb{R}^p, L_2(\Omega)]$ is given by

$$[Bu(t)](x) = \sum_{j=1}^{p} \beta_{x_j^c}(x) u_j(t).$$

For instance, let

$$0 < \varepsilon < \inf_{1 \leq j \leq p} \inf_{x \in \partial\Omega} \|x_j^c - x\|_{\mathbb{R}^n}$$

such that the closed ball $\sigma_\varepsilon[x_j^c]$ of radius ε centered at x_j^c is contained in Ω for each $j = 1, \ldots, p$, and let $\mu_\varepsilon > 0$ be the usual measure of $\sigma_\varepsilon[x_j^c]$. Now set

$$\beta_{x_j^c}(x) = \begin{cases} \mu_\varepsilon^{-1}; & \text{if } x \in \sigma_\varepsilon[x_j^c], \\ 0; & \text{otherwise.} \end{cases}$$

Hence

$$\langle \beta_j;\phi_i\rangle_{\mathrm{L}_2(\Omega)} = \mu_\varepsilon^{-1} \int_{\sigma_\varepsilon[x_j^c]} \phi_i(x)\,\mathrm{d}x.$$

Therefore the approximate (N-modal) representation for the input transformation in (11) is given by

$$\mathrm{B}_N = \mathrm{B}_N(\mathbf{x}^c) = \mu_\varepsilon^{-1} \begin{bmatrix} \int_{\sigma_\varepsilon[x_1^c]} \phi_1(x)\,\mathrm{d}x & \ldots & \int_{\sigma_\varepsilon[x_p^c]} \phi_1(x)\,\mathrm{d}x \\ \vdots & & \vdots \\ \int_{\sigma_\varepsilon[x_1^c]} \phi_N(x)\,\mathrm{d}x & \ldots & \int_{\sigma_\varepsilon[x_p^c]} \phi_N(x)\,\mathrm{d}x \end{bmatrix} \in \mathrm{M}[\mathbb{R}^p, \mathbb{R}^N].$$

In a similar fashion, suppose the output transformation C in (8) depends on a vector $\mathbf{x}^s = (x_1^s, \ldots, x_m^s) \in \mathbb{R}^{nm}$ as follows. Let the output coefficients depend on $\mathbf{x}^s$ in the following way:

$$\gamma_k = \gamma_{x_k^s}; \quad x_k^s \in \Omega \subset \mathbb{R}^n$$

where $\mathbf{x}^s$ describes the sensors spatial location, such that $C = C(\mathbf{x}^s) \in \mathrm{B}lt[L_2(\Omega), \mathbb{R}^m]$ is given by

$$Cy(t) = (\langle y(t);\gamma_{x_1^s}\rangle_{\mathrm{L}_2(\Omega)}, \ldots, \langle y(t);\gamma_{x_m^s}\rangle_{\mathrm{L}_2(\Omega)}).$$

For instance, set

$$\gamma_{x_k^s}(x) = \begin{cases} \mu_\varepsilon^{-1}; & \text{if } x \in \sigma_\varepsilon[x_k^s] \\ 0; & \text{otherwise} \end{cases}$$

where $\sigma_\varepsilon[x_k^s]$ is defined as $\sigma_\varepsilon[x_j^c]$, with $\mathbf{x}^c$ replaced by $\mathbf{x}^s$. Therefore the appropriate (N-modal) representation for the output transformation in (12) is given by

$$C_N = C_N(\mathbf{x}^s) = \mu_\varepsilon^{-1}\begin{bmatrix} \int_{\sigma_\varepsilon[x_1^s]} \phi_1(x)\,dx & \dots & \int_{\sigma_\varepsilon[x_1^s]} \phi_N(x)\,dx \\ \vdots & & \vdots \\ \int_{\sigma_\varepsilon[x_m^s]} \phi_1(x)\,dx & \dots & \int_{\sigma_\varepsilon[x_m^s]} \phi_N(x)\,dx \end{bmatrix} \in M[\mathbb{R}^N, \mathbb{R}^m].$$

Before going further it is worth remarking on pointwise controllers and sensors. Consider a formal approach by letting $\varepsilon \to 0$. In such a case the input (or control) and output coefficients $\beta_{x_j^c}$ and $\gamma_{x_k^s}$ can be thought of as Dirac measures, that is

$$\beta_{x_j^c}(x) = \delta(x - x_j^c)$$

$$\gamma_{x_k^s}(x) = \delta(x - x_k^s)$$

thus supplying approximate representations for the input and output transformations of the following form.

$$B_N(\mathbf{x}^c) = \begin{bmatrix} \phi_1(x_1^c) & \dots & \phi_1(x_p^c) \\ \vdots & & \vdots \\ \phi_N(x_1^c) & \dots & \phi_N(x_p^c) \end{bmatrix}$$

$$C_N(\mathbf{x}^s) = \begin{bmatrix} \phi_1(x_1^s) & \dots & \phi_N(x_1^s) \\ \vdots & & \vdots \\ \phi_1(x_m^s) & \dots & \phi_N(x_m^s) \end{bmatrix}.$$

However the above formal approach leads to unbounded transformations, and $L_2(\Omega)$ is no longer an appropriate state space.

Illustrative examples of OCL and OSL problems. The idea behind the following examples is only to illustrate, in a formal and most simplified way, two optimal location problems. Questions of well-posedness are not addressed here, but discussed in the next section. The first example concerns OCL and OSL for optimal stochastic control, and the second one regards OSL for system identification.

Example 1. For simplicity consider a stochastic version of the approximate model in (11), (12).

$$da_N(t) = A_N a_N(t)\,dt + B_N(\mathbf{x}^c)[u(t)\,dt + dw(t)]$$

$$dz_N(t) = C_N(\mathbf{x}^s) a_N(t)\,dt + dv(t)$$

where $\mathbf{x}^c \in \Omega^p \subset \mathbb{R}^{np}$ and $\mathbf{x}^s \in \Omega^m \subset \mathbb{R}^{nm}$ are parameters characterizing the spatial location of controllers and sensors, respectively. Here $\{w(t); t \geq 0\}$ and $\{v(t); t \geq 0\}$ are independent Wiener processes in $\mathbb{R}^p$ and $\mathbb{R}^m$, with incremental covariance matrices $R_w \in M[\mathbb{R}^p]$ and $R_v \in M[\mathbb{R}^m]$, standing for input disturbance and observation noise, respectively. $\{u(t); 0 \leq t \leq T\}$ is an $\mathbb{R}^p$-valued second order stochastic control, that is

$$E\{\|u\|^2_{L_2(0,T;\mathbb{R}^p)}\} = E\left\{\int_0^T \|u(t)\|^2_{\mathbb{R}^p}\,dt\right\} < \infty$$

which depends only on the past observations $\{z_N(\tau); 0 \leq \tau \leq t\}$; and $a_N(0)$ is a zero mean Gaussian random variable in $\mathbb{R}^n$ with covariance matrix $P_0 \in M[\mathbb{R}^N]$, which is independent of $w(t)$ and $v(t)$. A simplified version for the Linear-Quadratic-Gaussian (LQG) problem is to find a stochastic control u, as above, which minimizes the cost

$$J(u) = E\{\|a_N(t)\|^2_{\mathbb{R}^N}\} + E\{\|a_N\|^2_{L_2(0,T;\mathbb{R}^N)}\} + E\{\|u\|^2_{L_2(0,T;\mathbb{R}^p)}\}$$

where the first two criteria characterize the accuracy in which the state can be driven to zero at the final time and along the whole trajectory, respectively, and the third one stands for the control energy. For simplicity, identity weighting matrices have been assumed for each criterion. According to the separation principle (e.g. see Davis, 1977) the solution $u_N = u_N(\mathbf{x}^c, \mathbf{x}^s)$ is given by

$$u_N(t) = -B_N^*(\mathbf{x}^c)Q_N(t)\hat{a}_N(t)$$

where the symmetric feedback control matrix $Q_N(t)$ in $M[\mathbb{R}^N]$ is the unique solution of the backward Riccati equation

$$\dot{Q}_N(t) = Q_N(t)B_N(\mathbf{x}^c)B_N^*(\mathbf{x}^c)Q_N(t) - Q_N(t)A_N - A_N^*Q_N(t) - I_N;\quad Q_N(T) = I_N$$

and $\hat{a}_N(t)$ denotes the Kalman–Bucy filtered estimate of the state $a_N(t)$

$$d\hat{a}_N(t) = [A_N - P_N(t)C_N^*(\mathbf{x}^s)R_v^{-1}C_N(\mathbf{x}^s) - B_N(\mathbf{x}^c)B_N^*(\mathbf{x}^c)Q_N(t)]\hat{a}_N(t)\,dt + P_N(t)C_N^*(\mathbf{x}^s)R_v^{-1}dz_N(t);\quad \hat{a}_N(0) = 0$$

where the error covariance $P_N(t) = E\{[a_N(t) - \hat{a}_N(t)][a_N(t) - \hat{a}_N(t)]^*\}$ in $M[\mathbb{R}^N]$ is the unique solution of the Riccati equation

$$\dot{P}_N(t) = A_N P_N(t) + P_N(t)A_N^* + B_N^*(\mathbf{x}^c)R_w B_N(\mathbf{x}^c) - P_N(t)C_N^*(\mathbf{x}^s)R_v^{-1}C_N(\mathbf{x}^s)P_N(t);\quad P_N(0) = P_0.$$

