An Essay in the Time of Corona

Carlos Kubrusly

1 Is There?

One hundred and twenty two days ago, I was dining with a friend in a empty restaurant: a piece of grilled rump and chips with cream cheese on pita bread. So simple a meal, and yet so far way now. The world was already at the dawn of a new age of perplexity. After taking my friend to her place, I headed to mine into lockdown. This was on a Sunday, March 15 of the virulent year of 2020; it was just four months before but it seems four long decades ago. This is a COVID 19 pandemic scenario.

At that time I was thinking on revisiting tensor products, a subject on which I had written some papers a dozen years before in connection with transferring properties from a pair of Hilbert-space operators to their tensor product. It was quite a fashionable subject at the time. So perhaps this Corona social distance enforcement and consequent home imprisonment might be a chance to give it a try. As an aftermath came the next few lines addressed to a wide audience.

There is a conventional protocol to build up a tensor product of Hilbert spaces where a reasonable crossnorm comes nicely and naturally from the factors' inner products. There is, however, an intriguing point: why is it, and where does it come from? As George Pólya [3] taught us: *is there an easier question to ask*? So, are there other ways to construct those tensor product spaces? Are they somehow equivalent, thus boiling down to the same thing? Yes, indeed. Here is the yellow brick road.

Mathematics Subjects Classification (2010): 46M05, 47A80, 15A63; Keywords Algebraic tensor product

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A. Wonders (ed.) *Math in the Time of Corona*, Mathematics Online First Collections,
https://doi.org/10.1007/16618_2020_10

2 Is This a Possible Start?

$$\begin{array}{ccc} \mathcal{X}\times\mathcal{Y} & \xrightarrow{\ \phi\ } & \mathcal{Z} \\ & {\scriptstyle\theta}\searrow & \uparrow{\scriptstyle\Phi} \\ & & \mathcal{T} \end{array}$$

These are all rooted on firm grounds, no abstract nonsense required: $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{T}$ are linear spaces (all over the same field), and a pair $(\mathcal{T}, \theta)$ is a tensor product of $\mathcal{X}$ and $\mathcal{Y}$ if (a) θ is a bilinear map whose range spans $\mathcal{T}$, and (b) for every bilinear map ϕ into any $\mathcal{Z}$ there is a linear transformation Φ for which the above diagram commutes. These are the axioms of tensor product whose definition can be rewritten as "a tensor product space $\mathcal{T}$ of $\mathcal{X}$ and $\mathcal{Y}$ has the universal property with respect to a bilinear map θ on the Cartesian product $\mathcal{X} \times \mathcal{Y}$" , and so θ factors every bilinear map ϕ through $\mathcal{T}$ thus "linearising ϕ by Φ". Enough is enough.

It is a rather clean start and it seems impressive to me how much follows from these axioms. Basically all properties of concrete tensor products (yes, they do exist) follow from such an abstract formulation. First of all if $(\mathcal{T}, \theta)$ and $(\mathcal{T}', \theta')$ are tensor products of $\mathcal{X}$ and $\mathcal{Y}$, then they are essentially (i.e., up to isomorphism) the same, and so it is usual to write $\mathcal{X} \otimes \mathcal{Y}$ for "the" tensor product space $\mathcal{T}$ of $\mathcal{X}$ and $\mathcal{Y}$. Also, the linear space $b[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$ of all bilinear maps ϕ from the Cartesian product $\mathcal{X} \times \mathcal{Y}$ to an arbitrary linear space $\mathcal{Z}$ is essentially equal (i.e., isomorphic) to the linear space $\mathcal{L}[\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z}]$ of all linear transformations Φ from the tensor product space $\mathcal{X} \otimes \mathcal{Y}$ to $\mathcal{Z}$, thus showing how and why tensor products linearise bilinear maps; and many more properties follow from the axioms including, of course, the dimension identity: $\dim \mathcal{X} \otimes \mathcal{Y} = \dim \mathcal{X} \cdot \dim \mathcal{Y}$.

3 Plus ça Change, Plus c'est la même Chose

It was said above that concrete tensor products do exist. In fact, the most common interpretations of tensor products are the so-called quotient space and linear maps of bilinear maps realisations. The former states that a quotient space $\mathcal{S}/\mathcal{M}$ can be made into a tensor product space of $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{S}$ is the free linear space generated by the Cartesian product $\mathcal{X} \times \mathcal{Y}$ of linear spaces $\mathcal{X}$ and $\mathcal{Y}$ (which is not a linear space itself) and $\mathcal{M}$ is an appropriate linear manifold of $\mathcal{S}$. The latter in its simplest form goes as follows.

