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Mean Square Stability Conditions for Discrete Stochastic Bilinear Systems

C. S. KUBRUSLY AND O. L. V. COSTA

Abstract—**Necessary and sufficient conditions for mean square stability are proved for the following class of nonlinear dynamical systems: finite-dimensional bilinear models, evolving in discrete-time, and driven by random sequences. The stochastic environment under consideration is characterized only by independence, wide sense stationarity, and second-order properties. Thus, we do not assume random sequences to be Gaussian, zero-mean, or ergodic. The probability distributions involved are allowed to be arbitrary and unknown. Limiting state moments are given in terms of the model parameters and disturbances moments.**

I. Introduction

SEVERAL aspects regarding structural properties of bilinear systems have been investigated in the current literature during the past decade. Fundamental questions on such a class of nonlinear dynamical systems, as well as practical and theoretical motivations for considering them, have been properly addressed in the surveys [1]–[3] concerning the continuous-time case. On the other hand, many real systems are naturally described by discrete-time bilinear models (e.g., see [4] and the references therein).

The stability problem for continuous-time bilinear systems operating in a stochastic environment has been considered by many authors and reviewed in [3]. However, the same problem for discrete-time systems has not received so much attention. Stability conditions for discrete-time stochastic nonlinear systems, including some particular cases of bilinear models, were presented in [5]. A brief account on the few papers dealing with the stability problem for discrete-time bilinear systems operating in a stochastic environment was given in [6], where sufficient conditions for mean square stability were established.

In this paper we obtain necessary and sufficient conditions for mean square stability of finite-dimensional discrete bilinear systems driven by random sequences. The paper is organized as follows. In Section II we pose the notation and basic results that will be used throughout the text. The model under consideration is described in Section III, where the stability problem is formulated. In Section IV we consider some auxiliary propositions for supporting the proofs of the main results, which will appear in Section V. There we present five stability lemmas that combined will supply necessary and sufficient conditions for mean square stability, as stated in Theorem 1. Our approach, which applies to a general class of discrete-time bilinear systems, was motivated by the earlier works for particular classes of continuous-time systems considered in [7] and [8].

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C. S. Kubrusly is with the Department of Research and Development, National Laboratory for Scientific Computation, LNCC/CNPq, Rio de Janeiro, Brazil, and the Department of Electrical Engineering, Catholic University—PUC/RJ, Rio de Janeiro, Brazil.

O. L. V. Costa is with the Department of Electrical Engineering, Imperial College of Science and Technology, University of London, London, England.

II. Notation and Conceptual Preliminaries

Let $\mathfrak{R}$ and $\mathcal{C}$ denote the real and complex fields, respectively, and $\mathcal{C}^n$ the n-dimensional complex Euclidean space. Let $\mathfrak{M}(\mathcal{C}^n, \mathcal{C}^m)$ denote the normed linear space of all m by n complex matrices. For simplicity we set $\mathfrak{M}(\mathcal{C}^n) = \mathfrak{M}(\mathcal{C}^n, \mathcal{C}^n)$. $\langle ; \rangle$ will stand for the usual inner product in $\mathcal{C}^n$, and $\| \;\|$ will denote either the standard (Euclidean) norm in $\mathcal{C}^n$ or the uniform induced norm in $\mathfrak{M}(\mathcal{C}^n)$. We shall use the superscripts $^-$, $'$, and $*$ for complex conjugate, transpose, and conjugate transpose (i.e., adjoint), respectively. Throughout this paper $f: \mathfrak{M}(\mathcal{C}^n) \to \mathcal{C}^{n^2}$ will denote a "stacking operator," which is defined as follows: for a given $H = [h_1 \cdots h_n] \in \mathfrak{M}(\mathcal{C}^n)$, with $h_k \in \mathcal{C}^n$ for each $k = 1, \cdots, n$,

$$f(H) = (h_1, \cdots, h_n).$$

Obviously, f is a topological isomorphism. With the Kronecker product $L \otimes K \in \mathfrak{M}(\mathcal{C}^{n^2})$ defined as usual for any $L, K \in \mathfrak{M}(\mathcal{C}^n)$, the following can be shown [9].

Remark 1: For any $L, K, H \in \mathfrak{M}(\mathcal{C}^n)$,

$$(L \otimes K)^* = L^* \otimes K^*,$$

$$f(LKH) = (H' \otimes L) f(K).$$

$L \geqq 0$ and $L > 0$ will be used if a self-adjoint [i.e., $L = L^* \in \mathfrak{M}(\mathcal{C}^n)$] matrix is nonnegative (i.e., $\langle Ly; y \rangle \geqq 0$, $\forall\, y \in \mathcal{C}^n$) or positive (i.e., $\langle Ly; y \rangle > 0$, $\forall\, y \neq 0 \in \mathcal{C}^n$), respectively. We set $\mathfrak{M}(\mathcal{C}^n)^+ = \{L \in \mathfrak{M}(\mathcal{C}^n): L = L^* \geqq 0\}$ the convex closed cone

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of all nonnegative matrices in $\mathcal{M}(\mathbb{C}^n)$. If $L \in \mathcal{M}(\mathbb{C}^n)^-$, then there exists a unique $L^{1/2} \in \mathcal{M}(\mathbb{C}^n)^+$ such that $(L^{1/2})^2 = L$. Notice that $\|L^{1/2}\| = \|L\|^{1/2}$ for every $L \in \mathcal{M}(\mathbb{C}^n)^+$. Since any element in $\mathcal{M}(\mathbb{C}^n)$ has a Cartesian decomposition (cf. [10, p. 376]), and every self-adjoint element in $\mathcal{M}(\mathbb{C}^n)$ can be decomposed in positive and negative parts (cf. [10, p. 464]), the following is readily verified.