The optimal cost is then given by

$$J[u_N(\mathbf{x}^c, \mathbf{x}^s)] = \mathrm{tr}[P_N(T)] + \int_0^T \mathrm{tr}[P_N(t)]\,dt + \int_0^T \mathrm{tr}[Q_N(t)P_N(t)C_N^*(\mathbf{x}^s)R_v^{-1}C_N(\mathbf{x}^s)P_N(t)]\,dt.$$

An example of an OCL and OSL problem (for a fixed number of sensors and controllers) is to select $(\mathbf{x}^c, \mathbf{x}^s) \in \Omega^p \times \Omega^m \subset \mathbb{R}^{np} \times \mathbb{R}^{nm}$ which minimizes the cost $J[u_N(\mathbf{x}^c, \mathbf{x}^s)]$ of the above described optimal (closed-loop) control strategy.

Example 2. Let

$$w = Bu = B(\mathbf{x}^c)u \in W \subset L_2(0, T;\ L_2(\Omega))$$

denote the (transformed) input for the model in (1)–(8), where W stands for an admissible class of inputs (or controls), which includes the possible controllers spatial configurations $\mathbf{x}^c$. According to the N-modal approximation scheme set

$$w_N = B_N u = B_N(\mathbf{x}^c)u \in W_N \subset L_2(0, T; \mathbb{R}^N)$$

with W_N standing for the associated class of admissible truncated (transformed) inputs w_N. For simplicity consider again the approximate model in (11), (12) with the following further simplifications: $a_N(0) = 0$ and $v = 0$. Then the model observation $z_N = z_N(\lambda, \mathbf{x}^s, w) \in L_2(0, T; \mathbb{R}^m)$ is given by

$$z_N(t) = C_N(\mathbf{x}^s)\int_0^t e^{A_N(\lambda)(t-\tau)}w_N(\tau)\,d\tau \tag{13}$$

where $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ is a parameter to be identified in $A_N = A_N(\lambda) = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \in M[\mathbb{R}^N]$, and $\mathbf{x}^s \in \Sigma \subset \Omega^m \subset \mathbb{R}^{nm}$ characterizes the sensors spatial location. Here Σ stands for the admissible location strategies (e.g. Σ may be a finite set properly defined to avoid sensor clustering). Now let $z_S = z_S(\mathbf{x}^s, w) \in L_2(0, T; \mathbb{R}^m)$ be the observed output at $\mathbf{x}^s \in \Sigma$ of a real system driven by $w \in W$, and consider the following criterion

$$J(\lambda) = \|z_S - z_N\|_{L_2(0,T;\mathbb{R}^m)}.$$

The class of models, given by the input–output maps in (13) for every $\lambda \in \mathbb{R}^N$, is said to be identifiable (with respect to the criterion under consideration) if there exists a unique $\lambda^0 \in \mathbb{R}^N$, for each $\mathbf{x}^s \in \Sigma$ and for all $w \in W$, such that the cost $J(\lambda)$ is minimized (e.g. see Kubrusly, 1984 and Crouch, Kubrusly and Pritchard, 1982).

Assume this is the case such that, for each $\mathbf{x}^s \in \Sigma$, the minimum cost $J[\lambda^0(\mathbf{x}^s)]$ does not depend on $w \in W$. An example of an OSL problem (for a fixed number of sensors) is to find $\mathbf{x}^s \in \Sigma$ which minimizes the identification performance $J[\lambda^0(\mathbf{x}^s)]$.

3. *A general review*

The purpose of this section is to present a brief review of recent OSL and/or OCL methods considered in the current literature. The bibliography mentioned here comprises over 50 widely available papers published in the last decade. The several contributions in the field are primary grouped according to the main goal for which OSL and OCL problems are developed (instead of using a chronological order), namely: System Identification, State Estimation and Optimal Control. Since the main goal behind an OSL problem may be any of the above mentioned, the following further abbreviations concerning the methods dealing with OSL will be adopted:

SLI: OSL for System Identification.
SLE: OSL for State Estimation.
SCL: OSL and OCL for Optimal Control.

SLI (Optimal Sensors Location for System Identification). The problem has received some attention in the DPS identification literature over the past decade. Some questions regarding the effects of either number or spatial location of sensors on the implementation of parameters estimation techniques have eventually been considered (e.g. see Seinfeld and Chen, 1971; Polis, Goodson and Wozny, 1973; Kubrusly, 1980; Carotenuto and Raiconi, 1980 and Kitamura and Taniguchi, 1981). Moreover, since identifiability of DPS depends on the number and position of measurement devices, the SLI problem can also be approached according to identifiability requirements (e.g. see Kitamura and Nakagiri, 1977; Nakagiri, 1983; Chavent, 1979a, 1979b, 1983; Courdesses and Amoroux, 1982; Courdesses, Amouroux and Polis, 1981a, 1981b and Polis, 1982). For a previous discussion on this topic, mainly based on observability arguments, see also Goodson and Polis, 1978. However only recently a few papers have appeared specifically on the SLI problem, where optimal location strategies have been proposed.

Le Pourchiet and Le Letty (1976) presented two algorithms, somewhat similar to each other, as an SLI procedure for deterministic DPS. The basic idea was to maximize, at each iteration, the identification error sensitivity (according to preestablished identifiability definitions) with respect to the location of a new sensor. The first algorithm concerns the improvement in the sensitivity criterion by adding a new sensor to the set of all sensors already located in previous iterations; and the second one also takes into account the location of the new sensor at the preceding iteration. Both algorithms stop when the placement of a new sensor adds no substantial improvement as far as the identification error sensitivity is concerned. It is worth emphasizing that in the above described approach it was not assumed an *a priori* fixed number of available sensors.

Sokollik (1976b) considered both the number and location of sensors, as well as the measurement times, for identifying DPS. The distributed model was approximated by a lumped one by using finite-differences. In this way both time and space domains were discretized with constant sampling rates. The optimal space-time net (i.e. the optimization of time and spatial location for the measurements) was given by minimizing the parameter estimate covariance, which was performed by the stochastic approximation schemes analysed in Sokollik (1974, 1976a).

Qureshi, Ng and Goodwin (1980) presented a method for designing optimal experiments for DPS identification with noisy observations. Besides the SLI, it was also considered the determination of boundary perturbations for identifying not necessarily linear systems. The optimization criterion to be maximized was the determinant of the Fisher information matrix associated to the parameters to be identified, which depends on both boundary perturbations and spatial location of the observation points. The design method was developed for hyperbolic and parabolic PDE.

Carotenuto and Raiconi (1981) considered the identification of the parameter appearing in an one-dimensional static (rather than dynamical) diffusion equation. They analysed the effects of an extra measurement point towards a possible improvement in the parameter estimate, and presented a criterion for selecting the location of it.

Rafajłowicz (1981) also presented a method for designing optimal experiments for the DPS identification problem, which comprises sensors location and determination of classes of random inputs. The SLI problem, for second order linear hyperbolic PDE observed through noisy measurements, was replaced by one of seeking an optimal probability measure corresponding to the position of sensors. The approach is similar to that considered in Qureshi, Ng and Goodwin (1980), where the determinant of the information matrix is maximized. However the information matrix was associated to the system eigenvalues rather than to the system parameters. Conditions for optimality of the experiment design, including an upper bound for the number of sensors, were derived (see also Rafajłowicz, 1983).

