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ON THE EXISTENCE, EVOLUTION, AND STABILITY OF INFINITE-DIMENSIONAL STOCHASTIC DISCRETE BILINEAR MODELS*

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Abstract. Sufficient conditions for mean-square stability of infinite-dimensional discrete bilinear models driven by Hilbert-space-valued random sequences are given in this paper. It is shown that the class of models under consideration can be properly defined as the uniform limit of finite-dimensional bilinear models. The stochastic stability problem is approached by analysing the evolution and the asymptotic behaviour of the state expectation and correlation sequences for such a limiting model.

Key Words—Asymptotic stability, bilinear dynamical systems, discrete-time systems, infinite-dimensional systems, stochastic environment.

1. Introduction

Bilinear systems, which hold several significant structural properties, have been in evidence for the past decade. Motivations for considering such a special class of nonlinear dynamical systems operating in a stochastic environment have been extensively discussed in the available literature (e.g. see the references in Kubrusly, 1986). The major part of it is related to continuous-time evolution and finite-dimensional models. However, on the one hand, some effort has already been made towards infinite-dimensional continuous-time bilinear models (e.g. see Zabczyk, 1979) and, on the other hand, many real systems are naturally described by discrete-time bilinear models (e.g. see Goka, Tarn and Zaborsky, 1973, and the references therein). Therefore, it seems opportune to attempt to an investigation of infinite-dimensional discrete-time bilinear models. These can be thought of either as an extension of finite-dimensional discrete-time models, or as a discrete version of continuous-time infinite-dimensional models (resulting, for instance, from usual discretization procedures).

Here we shall be focusing on the discrete evolution and asymptotic behaviour of bilinear systems operating in a stochastic environment, whose model is formally given by the following difference equation.

$$x_{i+1} = [A_0 + \sum_{k\geq 1} A_k < w_i; e_k >]x_i + u_i,$$

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where $\{A_k;\ k\geq 0\}$ is a sequence of bounded linear operators on some separable Hilbert space H, $\{e_k;\ k\geq 1\}$ is an orthonormal basis for H, and $\{u_i;\ i\geq 0\}$, $\{w_i;\ i\geq 0\}$ and $\{x_i;\ i\geq 0\}$ are H-valued random sequences. If H is finite-dimensional, then such a model characterizes precisely a finite-dimensional discrete bilinear system, since the above series is finite. For such a finite-dimensional case, the evolution of the state moments is easy to obtain under independence assumptions (e.g. see Kubrusly, 1986). This supplies a suitable framework to investigate mean-square stability, by analysing the asymptotic properties of the state moments (e.g. see Kubrusly, 1986; Kubrusly and Costa, 1985, and the references therein). On the other hand, if we set $A_k=0$ for every $k\geq 1$, then the above model is naturally reduced to a linear one. In this case the evolution of the state moments are easily obtained, even when H is infinite-dimensional, which enable us (at least in principle) to investigate their asymptotic behaviour. Actually, mean-square stability for infinite-dimensional discrete linear systems has already been properly addressed in the current literature (e.g. see Kubrusly, 1985; 1987; Zabczyk, 1975).

The main goal of the present paper is to supply sufficient conditions for mean-square stability for the general case, where H is infinite-dimensional and $\{A_k;\ k\geq 0\}$ is any uniformly bounded sequence. This will be achieved in Sec. 5. The background material that will be required for supporting the stability results is presented in Secs. 2 to 4. Notational preliminaries are given in Sec. 2. In Sec. 3 it will be shown that the infinite-dimensional model under consideration can be rigorously defined as the uniform limit of finite-dimensional models on the Hilbert space $\mathscr{H}$ of all second-order H-valued random variables. Section 4 deals with the evolution of the state expectation and correlation sequences, whose asymptotic properties will be analysed in Sec. 5.

2. Preliminaries

In this section we pose the notation and basic results that will be needed in the sequel. Throughout this paper we assume that H is a separable nontrivial Hilbert space, with $\|\ \|$ and $<;>$ standing for norm and inner product in H, respectively.