Remark 2: For any $L \in \mathcal{M}(\mathbb{C}^n)$ there exist $L_1, L_2, L_3, L_4 \in \mathcal{M}(\mathbb{C}^n)^+$, such that

$$L=(L_1-L_2)+\sqrt{-1}\,(L_3-L_4).$$

Let $\sigma(L) \subset \mathbb{C}$ denote the spectrum (i.e., the set of all eigenvalues) of $L \in \mathcal{M}(\mathbb{C}^n)$, and $r_o(L) = \max\{|\lambda|:\lambda \in \sigma(L)\}$ the spectral radius of $L \in \mathcal{M}(\mathbb{C}^n)$. The result below will be needed in the sequel.

Proposition 1: For any $L \in \mathcal{M}(\mathbb{C}^{n^2})$, the following assertions are equivalent:

(a) $r_o(L)<1$.

(b) $\|L^i f(Y)\| \to 0$ as $i \to \infty$; $\forall\, Y \in \mathcal{M}(\mathbb{C}^n)^+$.

Proof: It is well known (e.g., see [11]) that (a) holds if and only if

(c) $\|L^i y\| \to 0$ as $i\to\infty$; $\forall\, y \in \mathbb{C}^{n^2}$,

which is equivalent to

(d) $\|L^i f(Y)\| \to 0$ as $i\to\infty$; $\forall\, Y \in \mathcal{M}(\mathbb{C}^n)$,

since range $(f) = \mathbb{C}^{n^2}$. Hence, in particular, (a) implies (b). Now by Remark 2, any $Y \in \mathcal{M}(\mathbb{C}^n)$ can be decomposed as $Y = (Y_1 - Y_2) + \sqrt{-1}(Y_3 - Y_4)$, with $Y_j \in \mathcal{M}(\mathbb{C}^n)^+, j = 1, 2, 3, 4$. Therefore,

$$\|L^i f(Y)\| \le \sum_{j=1}^{4} \|L^i f(Y_j)\|$$

since f is linear. Thus, (b) implies (d), which is equivalent to (a). □

A Cauchy sequence $\{z(i);\ i \geq 0\}$ in a normed linear space Z [e.g., in $\mathbb{C}^n$ or $\mathcal{M}(\mathbb{C}^n)$] is Cauchy summable iff [6]

$$\sum_{i=0}^{\infty} \sup_{\nu\ge 0} \|z(i+\nu)-z(i)\| < \infty.$$

The next proposition states a deterministic stability result for Cauchy summable sequences through finite-dimensional linear systems, that will suffice our needs in Section V. It is a particular case of Lemma (L-1) in [6], which has been established for a class of separable nonlinear discrete systems evolving in a Banach space.

Proposition 2: Let $\{v(i);\ i \geq 0\}$ be any Cauchy summable sequence in $\mathbb{C}^n$, and consider a $\mathbb{C}^n$-valued sequence $\{y(i);\ i \geq 0\}$ as follows:

$$y(i+1)=Ly(i)+v(i)$$

where $L \in \mathcal{M}(\mathbb{C}^n)$. If $r_o(L) < 1$, then $\{y(i);\ i \geq 0\}$ is a Cauchy summable sequence, and

$$\lim_{i\to\infty} y(i)=(I-L)^{-1} \lim_{i\to\infty} v(i)$$

for any initial condition $y(0) \in \mathbb{C}^n$.

Finally, let $\mathrm{tr}(L)$ denote the trace of $L \in \mathcal{M}(\mathbb{C}^n)$ as usual, and recall the following (cf. [12, p. 96]).

Remark 3: tr: $\mathcal{M}(\mathbb{C}^n) \to \mathbb{C}$ is a linear functional; that is

(a) $\mathrm{tr}(\alpha K+\beta L)=\alpha\ \mathrm{tr}(K)+\beta\ \mathrm{tr}(L)$

for any $K, L \in \mathcal{M}(\mathbb{C}^n)$ and $\alpha, \beta \in \mathbb{C}$, with the following additional properties:

(b) $\mathrm{tr}(KL)=\mathrm{tr}(LK)$,

(c) $0\le L\neq 0 \Rightarrow 0<\mathrm{tr}(L) \in \mathbb{R}$.

III. Problem Formulation

Model Description: Consider a discrete-time dynamical system modeled by the following n-dimensional difference equation:

$$x(i+1)=\left[A_o+\sum_{k=1}^{p} \omega_k(i)A_k\right]x(i)+Bu(i);\quad x(0)=x_o \qquad (1)$$

where $A_k \in \mathcal{M}(\mathbb{C}^n)$ for each $k = 0, 1, \cdots, p$, and $B \in \mathcal{M}(\mathbb{C}^m, \mathbb{C}^n)$. Here $\{x(i);\ i \geq 0\}$ denotes the $\mathbb{C}^n$-valued random state sequence, and $\{u(i);\ i \geq 0\}$ and $\{w(i) = (\omega_1(i), \cdots, \omega_p(i));\ i \geq 0\}$ are random disturbances in $\mathbb{C}^m$ and $\mathbb{C}^p$, respectively, which may eventually be equal to each other or even mutually independent.