SLE (Optimal Sensors Location for State Estimation). Unlike the SLI, the SLE literature is plentiful (cf. Tzafestas, 1978) and several methods share some common aspects. For instance, every method discussed here considers white Gaussian observation noise when dealing with the (stochastic) filtering problem. Cannon and Klein (1971) and Caravani, Di Pillo and Grippo (1975) consider a dynamical equation without input disturbance, while the others always assume Gaussian input disturbances. Aidarous, Gevers and Installé (1975, 1978) are the only to consider a discrete-time observation process. The main characteristic of the majority of the SLE methods analysed here is the reduction of an infinite dimensional system to a finite dimensional one, by truncating the (infinite) Fourier expansion of either the state or the estimates in its first N terms (N-modal approximation), according to the increasing order of the partial differential operator eigenvalues. In this way the filtering procedure is applied either in a finite dimensional state space or in an infinite dimensional one, respectively. Concerning the latter case, when the state estimate error covariance appears explicitly in the performance index, such an approximation is applied on the covariance operator rather than on the estimate itself. In the light of the above introductory discussion, the SLE bibliography reviewed here can be gathered in three major groups.

Group I. Yu and Seinfeld, 1973; Caravani, Di Pillo and Grippo, 1975; Omatu, Koide and Soeda, 1978 and Sawaragi, Soeda and Omatu, 1978 treated the SLE problem in a somewhat similar fashion. The idea behind the approach used was to represent the state variable $y(t)$ as an infinite series of eigenfunctions of the partial differential operator modelling the DPS. This yields an equivalent model described by an ODE in the sequence $a(t)$, comprising the coefficients of that expansion. Such an infinite sequence is approximated by an N-dimensional vector $a_N(t)$, obtained by truncating it in its first N terms. This supplies the state N-modal approximation $y_N(t)$ (cf. Section 2). The state estimation problem is then approached by determining the finite dimensional estimate $\hat{a}_N(t)$ for the N-modal approximation estimate $\hat{y}_N(t)$. The SLE $\mathbf{x}^s$ is finally determined through $\hat{y}_N(t)$ by optimizing some appropriate criterion (cf. Fig. 1).

In Caravani, Di Pillo and Grippo (1975) the location of a single sensor for estimating the initial state in the one-dimensional heat equation was investigated. Homogeneous boundary conditions in the state were assumed, such that the DPS was excited only by the unknown initial conditions. The noisy sensor placement was performed by minimizing the maximum mean square error for the initial state.

The SLE was accomplished in Yu and Seinfeld (1973); Omatu, Koide and Soeda (1978) and Sawaragi, Soeda and Omatu (1978), by minimizing the trace of the estimate error covariance matrix at the final time. The effect of measurement location on observability, as an extension of Yu and Seinfeld (1971) to a wide class of linear DPS, was considered in Yu and Seinfeld (1975). A recursive algorithm was also proposed in this last reference, which determines the optimal location of one sensor in terms of the previously located sensors.

In Omatu and Seinfeld (1983) and Sawaragi, Soeda and Omatu (1978) some existence theorems concerning the solution of the SLE problem in infinite dimension were presented. Theorems establishing necessary and sufficient conditions for the SLE, before considering any state space approximation, were also presented.

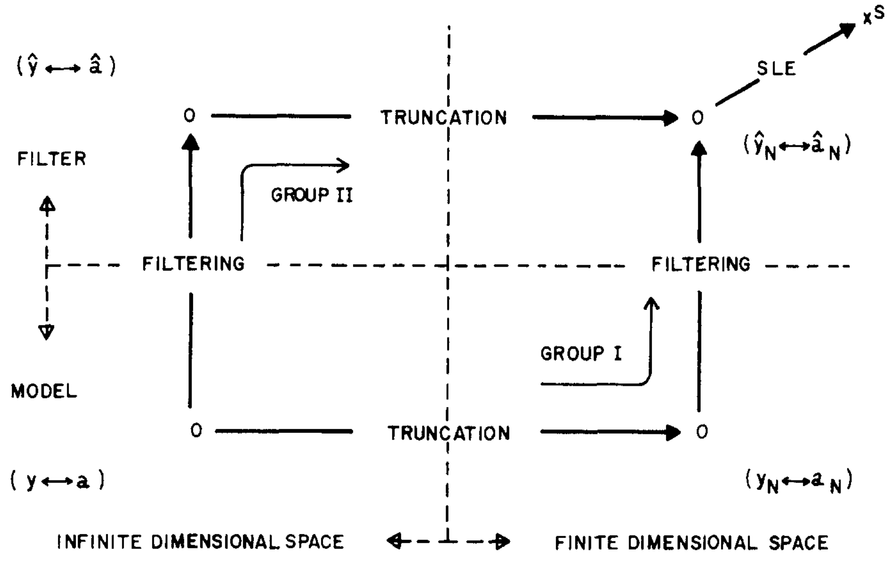

FIG. 1. N-Modal approximations for SLE.

Group II. The methods discussed above (Group I) used (either implicitly or explicitly) an N-modal approximation for the state $y(t)$ and so they applied finite dimensional filtering algorithms to $a_N(t)$. On the other hand, Bensoussan (1972); Aidarous, Gevers and Installé (1975, 1978); Amouroux, Babary and Malandrakis (1978); Kumar and Seinfeld (1978a); Curtain and Ichikawa (1978) and Nakamori *et al.* (1980) used a different approach. The idea behind this was to apply infinite dimensional filtering to the state $y(t)$, and then to represent the state estimate $\hat{y}(t)$ as an infinite series with the coefficients sequence $\hat{a}(t)$; which is truncated in its first N terms yielding the vector $\hat{a}_N(t)$ of the estimate *N-modal approximation* $\hat{y}_N(t)$.

In such methods this approximation procedure was actually applied only on the covariance operator, rather than in the state estimate itself. The great majority of the above mentioned papers faced the SLE problem by minimizing a cost function given in terms of the trace of the N-modal approximation for the estimate error covariance operator; thus supplying the SLE x^s (cf. Fig. 1).

A theoretical treatment for the SLE problems was proposed in Bensoussan (1972) by using functional analysis techniques based on the Lions (1968) approach to control theory for DPS. The existence of solutions for the SLE problem, as well as necessary conditions for optimality, were established. This was achieved by formulating the SLE problem as one of optimal control on the Riccati equation describing the evolution of the estimate error covariance operator.

In Aidarous, Gevers and Installé (1975) the location of a single sensor was initially considered, and the procedure was then extended to cover the case of several sensors. They assumed discrete-time observations. The SLE problem was approached by minimizing the spatial integral of the N-modal approximation for the estimate error covariance. In Aidarous, Gevers and Installé (1978) the existence of solutions for the SLE problem was proved, and also the location algorithm convergence, for the method presented in the earlier reference (1975).

In Amouroux, Babary and Malandrakis (1978) a weighting function for the terms in the trace of the error covariance N-modal approximation was used. This was done in order to increase the accuracy for the first coefficients of the state N-modal approximation.

In Kumar and Seinfeld (1978a) the computational problem concerning the minimization of the integral of the trace of the estimate error covariance matrix was overcome. That matrix was replaced by an upper bound of it, given in terms of the covariance matrix associated to the free system. Also analysed were the effects of the observation noise covariance, and initial and boundary conditions covariances, on the SLE problem.

The filtering problem was approached in Curtain and Ichikawa (1978) by using abstract evolution equations in Hilbert space. As in Bensoussan (1972), the SLE problem was rigorously treated as one of optimal control, where the control variable characterizes the sensors location. Opposite to this approach, a mild evolution operators approach was used for considering existence theorems for SLE, as well as necessary conditions for optimality.

In Nakamori *et al.* (1980), as in Bensoussan (1972) and Curtain and Ichikawa (1978), the SLE problem was approached as one of deterministic optimal control, whose basic cost function was given by the trace of the estimate error covariance operator and by a further term standing for the control cost. Semigroup theory was used as in Curtain and Ichikawa (1978). An existence theorem and sufficient conditions for optimality were established by using a sensitivity criterion given by the trace of the information operator; which can be thought of as an extension of the Fisher information matrix to infinite dimensional spaces. The computational effort in connection with the above criterion was claimed to be smaller compared with that required for the trace of the filter covariance. For implementation an N-modal approximation was suggested for that information operator.

Group III. Cannon and Klein (1970, 1971); Klein (1971); Ewing and Higgins (1971); Chen and Seinfeld (1975); Kumar and Seinfeld (1978b) and Morari and O'Dowd (1980), also investigated the SLE by considering the estimation problem in an infinite dimensional space. However in each one of the above papers a somewhat specific characteristic was presented, which suggests a separate review rather than an inclusion in the previous groups.

The heat equation in one-dimensional spatial domain and without a forcing term was considered in Cannon and Klein (1970, 1971). Although the DPS was supposed to operate in a deterministic environment, uncertainties were allowed in the initial and boundary conditions, as well as in the observation process. The location of a single transducer, which was assumed to average the measurements over a small neighborhood in the spatial domain, was investigated. The theory behind the method applies analytical arguments for establishing an (upper bound) estimate for the state, which was used to supply estimates for the error between the state itself and numerical approximations of it. The SLE was then accomplished by minimizing these error estimates. The same approach was also considered in Klein (1971).