Nuclear operators. Let $B[X,Y]$ denote the Banach space of all bounded linear transformations of a Banach space X into a Banach space Y, and set $B[X]=B[X,X]$. We shall use the same symbol $\|\ \|$ to denote the uniform induced norm in $B[X,Y]$. Let $C^*\in B[H]$ be the adjoint of $C\in B[H]$ and set $|C|=(C^*C)^{1/2}\in B[H]^+$, where $B[H]^+=\{C\in B[H]:\ 0\leq C=C^*\}=\{C\in B[H]:\ C=|C|\}$ is the closed convex cone of all self-adjoint nonnegative operators on H. The class of all compact operators from $B[H]$ will be denoted by $B_\infty[H]$. If $C\in B_\infty[H]$ (or equivalently, $|C|\in B_\infty[H]$), let $\{\lambda_k\geq 0;\ k\geq 1\}$ be the nonincreasing nonnegative null sequence made up of all singular values of C (i.e. eigenvalues of $|C|$), each nonzero one counted according to its multiplicity as an eigenvalue of $|C|$. Set $\|C\|_1=\sum_{k=1}^\infty\lambda_k$ and let $B_1[H]=\{C\in B_\infty[H]:\ \|C\|_1<\infty\}$ be the class of all nuclear operators on H. $\|\ \|_1$ is a norm in $B_1[H]$, $\|C\|\leq\|C\|_1$ for every $C\in B_1[H]$, and $(B_1[H],\ \|\ \|_1)$ is a Banach space. Set $B_1[H]^+=B_1[H]\cap B[H]^+$. For any $f,g\in H$, define the outer product operator $(f\circ g)\in B_1[H]$ as follows: $(f\circ g)h=<h;g>f$ for all $h\in H$, so that $(f\circ f)\in B_1[H]^+$. The above standard concepts may be found in Gohberg and Krein (1969), Schatten (1970) and Weidmann (1980).

H-valued random variables. Let $\mathcal{H}$ denote the Hilbert space of all second-order H-valued random variables, whose inner product $<;>_{\mathcal{H}}$ is given by $<x;y>_{\mathcal{H}}=\varepsilon\{<x;y>\}$ for every $x,y\in\mathcal{H}$, where ε stands for expectation of scalar-valued random variables. Let $\|\ \|_{\mathcal{H}}$ denote the norm induced by $<;>_{\mathcal{H}}$, so that $\|x\|_{\mathcal{H}}^2=\varepsilon\{\|x\|^2\}<\infty$ for every $x\in\mathcal{H}$. As usual (cf. Kubrusly, 1987), the expectation $E\{x\}\in H$ and the correlation operator $\mathcal{E}\{x\circ y\}\in B_1[H]$ are (uniquely) defined for every $x,y\in\mathcal{H}$ by the formulas: $<E\{x\};g>=\varepsilon\{<x;g>\}$ and $<\mathcal{E}\{x\circ y\}f;g>=\varepsilon\{<f;y><x;g>\}$ for every $f,g\in H$. A random sequence $\{x_i\in\mathcal{H};\ i\geq 0\}$ is correlation stationary if there exists $Q\in B_1[H]^+$ such that $\mathcal{E}\{x_i\circ x_1\}=Q$ for every $i\geq 0$.

Remark 1: Recall that $E\{\ \}: \mathcal{H}\to H$ is a linear map, $E\{Cx\}=CE\{x\}$, and $\mathcal{E}\{x\circ x\}\in B_1[H]^+$, for every $x\in\mathcal{H}$ and $C\in B[H]$. The following basic results, which are readily verified, will also be needed in the sequel. For any $u,v,x,y\in\mathcal{H}$ and $C,D\in B[H]$,

(a) $$\mathcal{E}\{(u+v)\circ(x+y)\} = \mathcal{E}\{u\circ x\} + \mathcal{E}\{u\circ y\} + \mathcal{E}\{v\circ x\} + \mathcal{E}\{v\circ y\},$$
$$C\mathcal{E}\{x\circ y\}D^* = \mathcal{E}\{Cx\circ Dy\} = \mathcal{E}\{Dy\circ Cx\}^*,$$

(b) $$\|\mathcal{E}\{x\circ y\}\|_1 \leq \|x\|_{\mathcal{H}}\|y\|_{\mathcal{H}},\quad \|E\{x\}\|^2 \leq \|x\|_{\mathcal{H}}^2=\|\mathcal{E}\{x\circ x\}\|_1,$$
$$\|Cx\|_{\mathcal{H}} \leq \|C\|\ \|x\|_{\mathcal{H}},$$

(c) $$\sum_{k,l=m}^{p} |<(E\{x\}\circ E\{x\})e_l;e_k>| \leq \sum_{k,l=m}^{p} |<(\mathcal{E}\{x\circ x\}e_l;e_k>|,$$

for any integers $1\leq m\leq p$, and for any orthonormal basis $\{e_k;\ k\geq 1\}$ for H. Moreover, if a given sequence $\{x_n\in\mathcal{H};\ n\geq 1\}$ converges in $(\mathcal{H},\ \|\ \|_{\mathcal{H}})$ to $x\in\mathcal{H}$, then

(d) $$\lim_{n\to\infty}\|E\{x_n\}-E\{x\}\| = \lim_{n\to\infty}\|\mathcal{E}\{x_n\circ x_n\}-\mathcal{E}\{x\circ x\}\|_1$$
$$= \lim_{n\to\infty}\|\mathcal{E}\{x_n\circ y\}-\mathcal{E}\{x\circ y\}\|_1 = 0,\quad \forall y\in\mathcal{H}.$$

Remark 2: For any family $\{x_\xi\in\mathcal{H};\ \xi\in\Xi\neq\emptyset\}$ set $\mathcal{I}_{\{x_\xi;\xi\in\Xi\}}=\{y\in\mathcal{H}:\ y$ is independent of $\{x_\xi\in\mathcal{H};\ \xi\in\Xi\neq\emptyset\}\}$. In particular, $\mathcal{I}_x=\{y\in\mathcal{H}:\ y$ is independent $x\in\mathcal{H}\}$. The following well-known independence properties (e.g. see Neveu, 1965) will be required in the next sections.