Assumption 1: x_o is a second-order random vector independent of $\{(w(i), u(i));\ i \geq 0\}$, which is an independent second-order wide sense stationary random sequence in $\mathbb{C}^{m+p}$.

Problem Statement: Let E denote expectation as usual, and set

$$q(i)=E\{x(i)\},$$

$$Q_\nu(i)=E\{x(i+\nu)x(i)^*\},$$

$$Q(i)=Q_o(i)$$

in $\mathbb{C}^n$, $\mathcal{M}(\mathbb{C}^n)$, and $\mathcal{M}(\mathbb{C}^n)^+$, respectively, for each $i, \nu \geq 0$. In Section V we shall investigate necessary and sufficient conditions on the model $([A_o + \Sigma_{k=1}^{p} \omega_k(i)A_k], B)$ described in (1), to ensure that $\{q(i);\ i \geq 0\}$ and $\{Q_\nu(i);\ i \geq 0\}$ converge for any admissible initial condition x_o and input disturbance $\{u(i);\ i \geq 0\}$, and their limits do not depend on x_o. So we define the following.

Definition 1: The model (1) is mean square stable (MSS) if, for any initial condition x_o and input disturbance $\{u(i);\ i \geq 0\}$ satisfying Assumption 1, there exists $q \in \mathbb{C}^n$ and $Q \in \mathcal{M}(\mathbb{C}^n)^+$ independent of x_o, such that

(a) $\|q(i)-q\| \to 0$ as $i\to\infty$,

(b) $\|Q(i)-Q\| \to 0$ as $i\to\infty$.

Definition 2: The second-order state sequence $\{x(i);\ i \geq 0\}$ is asymptotically wide sense stationary (AWSS) if there exists $q \in \mathbb{C}^n$ and $Q_\nu \in \mathcal{M}(\mathbb{C}^n)$, for each $\nu \geq 0$, such that

$$\|q(i)-q\| \to 0 \quad \text{as } i\to\infty,$$

$$\|Q_\nu(i)-Q_\nu\| \to 0 \quad \text{as } i\to\infty.$$

The model (1) is AWSS if $\{x(i);\ i \geq 0\}$ is AWSS for any x_o and $\{u(i);\ i \geq 0\}$ as in Assumption 1, and the limits q and Q_ν do not depend on x_o.

Remark 4: Notice that the existence of $q(i)$ and $Q_\nu(i)$, for each $i \geq 0$, is ensured by Assumption 1. Hence, $\{x(i);\ i \geq 0\}$ is actually a second-order random sequence. Moreover, if the sequence $\{Q(i) = Q_o(i) \in \mathcal{M}(\mathbb{C}^n)^+;\ i \geq 0\}$ converges, then the limit $Q = Q_o \in \mathcal{M}(\mathbb{C}^n)^+$, since $\mathcal{M}(\mathbb{C}^n)^+$ is closed in $\mathcal{M}(\mathbb{C}^n)$.

IV. Auxiliary Results

First, let us pose some further notation, concerning the model (1), that will be required in the sequel. Regarding the disturbances $\{w(i);\ i \geq 0\}$ and $\{u(i);\ i \geq 0\}$, set

$$\rho_k=E\{\omega_k(i)\}$$

$$\gamma_{kl}=E\{\omega_k(i)\overline{\omega_l(i)}\}-E\{\omega_k(i)\}\overline{E\{\omega_l(i)\}}$$

in $\mathbb{C}$ for each $k, l = 1, \cdots, p$, such that $\gamma_{kl} = \overline{\gamma_{lk}}$, and

$$r=E\{u(i)\}$$

$$r_k=E\{\overline{\omega_k(i)}u(i)\}$$

$$R=E\{u(i)u(i)^*\}$$

in $\mathbb{C}^m$, $\mathbb{C}^m$, and $\mathfrak{M}(\mathbb{C}^m)^+$, respectively, for every $i \geqq 0$. Moreover, set

$$F=E\left\{A_o+\sum_{k=1}^{p}\omega_k(i)A_k\right\}=A_o+\sum_{k=1}^{p}\rho_kA_k$$

$$A=\bar{F}\otimes F+\sum_{k,l=1}^{p}\gamma_{kl}\bar{A}_l\otimes A_k$$

in $\mathfrak{M}(\mathbb{C}^n)$ and $\mathfrak{M}(\mathbb{C}^{n^2})$, respectively. Now define linear operators $T, T^{\#}:\mathfrak{M}(\mathbb{C}^n)\rightarrow\mathfrak{M}(\mathbb{C}^n)$ as follows:

$$T(L)=\sum_{k,l=1}^{p}\gamma_{kl}A_kLA_l^*$$

$$T^{\#}(L)=\sum_{k,l=1}^{p}\gamma_{lk}A_k^*LA_l$$

for all $L\in\mathfrak{M}(\mathbb{C}^n)$. Finally, let $P, V:\mathbb{C}^n\rightarrow\mathfrak{M}(\mathbb{C}^n)$ be transformations given by

$$P(y)=\left[A_oyr^*+\sum_{k=1}^{p}A_kyr_k^*\right]B^*$$

$$V(y)=P(y)+P(y)^*+BRB^*$$

for all $y\in\mathbb{C}^n$. The following propositions comprise the basic auxiliary results for establishing the stability lemmas of Section V.