In Chen and Seinfeld (1975) the optimality criterion was given by the space-time integral of the trace of the estimate error covariance. The spatial domain was *a priori* discretized in order to avoid a possible sensors clustering in a small region. The SLE problem was then approached as one of optimal control in which: (1) the state dynamics was given by the matrix PDE describing the estimate error covariance evolution, and (2) the control variables were characterized by a Boolean vector indicating either the presence or absence of sensors over the discete spatial domain. Although it was not considered a finite dimensional approximation for the state space, the algorithm developed for sensor location requires at each iteration the resolution of two matrix PDE (the covariance evolution and its adjoint), whose solution method was not discussed in this reference. This method has been applied in Kumar and Seinfeld (1978b) for state estimation in tubular chemical reactors, where they have used orthogonal collocation techniques for reducing the infinite dimensional system to a finite dimensional one.

The SLE problem was investigated by Morari and O'Dowd (1980) who assumed that the DPS is driven by nonstationary input disturbances. Their approach was based on the Goodson–Klein observability criterion (cf. Goodson and Klein, 1970), whose conditions may not be satisfied for a limited number of sensors. In this way an SLE method was proposed to minimize the information loss associated to the nonobservable subspace. This optimality criterion is given by the spatial integral of the trace of the estimate error covariance operator. Such an error is caused mainly by the lack of observability due to the presence of nonstationary input disturbances. Although the theory was not developed in a finite dimensional space, they used state approximations for example implementation, including experimental results.

OCL (*Optimal Controllers Location*). The OCL problem has been investigated mainly by the French school. Lions (1972); Amouroux (1973); Amouroux and Babary (1973, 1975, 1978, 1979); Aidarous (1975); Aidarous, Gevers and Installé (1976) and Burger (1975, 1976) considered the optimal location of control points (actuators) for DPS. Generally these methods presented several common characteristics, applied to somewhat similar mathematical models. For instance, those which considered feedback control assumed that the observation points (i.e. the sensors location) were *a priori* determined, as opposed to the SCL methods discussed later in this section.

Like the SLE methods in Group I, the idea behind the OCL approach used in Aidarous (1975); Aidarous, Gevers and Installé (1976); Amouroux (1973) and Amouroux and Babary (1973, 1975, 1978, 1979) lies in truncating the coefficients sequence $a(t)$ of the eigenfunction series expansion for the state $y(t)$. After that, the optimal control strategy is determined for the system N-modal approximation. In this way the optimal control problem, for the state N-modal approximation $y_N(t)$, is approached in a finite dimensional state space in terms of the truncated vector $a_N(t)$. The same approximation technique was also used in Burger (1975, 1976), where the OCL problem was faced from a rather different point of view.

The results presented in Amouroux (1973) and Amouroux and Babary (1973) are related to the concepts of controllability and reachability, rather than to the optimal control problem. The main goal was to maximize, over all possible control points, the volume of a hyperellipsoid in the state space comprising the reachable states for bounded pointwise controls. The OCL was formulated according to the necessary and sufficient conditions for reachability of the truncated system.

The optimality criterion in Amouroux and Babary (1978, 1979) was given by the overall control energy and by the state accuracy at the final time. They also reviewed other two OCL procedures besides the state truncation one: an iterative method using gradient-like algorithms, and a parametrization method using N-modal approximation for the distributed control. The pointwise controllers considered in the first reference was extended to the case of zones of action in the second, where both approaches were compared. In (1975) they considered several performance criteria for the optimal pointwise control problem. For instance, the minimization of the truncated optimal control and the final state error norms, were discussed among others.

Differently from Amouroux and Babary, the approach in Aidarous, Gevers and Installé (1976) considered discrete-time observations and stochastic (Gaussian) input disturbances. As far as the optimality criterion is concerned, they minimized a mixed cost function comprising the overall control energy and state accuracy along the whole trajectory. An interesting analysis on the duality between the SLE and OCL problems, according to this paper and an earlier (1975) one, was also presented here.

In Burger (1976) the OCL was discussed from two points of view: zones of action and pointwise controllers. After using a state N-modal approximation, it was then assumed that the system is static (rather than dynamical), thus referring the OCL problem to an ODE model approximated by an algebraic equation. A geometrical approach was considered, by using orthogonal projection arguments, for minimizing the distance between the desired 'state' (for the static system) and the reachability linear subspace.

Theoretical aspects regarding the OCL for pointwise controls was investigated to some extent in chapter 4 of Lions (1972) for deterministic DPS. A rigorous abstract approach for establishing the existence of optimal position for Dirac measures was considered.

SCL (*Optimal Sensors and Controllers Location*). The SCL problem refers to the optimal location of both sensors and controllers, generally for closed-loop optimal control problems in DPS. In case of feedback control, such a combined procedure involving OSL and OCL problems may eventually concern state estimation as well (and hence SLE as a special case of OSL).

Such a problem has been investigated by Amouroux, Di Pillo and Grippo (1976); El Jai (1977); Ichikawa and Ryan (1977, 1979); Courdesses (1978); Malandrakis (1979) and Omatu and Seinfeld (1983). Instead of the author by author review procedure used so far, it seems more appropriate to review the above SCL literature according to the main characteristics used to face the problem. This is motivated by the several common points shared by the subsets of the above mentioned set of papers.

Concerning the environment in which the DPS was supposed to evolve, Ichikawa and Ryan (1977, 1979); Malandrakis (1979) and Omatu and Seinfeld (1983) considered (Gaussian) disturbances corrupting the control action, and all the papers up to Courdesses (1978) assumed observation (Gaussian) noise corrupting the measurements. A completely deterministic formulation was considered in Courdesses (1978). In every of the above mentioned papers continuous time operation was assumed, and El Jai (1977) was the only one to consider open-loop control and a variable number of sensors and controllers. Pointwise controls were assumed in Courdesses (1978) and Malandrakis (1979).

For the SCL methods applied to stochastic DPS in the above references the optimal control strategy was given according to the separation principle, after performing the state estimation. The stochastic regulator problem for evolution equations was considered in Ichikawa and Ryan (1977, 1979) and Omatu and Seinfeld (1983) by using the semigroup approach. In Amouroux, Di Pillo and Grippo (1976) the filtering procedure was applied in a finite dimensional space by considering a state N-modal approximation. On the other hand, Ichikawa and Ryan (1977, 1979) and Malandrakis (1979) applied infinite dimensional filtering and used N-modal approximation for operators associated to the LQG (Linear-Quadratic-Gaussian) optimal control problem (i.e. they used N-modal approximation for the feedback and estimate error covariance operators). In a similar fashion, the deterministic approach considered in Courdesses (1978) involved N-modal approximation in connection to the linear-quadratic (deterministic) optimal control problem. In the open-loop approach presented in El Jai (1977) the pointwise OSL was implemented for estimating the initial state, and the OCL for reaching a desired final state using minimum energy controls.

The optimality criterion for the method presented in Amouroux, Di Pillo and Grippo (1976) was given by the minimization of the state estimate error at the final time and the overall control energy. Several possible criteria and practical considerations for the SCL problem, including the sensors and controllers number optimization, were discussed in El Jai (1977). In the other papers the cost functional to be minimized comprised three terms: final state accuracy, state accuracy along the whole trajectory, and the overall control energy. The existence of an optimal location was established in Ichikawa and Ryan (1977, 1979), where a comparative analysis involving either separate or simultaneous location of sensors and controllers was also presented. A similar approach was presented in Omatu and Seinfeld (1983), where necessary and sufficient conditions for the optimal location were derived.

The OSL problem for deterministic closed-loop control was also considered by Koivo and Kruh (1969). Such an approach, which was one of the first to appear, was quite different from those described above, since the control was supposed to act only on a fixed boundary point. Therefore this characterized an OSL problem for closed-loop control, rather than an SCL problem. The OSL problem towards feedback control was also discussed by Goodson and Klein (1970); Yu and Seinfeld (1971) and Sakawa (1975) as an observability matter. Some theoretical aspects regarding the existence of solutions for a particular OSL problem in feedback control for deterministic DPS were presented by Lions (1972). As in Koivo and Kruh (1969), the OSL problem for closed-loop control of temperature distribution was also considered by Kaizer (1971). Further applications involving the SCL problem for DPS were investigated by Lee, Koppel and Lim (1973) as well.

