STOCHASTIC APPROXIMATION ALGORITHMS AND APPLICATIONS

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ABSTRACT

This study presents the conditions of applicability of stochastic approximation algorithms that minimize a mean-square error criterion for identification of a linear discrete-time stationary system without dynamical numerator. The acceleration of the convergence is discussed. Then a tentative is outlined to overcome the previous requirement of states accessibility.

I – INTRODUCTION

The first part of the paper concerns the identification of a completely unknown functional form of a memoryless system where the input signals have unknown continuous probability density functions [1].

The second part applies this technique to identify a stationary linear discrete-time system and shows that the convergence is still obtained [2], but that the accelerated convergence is not anymore a consequence of the first part. Finally, a suggestion is given to overcome the requirement of having measurable states to apply this technique.

II – IDENTIFICATION OF A MEMORYLESS SYSTEM

Given a stationary discrete-time and memoryless system, Fig. 1, where:

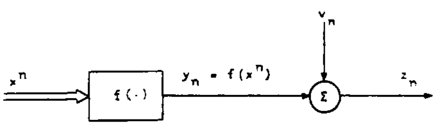

Figure 1 – Memoryless system

1) The inputs are sequences of independent and identically distributed m-vectors $x^n = [x_1^n \; x_2^n \ldots x_m^n]^T$, with real random components x_i^n, $\forall i \in I = [1,2,\ldots]$ such that $\|x^n\| < \infty$.

2) The real-valued continuous bounded function f(x) is defined over the set of possible sequences x^n.

3) v^n is a zero-mean white noise process with covariance $E[v_n v_k] = \psi_v^2(n)\,\delta_K(n-k) < \infty$, independent of x^n.

Let define $\hat{f}(x) \triangleq c\,\phi^T(x)$ as a linear estimate of f(x) based on the components $\phi_i(x)$ of $\phi(x) = [\phi_1(x) \ldots \phi_n(x)]^T$ which are preselected functions, linearly independent, real-valued, continuous and bounded over the set of possible sequences x^n, $\forall\, i = 1,2,\ldots,N < \infty$.

Saridis et al. [1] proved that a stochastic approximation algorithm to estimate a real vector c which minimizes a cost function $J(c) \triangleq E[\|f(x) - \hat{f}(x)\|^2]$ is:

$$c^{n+1} = \frac{1-\lambda_n}{1-\mu_n}c^n + \frac{M_n\phi(x^n)}{(1-\mu_n)/\mu_n + \phi^T(x^n)M_n\,\phi(x^n)} \times \left[\frac{\lambda_n}{\mu_n}z_n - \frac{1-\lambda_n}{1-\mu_n}\phi^T(x^n)c^n\right] \tag{1}$$

Where:

$$M_{n+1} = \frac{1}{1-\mu_n}\left[M_n - \frac{M_n\phi(x^n)\phi^T(x^n)M_n}{(1-\mu_n + \phi^T(x^n)M_n\,\phi(x^n)}\right] \tag{2}$$

λ_n and μ_n are sequences of real numbers which satisfy conditions of the following type:

$$\gamma_n \in (0,1),\ \forall n \in I \quad ;$$

$$\sum_{n=1}^{\infty} \gamma_n^2 < \infty$$

$$\prod_{n=1}^{\infty} (1-\gamma_n) = 0 \tag{3}$$

The initial conditions of (1) and (2) are such that $\|c^1\| < \infty$, $\|M_1\| < \infty$ and det $M^{-1} \neq 0$ where M_1 is an NxN symmetrical matrice.

The sequence c^n resulting of (1) is such that

$$P\,[\lim_{n\to\infty} c^n = c] = 1$$

The accelerated convergence is obtained by minimization of an upper bound of the mean-square-error of c^n,

$$E[\|c^n - c\|^2_{M^{-1}M^{-1}}]\ ,\ \forall n \in I.$$

This results in choosing the sequences λ_n and μ_n having the properties (3) presented below:

$$\lambda_n = \frac{\zeta_n^2}{\zeta_n^2 + \sigma_n^2} \quad \text{and} \quad \zeta_{n+1}^2 = \frac{\zeta_n^2\,\sigma_n^2}{\zeta_n^2 + \sigma_n^2}$$

where:

$$\zeta_n^2 \triangleq E[\| r^n - r \|^2] \qquad (4)$$

$$r \triangleq E[z\ \phi(x)]$$

$$r^n = M_n^{-1} c_n$$

$$\sigma_n^2 \triangleq E[\| z_n\ \phi(x^n) - r \|^2] \qquad (5)$$

If $\psi_v^2(n) = \psi_v^2$, then

$$E[\| z_n \phi(x^n) - r \|^2] \triangleq \sigma^2,\ \forall n \in I \qquad (6)$$

$$\text{Hence } \lambda_n = \frac{1}{n + \sigma^2/\zeta_1^2} \quad \text{and} \quad \zeta_n^2 = \frac{\sigma^2}{(n-1) + \sigma^2/\zeta_1^2}$$

$$\text{with } \zeta_1^2 \triangleq E[\| r^1 - r \|^2] \qquad (7)$$

Similarly for μ_n we have:

$$\mu_n = \frac{1}{n + \tau^2/\eta_1^2} \quad \text{and} \quad \eta_{n+1}^2 = \frac{\tau^2}{(n-1) + \tau^2/\eta_1^2}$$

where:

$$\eta_n^2 \triangleq E[\| M^{-1} - M^{-1} \|^2] \qquad (8)$$

$$M^{-1} \triangleq E[\phi(x)\ \phi^T(x)]\ ,\quad \det M^{-1} \neq 0$$

$$\text{and } \tau^2 \triangleq E[\| \phi(x)\ \phi^T(x) - M^{-1} \|^2] \qquad (9)$$

The equations (4) – (9) are different from Saridis' original work where these relations are defined as upper bounds.

Two necessary conditions to obtain the results concerning the accelerated convergence were that x^n and v^n be mutually independent.

III – IDENTIFICATION OF A LINEAR SYSTEM

Given now a stationary, discrete-time, linear system Fig. 2:

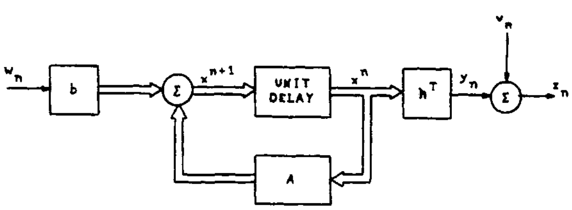

Figure 2 – Linear system

$$x^{n+1} = Ax^n + bw_n \qquad (10)$$

$$Z_n = h^T x^n + v_n \qquad (11)$$

where:

$$A = \begin{bmatrix} 0 & \vdots & I \\ \hdashline & a^T & \end{bmatrix}$$

$$x^n = [x_1^n\ x_2^n \dots x_m^n]^T\ ,\ \forall n \in I\ ;\quad b = [b_1\ b_2 \dots b_m]^T$$

is a known vector with $\| b \| < \infty$; h is an m-vector $h = [1\ 0 \dots \dots 0]^T$; $a = [a_1\ a_2 \dots a_m]^T$ is an unknown vector with $\| a \| < \infty$, w_n and v_n are zero-mean white noise independent processes with covariances

$$E[w_n w_k] = \psi_w^2\ \delta_k(n-k) < \infty$$

and

$$E[v_n v_k] = \psi_v^2\ \delta_k(n-k) < \infty$$

The identification problem consists in determining the vector x^n using the observed z_n.

From (10) and (11) we obtain:

$$z_{m+n} = a^T x^n + \nu_{m+n} \qquad (12)$$

where:

$$\nu_{m+n} \triangleq b^T\ w^{m+n-1} + v_{m+n}$$

$$w^{m+n-1} \triangleq [w_{m+n-1}\ w_{m+n-2}\ \cdots\ w_n]^T$$

The solution of (12) is analogous to the problem of section II with the fundamental advantage that f(x) is now a linear combination of x.

The equation (12) can be represented by:

$$z_{m+n} = f(x^n) + \nu_{m+n}$$

where:

$f(x^n) = a^T x^n$ as shown in Fig. 3.

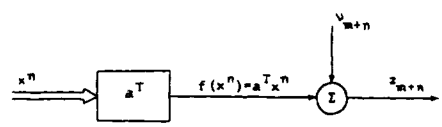

Figure 3 – System under consideration in (12)

It is easy to prove [3], [4] that:

1) x^n are identically distributed
2) $\| x^n \| < \infty$
3) ν_{m+n} is a zero-mean random variables independent of x^n with covariance $E[\nu_{m+n}\ \nu_{m+k}] < \infty$.