Proposition 3: Consider Assumption 1. For every $i\geqq 0$,

(a) $q(i+1)=Fq(i)+Br$,

(b) $Q(i+1)=FQ(i)F^*+T[Q(i)]+V[q(i)]$,

(c) $Q_\nu(i)=F^\nu Q(i)+\sum_{j=0}^{\nu-1}F^jBrq(i)^*; \quad \forall\,\nu\geqq 1.$

Proposition 4: For any $L\in\mathfrak{M}(\mathbb{C}^n)$,

(a) $f[FLF^*+T(L)]=Af(L)$,

(b) $f[F^*LF+T^{\#}(L)]=A^*f(L)$.

Proposition 5: For any $K, L\in\mathfrak{M}(\mathbb{C}^n)$,

(a) $T(L)^*=T(L^*), \quad T^{\#}(L)^*=T^{\#}(L^*)$.

(b) $L\in\mathfrak{M}(\mathbb{C}^n)^+ \Rightarrow T(L), \quad T^{\#}(L)\in\mathfrak{M}(\mathbb{C}^n)^+$.

(c) $\mathrm{tr}[T^{\#}(L)K]=\mathrm{tr}[LT(K)], \quad \mathrm{tr}[T(L)K]=\mathrm{tr}[LT^{\#}(K)]$.

Comments: For a proof of Proposition 3 under Assumption 1 see [6], where the independence argument within Assumption 1 is discussed in detail. Proposition 4 is readily verified by the definitions of A, T, $T^{\#}$, and Remark 1. Proposition 5(a) is trivial from the definition of T and $T^{\#}$. That T is $\mathfrak{M}(\mathbb{C}^n)^+$-invariant, thus resulting Proposition 5(b), has been shown in [6]. Proposition 5(c) is straightforward by Remark 3(a), (b).

Proposition 6:

$$r_\sigma(A)<1\Rightarrow r_\sigma(F)<1.$$

Proof: Consider the following sequence in $\mathfrak{M}(\mathbb{C}^n)$:

$$f[X(i+1)]=Af[X(i)]; \quad f[X(0)]\in\mathbb{C}^{n^2}.$$

It is readily verified by induction that

$$X(i)=F^iX(0)F^{*i}+\sum_{j=0}^{i-1}F^{i-j-1}T[X(j)]F^{*i-j-1}$$

for every $i\geqq 1$. Hence,

$$\langle X(i)y;\ y\rangle=\langle F^iX(0)F^{*i}y;\ y\rangle+\sum_{j=0}^{i-1}\langle F^{i-j-1}T[X(j)]F^{*i-j-1}y;\ y\rangle$$

for every $i\geqq 1$ and all $y\in\mathbb{C}^n$. Moreover, by Proposition 5(b) it follows by induction that $X(i)\in\mathfrak{M}(\mathbb{C}^n)^+$, and so $T[X(i)]\in\mathfrak{M}(\mathbb{C}^n)^+$, for every $i\geqq 0$ whenever $X(0)\in\mathfrak{M}(\mathbb{C}^n)^+$. Then, for $X(0)=I$,

$$\|X(i)^{1/2}y\|^2=\|F^{*i}y\|^2+\sum_{j=0}^{i-1}\|T[X(j)]^{1/2}F^{*i-j-1}y\|^2$$

for every $i\geqq 1$ and all $y\in\mathbb{C}^n$. Now, by Proposition 4(a), we have

$$f[X(i+1)]=Af[X(i)]; \quad f[X(0)]\in\mathbb{C}^{n^2}.$$

Therefore, as it is well known (e.g., see [11]), $r_\sigma(A)<1$ if and only if

$$\|f[X(i)]\|=\|A^if[X(0)]\|\rightarrow 0 \quad \text{as } i\rightarrow\infty; \quad \forall\, f[X(0)]\in\mathbb{C}^{n^2},$$

which implies that

$$\|X(i)\|=\|f^{-1}(f[X(i)])\|\rightarrow 0 \quad \text{as } i\rightarrow\infty; \quad \forall\, X(0)\in\mathfrak{M}(\mathbb{C}^n),$$

since f is a topological isomorphism. Hence,

$$\|X(i)^{1/2}y\|\leqq\|X(i)\|^{1/2}\|y\|\rightarrow 0 \quad \text{as } i\rightarrow\infty; \quad \forall\, X(0)\in\mathfrak{M}(\mathbb{C}^n),\ \forall\, y\in\mathbb{C}^n.$$

In particular, the above convergence holds true for $X(0)=I$, and so

$$\|F^{*i}y\|\rightarrow 0 \quad \text{as } i\rightarrow\infty; \quad \forall\, y\in\mathbb{C}^n,$$

which is equivalent to $r_\sigma(F^*)<1$, thus the desired result follows since $r_\sigma(F)=r_\sigma(F^*)$. □

Remark 5: The converse of Proposition 6 fails. That is,

$$r_\sigma(F)<1\nRightarrow \quad r_\sigma(A)<1.$$

For instance, set $\rho_k=0$, $\gamma_{kk}=1$ for every $k=1,\cdots,p$, and $\gamma_{kl}=0$ whenever $k\neq l$. Moreover, set $A_o=I/2$ and $A_k=I$ for every $k=1,\cdots,p$. Hence, $F=A_o=I/2$, and $A=\Sigma_{k=0}^{p}A_k\otimes A_k=(1/4+p)I$, such that $r_\sigma(F)=1/2$ and $r_\sigma(A)=(1/4+p)>1$. On the other hand, notice that $r_\sigma(F)<1$ if and only if $r_\sigma(\bar{F}\otimes F)<1$, since (e.g., see [9]) for any $K, L\in\mathfrak{M}(\mathbb{C}^n)$ $r_\sigma(K\otimes L)=r_\sigma(K)r_\sigma(L)$.