4. *A classification of methods*

Approximation methods are closely related to DPS analysis. As it has been commented on before (e.g. see Athans, 1970; Kubrusly, 1977; Polis and Goodson, 1976; Robinson, 1971), sooner or later one will be faced with approximation techniques (either for modelling or numerical and physical implementation) when dealing with any problem in DPS. For OCL and OSL problems it can be noticed from the previous section that N-modal approximation (also called truncation of eigenfunctions— or Fourier, or harmonic—expansion, as an approximation scheme resulting from the separation of variables technique) is certainly the most used for sensors and controllers location in DPS.

The purpose of this section is twofold. First of all some relevant characteristics of those methods for OCL and OSL in (dynamical) DPS which use, in one way or another, N-modal approximation schemes are summarized. Such methods are then classified according to the stage of the optimization procedure in which N-modal approximations are used.

Methods characteristics. Table 1 displays some models and methods characteristics for that part of the literature reviewed in the preceding section which uses N-modal approximation. The following notation has been adopted in Table 1, where the first four items concern the external action in the DPS.

- Input. The input (or forcing term) in the dynamic equation can be described either by a stochastic disturbance (w) and/or stochastic control (u), or by a deterministic control (u_d). Null input is denoted by (0).
- Observation noise. The presence or absence of noise corrupting the measurements is denoted by either (v) or (0), respectively.
- Boundary conditions (BC). They can be either homogeneous (H), or inhomogeneous; whose external action in the boundary can be described either by a stochastic (S) or by a deterministic (D) process.
- Initial conditions (IC). Both known and unknown IC are denoted by (D) or (S), whenever they are given by deterministic or stochastic processes, respectively. When an unknown IC is to be estimated, it is denoted by (E). Null IC are represented by (0).
- Number of located points at simulated examples (#). For SCL methods, the first number displayed concerns the OCL problem while the second one concerns the OSL problem.
- Approach (Appr). This points out whether the filtering procedure (when applied) is developed either in a finite ($F_{\mathbb{R}^N}$) or in an infinite dimensional space; the latter case presenting two possibilities, (F_{L_2}) or (F_{l_2}), according to filtering in $L_2(\Omega)$ or in l_2, respectively. The symbol (C) stands for control (or controllability) problems.
- Problem. The optimal location problem under consideration is characterized by the already given notation SLE, OCL, and SCL.

Methods classification. The diagram in Fig. 2 presents a classification of methods in the light of N-modal approximation schemes. The main classifying factor concerns the different stages of the optimization procedure in which such approximations (or truncations) are required. In addition to the notation already posed in this paper, the following has also been adopted in the diagram of Fig. 2.

$L_2 \leftrightarrow l_2$: Standing for the equivalent (infinite dimensional) system representation, either in $L_2(\Omega)$ or l_2, according to the eigenvector series expansion.

$l_2 \rightarrow \mathbb{R}^N$: Standing for the N-modal approximation; that is, the truncation of eigenvector series expansion in its first N terms.

Numbers between brackets concern the references mentioned in Table 1, and they point out the path which classifies the underlying method as follows:

Path Π_1: SCL methods using infinite dimensional filtering.
Path Π_2: SLE methods using infinite dimensional filtering.
Path Π_3: SCL methods using finite dimensional filtering.
Path Π_4: SLE methods using finite dimensional filtering.

TABLE 1. SUMMARY OF MODEL AND METHOD CHARACTERISTICS

Reference	Input	Obs. noise	BC	IC	#	Appr.	Problem
[1] Yu and Seinfeld (1973)	w	v	H	D	2	$F_{\mathbb{R}^N}$	
[2] Caravani, Di Pillo and Grippo (1975)	0	v	H	E	1	$F_{\mathbb{R}^N}$	SLE
[3] Omatu, Koide and Soeda (1978)	w	v	H	S	2	$F_{\mathbb{R}^N}$	(group I)
[4] Sawaragi, Soeda and Omatu (1978)	w	v	H	S	2	$F_{\mathbb{R}^N}$	
[5] Bensoussan (1972)	w	v	D,H	S	0	F_{L_2}	
[6] Aidarous, Gevers and Installé (1975)	w	v	H	S	2	F_{L_2}	
[7] Amouroux, Babary and Malandrakis (1978)	w	v	H	S	2	F_{l_2}	SLE
[8] Curtain and Ichikawa (1978)	w	v	S,H	D	2	F_{L_2}	(group II)
[9] Kumar and Seinfeld (1978)	w	v	H	S	2	F_{L_2}	
[10] Nakamori, Miyamoto, Ikeda and Sawaragi (1980)	w	v	D,H	0	1	F_{L_2}	
[11] Amouroux and Babary (1973)	u_d	0	H	D	1	C	
[12] Amouroux and Babary (1975)	u_d	0	H	D	1	C	
[13] Aidaraous, Gevers and Installé (1976)	u + w	v	H	S	1	C	OCL
[14] Amouroux and Babary (1978)	u_d	0	H	D	1	C	
[15] Amouroux and Babary (1979)	u_d	0	H	D	1	C	
[16] Amouroux, Di Pillo and Grippo (1976)	u	v	H	D	1/4	$F_{\mathbb{R}^N}/C$	
[17] El Jai (1977)	u	v	H	E	3/4	$F_{\mathbb{R}^N}/C$	
[18] Courdesses (1978)	u_d	0	H	D	3/1	C	SCL
[19] Ichikawa and Ryan (1979)	u + w	v	H	D	1/1	F_{L_2}/C	
[20] Malandrakis (1979)	u + w	v	D	S	2/1	F_{L_2}/C	
[21] Omatu and Seinfeld (1983)	u + w	v	S,H	S	2/2	F_{L_2}/C	

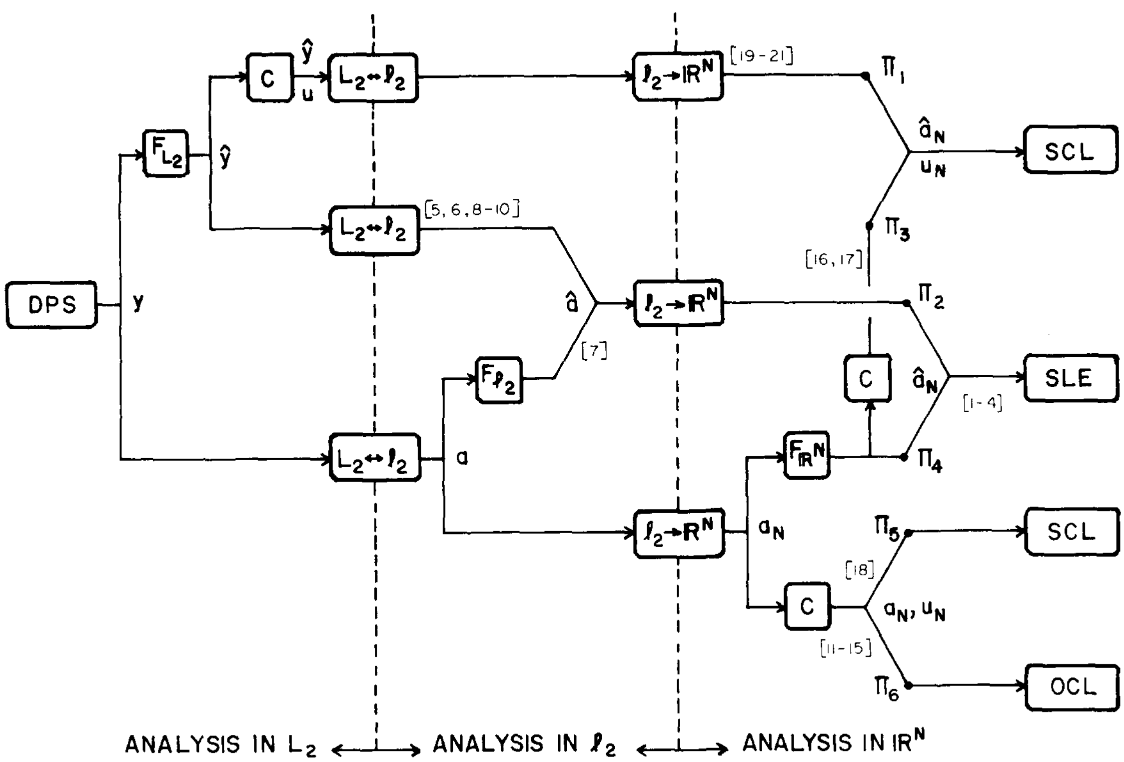

FIG. 2. A classification of OCL and OSL methods.

Path Π_5: SCL methods approached in a deterministic environment.

Path Π_6: OCL methods approached in a deterministic environment.

5. *Comments and concluding remarks*

Several remarks and some conclusions can be drawn from what has been discussed in the preceding sections. A brief selection of basic topics which deserve to be emphasized will be presented in this final section.