Suppose all linear spaces are complex (i.e., over the field $\mathbb{C}$) and let ψ: $\mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ be an arbitrary bilinear map (in this case it is called a bilinear form). Associated to each pair $(x, y) \in \mathcal{X} \times \mathcal{Y}$ consider a functional $x \otimes y$ defined by $(x \otimes y)(\psi) = \psi(x, y)$ for every bilinear form ψ. This is called a single tensor which is itself a linear functional (i.e., a linear form) on the linear space of bilinear forms, that is, $x \otimes y$: $b[\mathcal{X} \times \mathcal{Y}, \mathbb{C}] \to \mathbb{C}$ is linear; a linear map of bilinear maps. Consider the linear space $\mathcal{T}$ spanned by the collection of all single tensors, and define a bilinear map $\theta\colon \mathcal{X}\times\mathcal{Y}\to\mathcal{T}$ by setting $\theta(x,y)=x\otimes y$ for every $(x,y)\in\mathcal{X}\times\mathcal{Y}$. This supplies a tensor product $(\mathcal{T},\theta)$ of $\mathcal{X}$ and $\mathcal{Y}$. Now constrain the bilinear forms ψ to products of linear forms, say, $\psi(x,\,y)=\mu(x)\cdot\nu(y)$. Then to each $(x,y)\in\mathcal{X}\times\mathcal{Y}$ associate a single tensor given by $(x\otimes y)(\mu,\,\nu)=(x\otimes y)(\psi)=\psi(x,\,y)=\mu(x)\cdot\nu(y)$ for every pair of linear forms $\mu\colon\mathcal{X}\to\mathbb{C}$ and $\nu\colon\mathcal{Y}\to\mathbb{C}$. A tensor product is still obtained in this particular case. Thus write $\mathcal{X}\otimes\mathcal{Y}$ for the tensor product space $\mathcal{T}$ of $\mathcal{X}$ and $\mathcal{Y}$ spanned by the collection of all new single tensors as above.

As for the good old Hilbert-space case equip $\mathcal{X}$ and $\mathcal{Y}$ with inner products $\langle\,\cdot\,;\,\cdot\,\rangle_{\mathcal{X}}$ and $\langle\,\cdot\,;\,\cdot\,\rangle_{\mathcal{Y}}$. In this context, linear forms μ on $\mathcal{X}$ and ν on $\mathcal{Y}$ are identified with vectors u in $\mathcal{X}$ and v in $\mathcal{Y}$ by $\mu(\cdot)=\langle\,\cdot\,;u\rangle_{\mathcal{X}}$ for some $u\in\mathcal{X}$ and $\nu(\cdot)=\langle\,\cdot\,;v\rangle_{\mathcal{Y}}$ for some $v\in\mathcal{Y}$ according to the Riesz Representation Theorem in Hilbert space. Hence

$$(x\otimes y)(u,v)=\langle x\,;u\rangle_{\mathcal{X}}\langle y\,;v\rangle_{\mathcal{Y}}\quad\text{for every}\quad u\in\mathcal{X}\;\text{ and }\;v\in\mathcal{Y},$$

which is the conventional protocol to build a single tensor $x\otimes y$ in a Hilbert-space setting, and therefore is the initial step to build up a tensor product space $\mathcal{X}\otimes\mathcal{Y}$ of Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$. The more you change it the more it is the same thing.

4 There Is

The previous section title was originally coined by the French writer Jean-Baptiste Alphonse Karr [2]. But here it was stolen from Somerset Maugham's novel *Then and Now* [4]. As a matter of fact, the title is his entire Chapter 1. The plot is about a short period in the life of Niccolò Machiavelli in sixteen century Italy which had been written towards the end of Maugham's career. However, most of Maugham's novels have been written around the 1920s by the time of the American Pandemic [1] (or have they called it "little flu" as well? — then and now). If Machiavelli himself was alive and kicking during this time of Corona, then perhaps we would be safer than we are now on this side of the pond. Wanderings in the time of Corona cannot be confined to mathematics and virtual lectures (otherwise one most certainly would not be writing this but tied up in a Renaissance lunatic asylum). So rereading old books has become part of a new normal. Since a huge and sunny beach is just around the corner, perhaps rereading old books can share space with a contemplated new revival. I do not know that I will avoid a stroll by seaside before long.

References

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