V. Necessary and Sufficient Conditions for MSS

In this section we prove the main result of the paper, which is stated in Theorem 1 below. The proof of Theorem 1 is readily obtained by combining Lemmas 1-5, which will be established in the sequel.

Theorem 1: Consider a discrete-time dynamical system as described in (1), under Assumption 1. Let $F \in \mathfrak{M}(\mathbb{C}^n)$, $A \in \mathfrak{M}(\mathbb{C}^{n^2})$, and $T:\mathfrak{M}(\mathbb{C}^n) \to \mathfrak{M}(\mathbb{C}^n)$ be defined as in Section IV. The following assertions are equivalent.

(a) $r_\sigma(A) < 1$.

(b) For any $S > 0 \in \mathfrak{M}(\mathbb{C}^n)$, there exists a unique $G > 0 \in \mathfrak{M}(\mathbb{C}^n)$ such that

$$G = FGF^* + T(G) + S,$$

which is given by $G = f^{-1}[(I - A)^{-1}f(S)]$.

(c) The model (1) is AWSS in the sense of Definition 2.

(d) The model (1) is MSS in the sense of Definition 1.

(e) The model (1) is MSS in the sense of Definition 1(b) only.

Moreover, if the above holds then the limits in Definitions 1 and 2 are

$$q = (I - F)^{-1}Br,$$

$$Q = f^{-1}[(I - A)^{-1}f(V[q])],$$

$$Q_\nu = F^\nu Q + \sum_{j=0}^{\nu-1} F^j Brq^*; \qquad \forall\, \nu \geq 1,$$

in $\mathbb{C}^n$, $\mathfrak{M}(\mathbb{C}^n)^+$, and $\mathfrak{M}(\mathbb{C}^n)$, respectively.

Lemma 1: If $r_\sigma(A) < 1$, then (a) there exists a unique $G \in \mathfrak{M}(\mathbb{C}^n)$ such that

$$G = FGF^* + T(G) + S$$

for every $S \in \mathfrak{M}(\mathbb{C}^n)$. Moreover, (b) $G = f^{-1}[(I - A)^{-1}f(S)]$, and

$$S = S^* \Leftrightarrow G = G^*,$$

$$S \geq 0 \Rightarrow G \geq 0,$$

$$S > 0 \Rightarrow G > 0.$$

Proof: Since $r_\sigma(A) < 1$, there exists $(I - A)^{-1} = \Sigma_{i=0}^\infty A^i \in \mathfrak{M}(\mathbb{C}^{n^2})$ (cf. [13, p. 278]).

(a): For any $S \in \mathfrak{M}(\mathbb{C}^n)$, there exists $G \in \mathfrak{M}(\mathbb{C}^n)$ such that

$$f(G) = (I - A)^{-1}f(S)$$

in $\mathbb{C}^{n^2}$, since range $(f) = \mathbb{C}^{n^2}$. Hence,

$$f(S) = (I - A)f(G) = f[G - FGF^* - T(G)],$$

by Proposition 4(a), since f is linear. But this implies the existence result; that is

$$S = G - FGF^* - T(G)$$

since f is one-to-one. Now suppose there exists $K \in \mathfrak{M}(\mathbb{C}^n)$ such that $S = K - FKF^* - T(K)$. Then, for $L = G - K \in \mathfrak{M}(\mathbb{C}^n)$,

$$L = FLF^* + T(L).$$

Thus, by Proposition 4(a),

$$f(L) = Af(L).$$

Therefore, either $f(L) = 0$, which implies that $L = 0$ since f is linear, or $1 \in \sigma(A)$. Hence, $K = G$ since $r_\sigma(A) < 1$, which confirms the uniqueness result.

(b): By the above uniqueness result and Proposition 5(a), it follows that G is self-adjoint if and only if S is self-adjoint. Now consider the following sequence in $\mathfrak{M}(\mathbb{C}^n)$:

$$X(i+1) = FX(i)F^* + T[X(i)]; \quad X(0) = S.$$

According to Proposition 4(a) we have

$$f[X(i+1)] = Af[X(i)]; \quad f[X(0)] = f(S)$$

in $\mathbb{C}^{n^2}$. Hence, $f[X(i)] = A^i f(S)$ for every $i \geq 0$. Therefore,

$$G = f^{-1}[(I - A)^{-1}f(S)] = f^{-1}\left[\sum_{i=0}^\infty A^i f(S)\right]$$

$$= f^{-1}\left(\sum_{i=0}^\infty f[X(i)]\right) = \sum_{i=0}^\infty X(i)$$

since f is a topological isomorphism. By Proposition 5(b) it follows by induction that $X(i) \in \mathfrak{M}(\mathbb{C}^n)^+$ for every $i \geq 0$ whenever $X(0) \in \mathfrak{M}(\mathbb{C}^n)^+$, thus being $\{\Sigma_{i=0}^j X(i);\ j \geq 0\}$ a monotonically increasing sequence in $\mathfrak{M}(\mathbb{C}^n)^+$. That is,

$$0 \leq S = X(0) \leq \sum_{i=0}^{j} X(i) \leq \sum_{i=0}^{j+1} X(i) \leq G$$

for each $j \geq 0$, which concludes the proof of part (b). □

Lemma 2: If there exists $G > 0 \in \mathfrak{M}(\mathbb{C}^n)$ such that

$$G = FGF^* + T(G) + S$$

for some $S > 0 \in \mathfrak{M}(\mathbb{C}^n)$, then $r_\sigma(A) < 1$.