(1) Although this seems to be the first attempt to survey the several OCL and OSL methods for DPS, practical motivations for considering the problem were not addressed here. However such motivations can be found in the surveys by Kubrusly (1977); Polis and Goodson (1976); Ray (1975) and Robinson (1971), and books by Butkovskiy (1969); Ray and Lainiotis (1978); Ruberti (1978) and Wang (1964) mentioned in Section 1, in connection with identification, filtering, and control problems in DPS. See also the recent survey by Johnson (1983).

(2) Little literature has been written about OCL and OSL, compared with what has been published in either identification, filtering or control of DPS.

(3) In particular, more research is needed regarding the OSL problem for system identification (i.e. the SLI problem).

(4) Among the literature discussed here, Gaussian distribution has always been assumed, for both input disturbance and/or observation noise, when the DPS is supposed to operate in a stochastic environment. It would be interesting to have OCL and OSL strategies for arbitrary (and eventually unknown) probability distributions.

(5) Experimental results show that the location of sensors and/or controllers may be sensitive to the stochastic environment (e.g. to the statistics of the input disturbance, observation noise, and random initial and boundary conditions). For instance, experiments presented in Kumar and Seinfeld (1978a) have shown that the SLE depends on the covariance of the observation noise, as well as on the covariances of initial and boundary conditions. However a theoretical sensitivity analysis on the effects of the stochastic environment upon the OCL and OSL is still lacking.

(6) The great majority of the methods reviewed here apply to linear models. More effort towards OCL and OSL methods for nonlinear DPS should be attempted (for instance, by using the N-modal approximation theory considered by Banks and Kunish, 1982).

(7) As already remarked here, N-modal is the most used approximation technique in OCL and OSL. Unlike other areas in the DPS field (e.g. in DPS identification) finite-differences is not a very popular scheme, even among the methods which approximate the PDE to an ODE (or difference equation) thus reducing the DPS (modelled in an infinite dimensional state space) to an LPS (modelled in a finite dimensional state space).

(8) On the other hand, as in the whole DPS field, the question of when to use approximation techniques does not seem to have a final answer yet. According to Section 4, approximations have been applied either before or after optimization schemes. When the filtering problem was involved, it has been performed either in $L_2(\Omega)$, l_2 or $\mathbb{R}^N$; but the control problem was generally developed in $\mathbb{R}^N$. In any case the OCL and OSL strategies were always developed after applying approximation techniques.

(9) Only in a few papers (e.g. El Jai, 1977; Le Pourhiet and Le Letty, 1976), the optimal placement of a variable number of sensors and/or controllers has been considered. The problem of optimizing (i.e. minimizing) the number of sensors and/or controllers should receive more attention.

(10) More research is also needed towards OCL for boundary controls, and OSL for boundary measurements.

(11) The SCL problem, in connection with the design of finite dimensional compensators for DPS (e.g. Curtain, 1983a, b, 1984), also deserves more investigation.

(12) The problem of determining the best kind of sensors and/or controllers should also receive some attention. Fundamental questions in this area are: point or distributed devices? In the latter case, if a finite number of sensors and/or controllers act over neighborhoods on the spatial domain, which would be the best 'size' of these neighborhoods, and which would be the best type of distribution (e.g. uniform) for either the measurements collected by the sensors or the action supplied by the controllers?

(13) Simulation is certainly an important point towards DPS analysis (e.g. see Tzafestas, 1980). The simulated results presented in the OCL and OSL literature have generally been developed for DPS with one-dimensional spatial domain. Illustrative examples and experimental results considering two- or three-dimensional spatial domains would be welcome.

(14) Perhaps it is already time to have a comparison of effectiveness of some different OCL and OSL methods. The classification introduced in Section 4 can be viewed as a first step for a qualitative comparison. It can also be used as a starting point for further works towards a quantitative comparison, since some different approaches for solving the OCL and OSL problem have been grouped according to their main structural characteristic.

(15) Finally it is worth remarking on the N-modal approximation scheme again. Some models can survive truncation (i.e. the transformation $l_2 \rightarrow \mathbb{R}^N$ in Fig. 2). For instance, equivalent models for parabolic or diffusion equations which have spatially elliptical operators. However other equivalent models in l_2 for DPS cannot survive truncation. For instance, the finite propagation of wave travel is essentially destroyed in an N-modal approximation for a hyperbolic or wave equation. Therefore, before proceeding to approximate models, it is advisable to verify (i) whether any approximation is actually needed (one may decide to place sensors and controllers somewhere in the spatial domain, not necessarily in an optimal fashion, without approximating the DPS), and (ii) whether the truncation (or discretization procedure) does not destroy essential properties of the DPS, which may be inherent to infinite dimensional models.

Acknowledgements—The authors thank Prof. Ruth Curtain and one of the anonymous referees for bringing their attention to remarks (5) and (15), respectively.

References

Ahmed, N. U. and K. L. Teo (1981). *Optimal Control of Distributed Parameter Systems.* North-Holland, Amsterdam.

Aidarous, S. E. (1975). Optimal allocation of pointwise controllers in stochastic distributed systems. *Proc.* 1975 *IEEE Control and Decision Conf.*, Houston, Texas.

Aidarous, S. E., M. R. Gevers and M. J. Installé (1975). Optimal sensors allocation strategies for a class of stochastic distributed systems. *Int. J. Control,* **22**, 197.

Aidarous, S. E., M. R. Gevers and M. J. Installé (1976). Optimal pointwise discrete control and controllers location strategies for stochastic distributed systems. *Int. J. Control,* **24**, 493.

Aidarous, S. E., M. R. Gevers and M. J. Installé (1978). On the asymptotic behaviour of sensors' allocation algorithm in stochastic distributed systems. In A. Ruberti (Ed.), *Distributed Parameter Systems: Modelling and Identification*, Lecture Notes in Control & Inf. Sci., vol. 1. Springer-Verlag, Berlin.

Akhiezer, N. I. and I. M. Glazman (1981). *Theory of Linear Operators in Hilbert Space*, vols. I and II. Pitman, Boston.

Ames, W. F. (1972). *Nonlinear Partial Differential Equations in Engineering*, vols. I and II. Academic Press, New York.

Amouroux, M. (1973). Sur une méthode de détermination de l'emplacement optimal de points d'action pour une classe de systèmes linéaire à paramètres répartis. *C.R. Acad. Sci. Paris*, **277**, 193.

Amouroux, M. et J. P. Babary (1973). Sur la commandabilité de système linéaire à paramètres répartis et sons utilization pour la determination de points d'action optimaux. *R.A.I. R.O.*, **J-3**, 120.

Amouroux, M. and J. P. Babary (1975). Optimal pointwise control for a class of distributed parameter systems. *Proc. IFAC 6th World Congress*, Boston/Cambridge, Mass. p. 1-2.

Amouroux, M. and J. P. Babary (1978). On the optimal pointwise control and parametric optimization of distributed parameter systems. *Int. J. Control,* **28**, 789.

Amouroux, M. and J. P. Babary (1979). On optimization of zones of action for an optimal control problem for distributed parameter systems. *Int. J. Control,* **29**, 861.

Amouroux, M., J. P. Babary and C. Malandrakis (1978). Optimal location for linear stochastic distributed parameter systems. In A. Ruberti (Ed.), *Distributed Parameter Systems: Modelling and Identification*, Lecture Notes in Control & Inf. Sci., vol. 1. Springer-Verlag, Berlin.

Amouroux, M., G. Di Pillo and L. Grippo (1976). Optimal selection of sensors and control location for a class of distributed parameter systems. *Ric. Automatica*, **7**, 92.

Athans, M. (1970). Toward a practical theory of distributed parameter systems. *IEEE Trans Aut. Control*, **AC-15**, 245.

Aziz, A. K., J. W. Wingate and M. J. Balas, Eds. (1977). *Control Theory of Systems Governed by Partial Differential Equations.* Academic Press, New York.

Balakrishnan, A. V. (1980). *Applied Functional Analysis*, 2nd edn. Springer-Verlag, Berlin.

Banks, H. T. and K. Kunisch (1982). An approximation theory for nonlinear partial differential equations with applications to identification and control. *SIAM J. Control & Optimiz.*, **20**, 815.

Bencala, K. E. and J. H. Seinfeld (1979). Distributed parameter filtering: boundary noise and discrete observations. *Int. J. Syst. Sci.*, **10**, 493.

Bensoussan, A. (1971). *Filtrage Optimal des Systèmes Linéaires.* Dunod, Paris.

Bensoussan, A. (1972). Optimization of sensors location in a distributed parameter system. In R. F. Curtain (Ed.), *Stability of Stochastic Dynamical Systems*, Lecture Notes in Mathematics, vol. 294. Springer-Verlag, Berlin.