Comments:

The three above conditions are sufficient to apply the algorithms of section II. It is important to emphasize that: the two conditions x^n and ν_{m+n} mutually independent $\forall n$, are not satisfied in this case. This modification does not affect the convergence criterion but only the accelerated convergence. Saridis et al. [2] used a stochastic approximation algorithm to estimate recursively the vector a given by:

$$a^{n+1} = \frac{1-\lambda_n}{1-\mu_n} a^n + \frac{Q_n x^n}{(1-\mu_n)/\mu_n + x^{nT} Q_n x^n} \left[\frac{\lambda_n}{\mu_n} z_{m+n} - \frac{1-\lambda_n}{1-\mu_n} x^{nT} a^n \right] \qquad (13)$$

where:

$$Q_{n+1} = \frac{1}{1-\mu_n}\left[Q_n - \frac{Q_n\, x^n\, x^{nT}\, Q_n^T}{(1-\mu_n)/\mu_n + x^{nT}\, Q_n\, x^n}\right] \qquad (14)$$

λ_n and μ_n satisfy the conditions (3).

The initial conditions of (13) and (14) are such that $\| a^1 \| < \infty$, $\| Q_1 \| < \infty$ and $\det Q^{-1} \neq 0$ where Q_1 is an NxN symmetrical matrice.

The sequence a^n resulting of (13) is such that

$$P\ [\ \lim a^n = a\ ]\ = 1$$

As commented here, it is not anymore possible to accelerate the convergence of the algorithm using the results of section II because:

For two vectors p and q we have:

$$\begin{aligned} \| p \pm q \|^2 &= \| p \|^2 + \| q \|^2 \pm 2p^T q \\ &\leqslant \| p \|^2 + \| q \|^2 + 2\| p \|\ \| q \| \\ &\leqslant 2(\| p \|^2 + \| q \|^2) \end{aligned}$$

If p and q are two independent random vectors, with at least a zero-mean one, then:

$$E[\| p \pm q \|^2] = E[\| p \|^2] + E[\| q \|^2] \qquad (15)$$

But if they are not:

$$\begin{aligned} E[\| p \pm q \|^2] &= E[\| p \|^2] + E[\| q \|^2] \pm 2E[p^T q] \\ &\leqslant E[\| p \|^2] + E[\| q \|^2] + 2E[\| p \|\ \| q \|] \\ &\leqslant 2\,(E[\| p \|^2] + E[\| q \|^2]) \end{aligned}$$

This explains why we could use the equality (15) in section II and the impossibility of using:

$$E[\| p \pm q \|^2] \leqslant E[\| p \|^2] + E[\| q \|^2]$$

in section III as done in [2].

IV – SIMULTANEOUS IDENTIFICATION AND STATES ESTIMATION: A PARTICULAR CASE

A great disadvantage of the algorithm (13), (14) is the required accessibility of the states. To by-pass this difficulty it should be possible to try using a simultaneous iteration between the stochastic approximation algorithm presented in (13), (14) and a Kalman filter whose equations for the stationary monovariable case are given below [4]:

Gain equation:

$$k^n = R_n\, h / (h^T R_n h + \psi_v^2) \qquad (16)$$

A posteriori variance algorithm:

$$V_n = (I - k^n h^T) R_n \qquad (17)$$

A priori variance algorithm:

$$R_{n+1} = A_{n+1} V_n A_{n+1}^T + b\, \psi_w^2\, b^T \qquad (18)$$

Filter algorithm:

$$\hat{x}^{n+1} = A_{n+1}\, \hat{x}^n + k^{n+1}(z_{n+1} - h^T A_{n+1} \hat{x}^n) \qquad (19)$$

where:

A_{n+1} is the $(n+1)^{th}$ estimate of A.

The approach suggested above, fig.4 (next page), is applied to the particular case of a system given by the following equations:

$$x^{n+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.656 & 0.784 & -0.18 & 1 \end{bmatrix} x^n + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} w_n$$

$$z_n = [1 \quad 0 \quad 0 \quad 0]^T\, x^n + v_n$$

$$E[w_n] = E[v_n] = 0$$

$$E[w_n^2] = \psi_w^2 = 1$$

$$E[v_n^2] = \psi_v^2 = 0.25$$

where v_n and w_n have Gaussian distributions.