Proof: Consider the following discrete linear free system in $\mathbb{C}^{n^2}$.

$$y(i+1) = A^*y(i); \quad y(0) \in f(\mathfrak{M}(\mathbb{C}^n)^+). \tag{2}$$

From Proposition 4(a) we have

$$f^{-1}[y(i+1)] = F^*f^{-1}[y(i)]F + T^\#(f^{-1}[y(i)]);$$
$$f^{-1}[y(0)] \in \mathfrak{M}(\mathbb{C}^n)^+.$$

By Proposition 5(b) it follows by induction that $f^{-1}[y(i)] \in \mathfrak{M}(\mathbb{C}^n)^+$ for every $i \geq 0$. Hence, $y(i) \in f(\mathfrak{M}(\mathbb{C}^n)^+)$ for every $i \geq 0$. Now set $\phi: f(\mathfrak{M}(\mathbb{C}^n)^+) \subset \mathbb{C}^{n^2} \to \mathbb{R}$, such that

$$\phi(y) = \mathrm{tr}\,(f^{-1}(y)G) = \mathrm{tr}\,(G^{1/2}f^{-1}(y)G^{1/2})$$

with $G = FGF^* + T(G) + S > 0 \in \mathfrak{M}(\mathbb{C}^n)$ for some $S > 0 \in \mathfrak{M}(\mathbb{C}^n)$. It is readily verified that

i) ϕ is continuous, and

ii) $\phi(y) \to \infty$ as $\|y\| \to \infty$,

that is, ϕ is radially unbounded, since $\mathrm{tr}(f^{-1}(\cdot)G): \mathbb{C}^{n^2} \to \mathbb{C}$ is a composition of bounded linear (and so continuous) transformations, which is real-valued and radially unbounded on $f(\mathfrak{M}(\mathbb{C}^n)^+)$ whenever $G > 0$. Moreover ϕ is homogeneous and positive, such that

iii) $\phi(0) = 0$, and

iv) $\phi(y) > 0$; $\forall\, y \neq 0 \in f(\mathfrak{M}(\mathbb{C}^n)^+)$

since $0 \neq f^{-1}(y) \in \mathfrak{M}(\mathbb{C}^n)^+$ for all $y \neq 0 \in f(\mathfrak{M}(\mathbb{C}^n)^+)$, and $G > 0$, by Remark 3(c). Furthermore, by Remark 3(a), (b) and Proposition 5(c), we have

$$\begin{aligned}
&\mathrm{tr}\,(f^{-1}[y(i+1)]G) - \mathrm{tr}\,(f^{-1}[y(i)]G)\\
&= \mathrm{tr}\,(F^*f^{-1}[y(i)]FG + T^\#(f^{-1}[y(i)])G - f^{-1}[y(i)]G)\\
&= \mathrm{tr}\,[f^{-1}[y(i)](FGF^* + T(G) - G)]\\
&= -\mathrm{tr}\,(f^{-1}[y(i)]S) = -\mathrm{tr}\,(S^{1/2}f^{-1}[y(i)]S^{1/2})
\end{aligned}$$

for every $i \geq 0$. However, according to Remark 3(c), $\mathrm{tr}(S^{1/2}f^{-1}[y(i)]S^{1/2}) \geq 0$ for every $i \geq 0$, and $\mathrm{tr}(S^{1/2}f^{-1}[y(i)]S^{1/2}) = 0$ if and only if $f^{-1}[y(i)] = 0$, since $S^{1/2} > 0$ and $f^{-1}[y(i)] \geq 0$ for every $i \geq 0$. Thus, $\mathrm{tr}(S^{1/2}f^{-1}[y(i)]S^{1/2}) > 0$ whenever $y(i) \neq 0$, since f is an isomorphism. Hence,

$$\text{v)}\ \phi[y(i+1)]-\phi[y(i)]<0; \qquad \forall\, y(i)\neq 0 \in f(\mathfrak{M}(C^n)^+).$$

Therefore, by i)–v), ϕ is a Lyapunov function for the system in (2), and so (2) is an asymptotically stable system (e.g., see [14, p. 486]); that is,

$$\|A^{*i}y\|\to 0 \text{ as } i\to\infty; \quad \forall\, y\in f(\mathfrak{M}(\mathbb{C}^n)^+)$$

or equivalently,

$$\|A^{*i}f(Y)\|\to 0 \text{ as } i\to\infty; \quad \forall\, Y\in \mathfrak{M}(\mathbb{C}^n)^+.$$

Then the desired result follows by Proposition 1, since $r_\sigma(A) = r_\sigma(A^*)$. □

Lemma 3: If $r_\sigma(A) < 1$, then the model (1) is MSS in the sense of Definition 1, with

$$q=(I-F)^{-1}Br,$$

$$Q=f^{-1}[(I-A)^{-1}f(V[q])]$$

in $\mathbb{C}^n$ and $\mathfrak{M}(\mathbb{C}^n)^+$, respectively.