Bensoussan, A. (1978). Control of stochastic partial differential equations. In W. H. Ray and D. G. Lainiotis (Eds.), *Distributed Parameter Systems.* Marcel Dekker, New York.

Burger, J. (1975). Determination de l'emplacement optimal des actioneurs ponctuels dans un système regi par une équation aux derivée partielles linéaire. *C.R. Acad. Sci. Paris*, **280**, 161.

Burger, J. (1976). Emplacement optimal des actionneurs ponctuels ou par zones dans un système gouverné par une équation linéaire d'evolution du premier ordre. *R.A.I.R.O.*, **J-5**, 87.

Burger, J. et G. Chavent (1979). Identification de paramètre repartis dans les équations aux dérivés partielles. *R.A.I.R.O. Syst. Anal. & Control*, **13**, 115.

Butkovskiy, A. G. (1969). *Distributed Control System.* American Elsevier, New York.

Cannon, J. R. and R. E. Klein (1970). Optimal selection of measurement location in a conductor for approximate determination of temperature distribution. *Proc. Joint Aut. Control Conf.*, Atlanta, pp. 750–756.

Cannon, J. R. and R. E. Klein (1971). Optimal selection of measurement locations in a conductor for approximate determination of temperature distributions. *Trans. ASME, J. Dyn. Syst. Meas. & Control*, **93-G**, 193.

Caravani, P., G. Di Pillo and L. Grippo (1975). Optimal location of a measurement point in a diffusion process. *Proc. IFAC 6th World Congress*, Boston/Cambridge, Mass., p. 8-3.

Carotenuto, L. and G. Raiconi (1980). Identifiability and identification of a Galerkin approximation for a class of distributed parameter systems. *Int. J. Syst. Sci.*, **11**, 1035.

Carotenuto, L. and G. Raiconi (1981). On the identification of a random parameter function in the one-dimensional diffusion equation and related location. *Proc. 8th IFAC Triennial World Congress*, Kyoto, vol. II, pp. 76–81.

Chavent, G. (1979a). About the stability of the optimal control solution of inverse problems. In G. Anger (Ed.), *Inverse and Improperly Posed Problems in Differential Equations.* Academie-Verlag, Berlin.

Chavent, G. (1979b). Identification of distributed parameter systems: about the output least squares methods, its implementation, and identifiability. *Proc. 5th IFAC Symp. on Identification & Syst. Parameter Estim.*, Darmstadt, pp. 85–97.

Chavent, G. (1983). Local stability of the output least square parameter estimation technique. *Mat. Aplic. Comp.*, **2**, 3.

Chen, W. H. and J. H. Seinfeld (1975). Optimal location of process measurements. *Int. J. Control*, **21**, 1003.

Courant, R. and K. O. Friedrichs (1948). *Supersonic Flow and Shock Waves.* Interscience, New York.

Courant, R. and D. Hilbert (1962). *Methods of Mathematical Physics*, vol. II. Interscience, New York.

Courdesses, M. (1978). Optimal sensors and controllers allocation in output feedback control for a class of distributed parameter systems. *Proc. Symp. on Advances in Measurement & Control*, Athens, pp. 735–740.

Courdesses, M. and M. Amouroux (1982). Identifiabilité et identification des systèmes à paramètres repartis. *Proc. 3rd IFAC Symp. on Control of D.P.S.*, Toulouse, pp. XXI.1–XXI.6.

Courdesses, M., M. Amouroux and M. P. Polis (1981a). Sur des conditions d'identifiabilité des paramètres d'un système distribué de type parabolique. *R.A.I.R.O., Syst., Anal. & Control*, **15**, 347.

Courdesses, M., M. P. Polis and M. Amouroux (1981b). On identifiability of parameters in a class of parabolic distributed systems. *IEEE Trans Aut. Control*, **AC-26**, 474.

Crouch, P. E., C. S. Kubrusly and A. J. Pritchard (1982). Systems identification: some fundamental aspects. Control Theory Centre Report no. 104, Univ. of Warwick.

Curtain, R. F. (1975). A survey of infinite dimensional filtering. *SIAM Rev.*, **17**, 395.

Curtain, R. F. (1978). A semigroup approach to the L.Q.G. problem for infinite dimensional systems. *IEEE Trans Circ. & Syst.*, **CAS-25**, 713.

Curtain, R. F. (1983a). Compensators for infinite dimensional linear systems. *J. Franklin Inst.*, **315**, 331.

Curtain, R. F. (1983b). Tracking and regulation for distributed parameter systems. *Mat. Aplic. Comp.*, **2**, 199.

Curtain, R. F. (1984). Finite dimensional compensators for parabolic distributed systems with unbounded control and observation. *SIAM J. Control & Optimiz.*, **22**, 255.

Curtain, R. F. and A. Ichikawa (1978). Optimal location of sensors for filtering for distributed systems. In A. Ruberti (Ed.), *Distributed Parameter Systems: Modelling and Identification*, Lecture Notes in Control & Inf. Sci., vol. 1. Springer-Verlag, Berlin.

Curtain, R. F. and A. J. Pritchard (1977). *Functional Analysis in Modern Applied Mathematics.* Academic Press, New York.

Curtain, R. F. and A. J. Pritchard (1978). *Infinite Dimensional Linear Systems Theory*, Lecture Notes in Control & Inf. Sci., vol. 8. Springer-Verlag, Berlin.

Davis, M. H. A. (1977). *Linear Estimation and Stochastic Control.* Chapman and Hall, London.

Dunford, N. and J. T. Schwartz (1958). *Linear Operators—Part I: General Theory.* Interscience, New York.

El Jai, A. (1977). Sur la commande avec estimation de l'etat initial d'une classe de système a paramètres répartis, considérations pratiques. *R.A.I.R.O., Syst., Anal. & Control*, **11**, 313.

Ewing, D. J. and T. J. Higgins (1971). A sensor location algorithm for distributed parameter systems. *Proc. 9th Annual Allerton Conf. on Circ. and Syst. Theory*, Allerton.

Friedman, A. (1969). *Partial Differential Equations.* Holt, Rinehart and Winston, New York.

Fuhrmann, P. A. (1981). *Linear Systems and Operators in Hilbert Space.* McGraw-Hill, New York.

Garabedian, P. R. (1964). *Partial Differential Equations.* John Wiley, New York.

Goodson, R. E. and R. E. Klein (1970). A definition and some results of distributed systems observability. *IEEE Trans Aut. Control*, **AC-15**, 165.

Goodson, R. E. and M. P. Polis (1978). Identification of parameters in distributed systems. In W. H. Ray and D. G. Lainiotis (Eds.), *Distributed Parameter Systems.* Marcel Dekker, New York.

Gustafson, K. E. (1980). *Introduction to Partial Differential Equations and Hilbert Space Methods.* John Wiley, New York.

Helton, J. W. (1976). Systems with infinite dimensional state space: the Hilbert space approach. *Proc. IEEE*, **64**, 145.

Hille, E. and R. S. Phillips (1957). *Functional Analysis and Semigroups.* AMS Colloq. Publ., vol. XXXI, Providence.

Ichikawa, A. and E. P. Ryan (1977). Filtering and control of distributed parameter systems with point observations and inputs. *Proc. 2nd IFAC Symp. on Control of D.P.S.*, Coventry, pp. 347–357.

Ichikawa, A. and E. P. Ryan (1979). Sensor and controller location problems for distributed parameter systems. *Automatica*, **15**, 347.

John, F. (1981). *Partial Differential Equations*, 4th edn. Springer-Verlag, Berlin.

Johnson, T. L. (1983). Progress in modelling and control of flexible spacecraft. *J. Franklin Inst.*, **315**, 495.

Kaiser, K. W. (1971). A method of determining the heat-sensor configuration in temperature distribution controllers. *Proc. 1st IFAC Symp. on Control of D.P.S.*, Banff, pp. 1–3.

Kato, T. (1980). *Perturbation Theory for Linear Operators*, 2nd edn. Springer-Verlag, Berlin.

Kitamura, S. and S. Nakagiri (1977). Identifiability of spatially-varying and constant parameters in distributed systems of parabolic type. *SIAM J. Control & Optimiz.*, **15**, 785.

Kitamura, S. and Y. Taniguchi (1981). Parameter estimation of one- and two-dimensional heat conduction processes using a minicomputer. *Int. J. Control*, **34**, 955.

Klein, R. E. (1971). Non well-posed problems in the sense of Hadamard and their relationship to distributed parameter state reconstructor problems. *Proc. 1st IFAC Symp. on Control of D.P.S.*, Banff.