A program is developped to simulate this numerical example with the following initial conditions.

$$x^1 = \hat{x}^1 = a^1 = 0$$

$$Q_1 = R_1 = I$$

The sequences λ_n and μ_n chosen are:

$$\lambda_n = \frac{1}{n+0.5} \quad \text{and} \quad \mu_n = \frac{1}{n+1}$$

Using only the observations z_n of the simulated system we obtained the results given in Table 1 and plotted in Figure 5. In this particular case the sequence of estimates a^n converges numerically to the a priori known real vector a.

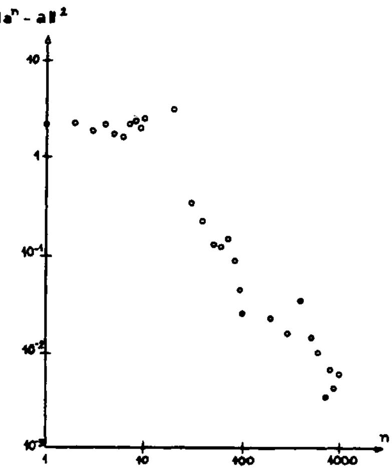

Figure 5 – Quadratic error obtained by simultaneous identification and estimation.

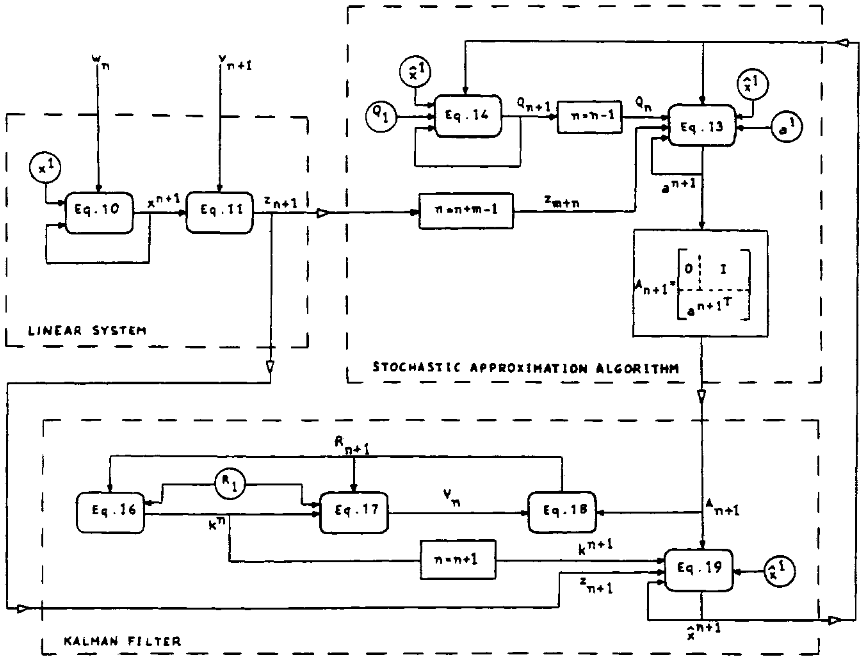

Figure 4 – Diagram of the simultaneous identification and state estimation

Table 1: Performance of simultaneous identification and estimation

Real Value: $a^T = [\,-0.656 \;\; 0.784 \;\; -0.18 \;\; 1\,]^T$					
n	$a^n = [a_1^n$	a_2^n	a_3^n	$a_4^n]^T$	$\lvert a^n - a\rvert^2$
100	–0.532	0.717	–0.260	1.026	2.7×10^{-2}
200	–0.543	0.781	–0.288	0.958	2.6×10^{-2}
300	–0.584	0.732	–0.282	1.015	1.9×10^{-2}
400	–0.498	0.743	–0.202	0.895	3.8×10^{-2}
500	–0.547	0.723	–0.216	0.997	1.7×10^{-2}
600	–0.637	0.741	–0.239	1.071	1.1×10^{-2}
700	–0.650	0.752	–0.200	1.048	3.7×10^{-3}
800	–0.654	0.797	–0.246	1.054	7.6×10^{-3}
900	–0.632	0.778	–0.236	1.030	4.7×10^{-3}
1000	–0.661	0.770	–0.233	1.075	8.7×10^{-3}

V – CONCLUSION

The relations to be satisfied for optimization of the accelerated convergence of the stochastic approximation algorithm are modified in the case of a stationary linear discrete-time system.

To avoid the exact measurement of the initial state at each step, a numerical tentative is made to use simultaneously the two recursive stochastic approximation and Kalman filter algorithms. Further research is under way to extend these results.

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