Proof

(a): If $r_\sigma(A) < 1$ then, by Proposition 6, $r_\sigma(F) < 1$. Hence, it follows the result in Definition 1(a) for the sequence $\{q(i) \in \mathbb{C}^n; i \geq 0\}$ in Proposition 3(a), which is Cauchy summable with limit $q = (I - F)^{-1}Br \in \mathbb{C}^n$, according to Proposition 2. Thus, for $P, V: \mathbb{C}^n \to \mathfrak{M}(\mathbb{C}^n)$ defined in Section IV, it also follows that $\{V[q(i)] \in \mathfrak{M}(\mathbb{C}^n); i \geq 0\}$ is a Cauchy summable sequence, since P is bounded and linear. Therefore, $\{f(V[q(i)]) \in \mathbb{C}^{n^2}; i \geq 0\}$ is also Cauchy summable since f is bounded and linear.

(b): Now consider the sequence $\{Q(i) \in \mathfrak{M}(\mathbb{C}^n)^+; i \geq 0\}$, such that

$$f(Q(i+1))=Af(Q(i))+f(V[q(i)])$$

by Propositions 3(b) and 4(a). Since $r_\sigma(A) < 1$ and $\{f(V[q(i)]) \in \mathbb{C}^{n^2}; i \geq 0\}$ is Cauchy summable, it follows by Proposition 2 that $\{f(Q(i)) \in \mathbb{C}^{n^2}; i \geq 0\}$ converges to $(I - A)^{-1}f(V[q]) \in \mathbb{C}^{n^2}$, where $f(V[q]) = \lim_{i\to\infty} f(V[q(i)])$ by continuity of f and V. Therefore, $\{Q(i) \in \mathfrak{M}(\mathbb{C}^n)^+; i \geq 0\}$ converges to $f^{-1}[(I - A)^{-1}f(V[q])] \in \mathfrak{M}(\mathbb{C}^n)^+$, since f^{-1} is continuous and $\mathfrak{M}(\mathbb{C}^n)^+$ is closed in $\mathfrak{M}(\mathbb{C}^n)$. Hence, the result in Definition 1(b). □

Lemma 4: If the model (1) is MSS in the sense of Definition 1(b), then $r_\sigma(A) < 1$.

Proof: From Propositions 3(b) and 4(a) we have

$$f(Q(i+1))=Af(Q(i))+f(V[q(i)])$$

for every $i \geq 0$, such that by induction,

$$f(Q(i))=A^i f(Q(0))+\sum_{j=0}^{i-1} A^{i-j-1}f(V[q(j)])$$

for every $i \geq 1$. Now, if the model (1) is MSS in the sense of Definition 1(b), then (since f is continuous)

$$f(Q(i))\to f(Q) \quad \text{as } i\to\infty$$

for any $Q(0) \in \mathfrak{M}(\mathbb{C}^n)^+$, and $f(Q) \in \mathbb{C}^{n^2}$ does not depend on $Q(0)$. In particular, by setting $Q(0) = 0$, we have

$$\sum_{j=0}^{i-1} A^{i-j-1}f(V[q(j)])\to f(Q) \quad \text{as } i\to\infty.$$

Therefore,

$$A^i f(Q(0))\to 0 \quad \text{as } i\to\infty$$

for any $Q(0) \in \mathfrak{M}(\mathbb{C}^n)^+$. Thus, $r_\sigma(A) < 1$, by Proposition 1. □

Lemma 5: The model (1) is MSS in the sense of Definition 1 if and only if it is AWSS in the sense of Definition 2.

Proof: Asymptotic wide sense stationarity trivially implies mean square stability (cf. Remark 4). By Proposition 3(c) we get

$$\|Q_\nu(i)-Q_\nu\|\leq \|F^\nu\|\|Q(i)-Q\|+\sum_{j=0}^{\nu-1}\|F^jBr\|\|q(i)-q\|$$

for every $\nu \geq 1$, with $Q_\nu \in \mathfrak{M}(\mathbb{C}^n)$ defined as in the theorem statement. Hence, the converse is also true. □

Remark 6: If we set $A_k = 0$ for every $k = 1, \cdots, p$, the model (1) is naturally reduced to a linear one:

$$x(i+1)=A_o x(i)+Bu(i); \quad x(0)=x_o. \tag{3}$$

For this particular case, the independence assumption may be relaxed to uncorrelatedness, according to the independence argument in [6], as follows.

Assumption 2: x_o is a second-order random vector uncorrelated with $\{u(i); i \geq 0\}$, which is an uncorrelated second-order wide sense stationary random sequence in $\mathbb{C}^m$.

Moreover, in this case, we have $F = A_o$, $T = 0$, and $A = \bar{A}_o \otimes A_o$, such that $r_\sigma(A) < 1$ if and only if $r_\sigma(A_o) < 1$ (cf. Remark 5). Therefore, Theorem 1 also gives necessary and sufficient conditions for mean square stability of discrete-time linear systems as in (3) with the following simplifications.

i) Assumption 1 replaced by the weaker Assumption 2.

ii) $A \in \mathfrak{M}(\mathbb{C}^{n^2})$ replaced by $A_o \in \mathfrak{M}(\mathbb{C}^n)$ in part (a) of Theorem 1.

iii) $F = A_o$ and $T = 0$ in part (b) of Theorem 1.

As one could expect, the equivalence between parts (a) and (b) of Theorem 1 is then reduced, in this particular case, to well-known stability conditions for time-invariant discrete linear systems (e.g., see [14, p. 487]).