Koivo, A. J. and P. Kruh (1969). On the design of approximately optimal feedback controllers for a distributed parameter system. *Int. J. Control*, **10**, 53.

Kubrusly, C. S. (1977). Distributed parameter system identification: a survey. *Int. J. Control*, **26**, 509.

Kubrusly, C. S. (1980). Linear DPS identification in a stochastic environment. *R.A.I.R.O. System Anal. & Control*, **14**, 343.

Kubrusly, C. S. (1984). On identifiability of linear dynamical systems. *Mat. Aplic. Comp.*, **3**, 219.

Kumar, S. and J. H. Seinfeld (1978a). Optimal location of measurements for distributed parameter estimation. *IEEE Trans Aut. Control*, **AC-23**, 690.

Kumar, S. and J. H. Seinfeld (1978b). Optimal location of measurements in tubular reactors. *Chem. Engng. Sci.*, **33**, 1507.

Le Pourhiet, A. and L. Le Letty (1976). Optimization of sensors location in distributed parameter system identification. *Proc. 4th IFAC Symp. on Identification & System Parameter Estimation*, Tibilisi, pp. 1581–1591.

Lee, M. H., L. B. Koppel and H. C. Lim (1973). Optimal sensor locations and controller settings for class of counter current processes. *Ind. Eng. Chem. Proc. Des. Devl.*, **12**, 36.

Leigh, J. R. (1980). *Functional Analysis and Linear Control Theory.* Academic Press, New York.

Lions, J. L. (1968). *Controle Optimal des Systèmes Gouvernés par des Équations aux Dérivées Partielles.* Dunod, Paris.

Lions, J. L. (1972). *Some Aspects of the Optimal Control of Distributed Parameter Systems.* CBMS Regional Conf. Series in Appl. Math., vol. 6. SIAM, Philadelphia.

Lions, J. L. (1978). Optimal control of deterministic distributed parameter system. In W. H. Ray and D. G. Lainiotis (Eds.), *Distributed Parameter Systems.* Marcel Dekker, New York.

Lions, J. L. (1980). On the foundations of the optimal control of distributed systems. In K. Irack, K. Malanowski and S. Walukiewicz (Eds.), *Optimization Techniques—Part I*, Lecture Notes in Control & Inf. Sci., vol. 22. Springer-Verlag, Berlin.

Malandrakis, C. G. (1979). Optimal sensor and controller allocation for a class of distributed parameter systems. *Int. J. Syst. Sci.*, **10**, 1283.

Mikhlin, S. G. (1970). *Mathematical Physics: An Advanced Course.* North-Holland, Amsterdam.

Morari, M. and J. O'Dowd (1980). Optimal sensors location in the presence of nonstationary noise. *Automatica*, **16**, 463.

Nakagiri, S-I. (1983). Identifiability of linear systems in Hilbert spaces. *SIAM J. Control & Optimiz.*, **21**, 501.

Nakamori, Y., S. Miyamoto, S. Ikeda and Y. Sawaragi (1980). Measurement optimization with sensitivity criteria for distributed parameter systems. *IEEE Trans Aut. Control*, **AC-25**, 889.

Naylor, A. W. and G. R. Sell (1982). *Linear Operator Theory in Engineering and Science*, 2nd edn. Springer-Verlag, Berlin.

Omatu, S., S. Koide and T. Soeda (1978). Optimal sensor location problem for linear distributed parameter systems. *IEEE Trans Aut. Control*, **AC-23**, 665.

Omatu, S. and J. H. Seinfeld (1983). Optimization of sensor and actuator location in a distributed parameter system. *J. Franklin Inst.*, **315**, 407.

Phillipson, G. A. (1971). *Identification of Distributed Systems.* American Elsevier, New York.

Polis, M. P. (1982). The distributed system parameter identification problem: a survey of recent results. *Proc. 3rd IFAC Symp. on Control of D.P.S.*, Toulouse, pp. SP.45–SP.58.

Polis, M. P. and R. E. Goodson (1976). Parameter identification in distributed systems: a synthesizing overview. *Proc. IEEE*, **64**, 45.

Polis, M. P., R. E. Goodson and M. J. Wozny (1973). On parameter identification of distributed systems using Galerkin's criterion. *Automatica*, **9**, 56.

Pritchard, A. J. (1979). Infinite-dimensional linear system theory. *Proc. IEE*, **126**, 641.

Qureshi, Z. H., T. S. Ng and G. C. Goodwin (1980). Optimum experimental design for identification of distributed parameter

systems. *Int. J. Control*, **31**, 21.
Rafajłowicz, E. (1981). Design of experiments for eignvalue identification in distributed-parameter systems. *Int. J. Control*, **34**, 1079.
Rafajłowicz, E. (1983). Optimal experimental design for identification of linear distributed-parameter systems: frequency domain approach. *IEEE Trans Aut. Control*, **AC-28**, 806.
Ray, W. H. (1975). Distributed parameter state estimation algorithms and applications—a survey. *Proc. IFAC 6th World Congress*, Boston/Cambridge, Mass., p. 8-1.
Ray, W. H. (1978). Some recent applications of distributed parameter systems theory—a survey. *Automatica*, **14**, 281.
Ray, W. H. and D. G. Lainiotis, Eds. (1978). *Distributed Parameter Systems—Identification, Estimation and Control.* Marcel Dekker, New York.
Robinson, A. C. (1971). A survey of optimal control of distributed parameter systems. *Automatica*, **7**, 371.
Ruberti, A., Ed. (1978). *Distributed Parameter Systems: Modelling and Identification*, Lecture Notes in Control & Inf. Sci., vol. 1. Springer-Verlag, Berlin.
Sakawa, Y. (1975). Observability and related problems of partial differential equations of parabolic type. *SIAM J. Control*, **13**, 14.
Sawaragi, Y., T. Soeda and S. Omatu (1978). *Modelling, Estimation and their Applications for Distributed Parameter Systems*, Lecture Notes in Control & Inf. Sci., vol. 11. Springer-Verlag, Berlin.
Schechter, M. (1977). *Modern Methods in Partial Differential Equations.* McGraw-Hill, New York.
Seinfeld, J. H. and W. H. Chen (1971). Estimation of parameters in partial differential equations from noisy experimental data. *Chem. Engng. Sci.*, **26**, 753.
Showalter, R. E. (1977). *Hilbert Space Methods for Partial Differential Equations.* Pitman, New York.
Sokollik, F. (1974). Identifikation von Systemen mit verteilten Parametern. *Messen. Steuern. Regeln.*, **17**, 271.
Sokollik, F. (1976a). Zur Parameterschätzung von Systemen mit Verteilten Parametern mittels on-Line-Verfahren. *Messen Steuern. Regeln.*, **19**, 84.
Sokollik, F. (1976b). Zur Optimalen Wahl der Meßorte bei der Parameterschätzung von Systemen mit verteilten Parametern. *Messen. Steuern. Regeln.*, **19**, 194.
Stavroulakis, P., Ed. (1983). *Distributed Parameter Systems Theory: Control and Estimation*, Parts I/II, Benchmark Papers in Elec. Eng. & Comp. Sci., vol. 26/27. Hutchinson Ross, Stroudsburg.
Treves, F. (1975). *Basic Linear Partial Differential Equations.* Academic Press, New York.
Tzafestas, S. G. (1978). Distributed parameter state estimation. In W. H. Ray and D. G. Lainiotis (Eds.), *Distributed Parameter Systems.* Marcel Dekker, New York.
Tzafestas, S. G., Ed. (1980). *Simulation of Distributed Parameter and Large-Scale Systems.* North-Holland, Amsterdam.
Tzafestas, S. G., Ed. (1982). *Distributed Parameter Control Systems.* Pergamon Press, Oxford.
Tzafestas, S. G. and P. Stavroulakis (1983). Recent advances in the study of distributed parameter systems. *J. Franklin Inst.*, **315**, 285.
Wang, P. K. C. (1964). Control of distributed parameter systems. In C. T. Leondes (Ed.), *Advances in Control Systems*, vol. 1. Academic Press, New York.
Weidmann, J. (1980). *Linear Operators in Hilbert Spaces.* Springer-Verlag, Berlin.
Weinberger, H. F. (1965). *A First Course in Partial Differential Equations.* Ginn-Xerox, Lexington.
Yosida, K. (1980). *Functional Analysis*, 6th edn. Springer-Verlag, Berlin.
Yu, T. K. and J. H. Seinfeld (1971). Observability of a class of hyperbolic distributed parameter systems. *IEEE Trans Aut. Control*, **AC-16**, 495.
Yu, T. K. and J. H. Seinfeld (1973). Observability and optimal measurement location in linear distributed parameter systems. *Int. J. Control*, **18**, 785.