Remark 7: By Lemmas 1, 3, and 4 it follows that $Q \in \mathfrak{M}(\mathbb{C}^n)^+$ is the only solution of

$$Q=FQF^*+T(Q)+V(q)$$

as has been commented on before in [6]. Also notice that if $G > 0$ in Lemma 1(b), it does not necessarily follow that $S \geq 0$, even for the linear case where $T = 0$. Or equivalently, $V(q) \in \mathfrak{M}(\mathbb{C}^n)$ may not lie in $\mathfrak{M}(\mathbb{C}^n)^+$, although all other terms in the above equation lie in $\mathfrak{M}(\mathbb{C}^n)^+$. To illustrate this, set $n = m = 2$, $p = 1$, $\rho_1 = 0$, $\gamma_{11} = 1$, $B = I$, $r = (1, 0)$, and

$$F=A_o=\frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad R=\frac{1}{6}\begin{bmatrix}7 & 0\\ 0 & 1\end{bmatrix}$$

such that $R - rr^* = I/6$. Now consider two cases.

i) $A_1 = 0$ (i.e., the linear case), such that $T = 0$ and $r_\sigma(A) = r_\sigma(A_o)^2 = 1/4$.

ii) $A_1 = I/2$, such that $T(Q) = Q/4$ and $r_\sigma(A) = 1/2$. In this case assume that $\{w(i); i \geq 0\}$ and $\{u(i); i \geq 0\}$ are uncorrelated, such that $r_1 = 0$.

In both cases we have $P(q) = A_o qr^* = A_o(I - A_o)^{-1}rr^*$. Hence,

$$V(q)=P(q)+P(q)^*+R=\frac{1}{6}\begin{bmatrix}11 & 4\\ 4 & 1\end{bmatrix}\notin \mathfrak{M}(\mathbb{C}^n)^+.$$

Notice that, for cases i) and ii) we have, respectively,

$$\text{i)}\ Q=\frac{2}{9}\begin{bmatrix}9 & 4\\ 4 & 3\end{bmatrix}; \qquad \text{ii)}\ Q=\frac{1}{6}\begin{bmatrix}17 & 8\\ 8 & 7\end{bmatrix}.$$

VI. Conclusion

In this paper we have established necessary and sufficient conditions for mean square stability for a wide class of bilinear systems. These were supposed to evolve in discrete-time and to operate in a second-order stochastic environment, under wide sense stationarity and independence assumptions only.

The main result was formulated in Theorem 1. There it was given two necessary and sufficient conditions for mean square stability, which were stated in parts (a) and (b). In parts (c), (d), and (e) it was shown that mean square stability is equivalent to asymptotic wide sense stationarity, which is equivalent to correlation convergence for any initial condition. Formulas for computing the limits of state mean and correlations sequences were also presented. The above-mentioned result was proved in Lemmas 1-5, by using the auxiliary propositions of Sections II and IV.

As we have commented in Section I, there are only a few papers on the stochastic stability for discrete-time bilinear systems. The general class of models described in Section III was also considered in [6], where sufficient conditions for mean square stability were investigated. We notice that those conditions may eventually be easier verified, in some practical cases, than the criteria in Theorem 1. This may happen because the conditions in [6] do not require the computation of the spectral radius of a linear combination of Kronecker products in $\mathfrak{M}(\mathcal{C}^{n^2})$, nor the analysis of Lyapunov-type operator equations. On the other hand the sufficient conditions in [6] are not necessary ones, and in this sense Theorem 1 delivers a complete theoretical answer to the problem under consideration, by supplying necessary and sufficient conditions. Also notice that the hypothesis of Lemma 2 can be independently applied as a sufficient condition for mean square stability, by choosing a suitable $S > 0$.

Finally, it is worth remarking that discrete-time stochastic stability is a fundamental property for developing recursive system identification techniques. Therefore, by establishing the weakest stability conditions as proposed in Theorem 1, we have also enlarged the class of identifiable bilinear structures that depend on mean square stability conditions (e.g., see [15], [16]).

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C. S. Kubrusly was born in Rio de Janeiro, Brazil, in November 1947. He received the diploma in electrical engineering and the M.Sc. degree from the Catholic University of Rio de Janeiro in 1971 and 1973, respectively, and the Ph.D. degree from the University of Warwick in 1976.

Since 1976 he has been with the Department of Electrical Engineering, Catholic University of Rio de Janeiro, where he is presently an Associate Professor lecturing on applied mathematics. In 1980 he also joined the Department of Research and Development, National Laboratory for Scientific Computation, LNCC/CNPq, Rio de Janeiro, where he is a Senior Researcher in mathematical system theory. His interests include identification and stability of stochastic systems.

Dr. Kubrusly is an Associate Editor of *Matemática Aplicada e Computacional*, the journal of the Brazilian Society for Computational and Applied Mathematics.

Oswaldo L. V. Costa was born in Rio de Janeiro, Brazil, on June 19, 1959. He received the B.Sc. and M.Sc. degrees, both in electrical engineering from the Catholic University of Rio de Janeiro, in 1981 and 1983, respectively.

From August 1982 to July 1983 he worked as a Research Assistant for the Catholic University of Rio de Janeiro. He is currently a Ph.D. student in the Control Section of the Department of Electrical Engineering, Imperial College of Science and Technology, University of London, London, England.