(a) If $x\in\mathcal{I}_y$ then, for every measurable functionals $\phi,\psi:H\to\mathbb{C}$,

$$\varepsilon\{\phi(x)\ \psi(y)\} = \varepsilon\{\phi(x)\}\ \varepsilon\{\psi(y)\}.$$

(b) If $\{y_v\in\mathcal{H};\ v\in\Upsilon\neq\emptyset\}$ is independent of $\{x_\xi\in\mathcal{H};\ \xi\in\Xi\neq\emptyset\}$ then, for any finite subset $\{x_{\xi_k};\ 1\leq k\leq m\}$ of $\{x_\xi;\ \xi\in\Xi\}$, and for every measurable map $N:H^m\to H$,

$$N(x_{\xi_1},\ \cdots,\ x_{\xi_m}) \in \mathcal{I}_{\{y_v;\ v\in\Upsilon\}}.$$

3. Model description

The purpose of this section is to properly pose the infinite-dimensional bilinear model that has been formally introduced in Sec. 1. This will be achieved in Lemma 1. We begin with an auxiliary result that will suffice our needs.

Proposition 1. Let $w \in \mathscr{H}$. If $\{e_k;\ k\geq 1\}$ is the orthonormal basis for H made up of all eigenvectors of $\mathscr{E}\{w\circ w\}\in B_1[H]^+$, and $\{A_k\in B[H];\ k\geq 0\}$ is uniformly bounded, then

$$\{\mathscr{A}_w(n) \overset{\text{def.}}{=} A_0 + \sum_{k=1}^{n} A_k<w;e_k>:\ \ \mathscr{I}_w\to\mathscr{H};\ \ n\geq 1\}$$

converges uniformly. Equivalently, there exists a map $\mathscr{A}_w$: $\mathscr{I}_w\to\mathscr{H}$ such that

$$\sup_{0\neq v\in\mathscr{I}_w} \|\mathscr{A}_w(n)v-\mathscr{A}_w v\|_{\mathscr{X}}/\|v\|_{\mathscr{X}} \to 0 \quad \text{as} \quad n\to\infty,$$

which is bounded, homogeneous and additive. That is: $\sup_{0\neq v\in\mathscr{I}_w}\|\mathscr{A}_w v\|_{\mathscr{X}}/\|v\|_{\mathscr{X}}<\infty$, $\mathscr{A}_w \alpha v=\alpha\mathscr{A}_w v$ for every $\alpha\in\mathbb{C}$ and $v\in\mathscr{I}_w$, and for each $i\geq 1$

$$\mathscr{A}_w\left(\sum_{j=0}^{i} v_j\right) = \sum_{j=0}^{i}\mathscr{A}_w v_j,$$

whenever $v_j\in\mathscr{I}_w$ for every $j=0,\ 1,\ \cdots,\ i$ and $(\sum_{j=0}^{k} v_j)\in\mathscr{I}_w$ for every $k=1,\ \cdots,\ i$.

Proof. From Remark 2(a), we have

$$<A_k<w;e_k>v;\ A_l<w;e_l>v>_{\mathscr{X}} = <\mathscr{E}\{w\circ w\}e_l;e_k>\ <A_k v;A_l v>_{\mathscr{X}}$$

for every $k,l\geq 1$, whenever $v\in\mathscr{I}_w$. Hence, with $\lambda_k\geq 0$ standing for the eigenvalue of $\mathscr{E}\{w\circ w\}$ associated with the eigenvector e_k for each $k\geq 1$, we get

$$\begin{aligned}\|\sum_{k=m}^{p} A_k<w;e_k>v\|_{\mathscr{X}}^2 &= \sum_{k,l=m}^{p} <A_k<w;e_k>v;\ A_l<w;e_l>v>_{\mathscr{X}}\\ &\leq \sup_{m\leq k\leq p}\|A_k\|^2\|v\|_{\mathscr{X}}^2\sum_{k=m}^{p}\lambda_k\end{aligned}$$

for all $v\in\mathscr{I}_w$ and for any $1\leq m\leq p$, according to Remark 1(b). Therefore,

$$\begin{aligned}&\sup_{\nu\geq 1}\sup_{0\neq v\in\mathscr{I}_w} \|\mathscr{A}_w(n+\nu)v-\mathscr{A}_w(n)v\|_{\mathscr{X}}/\|v\|_{\mathscr{X}}\\ &\quad\leq \sup_{k\geq 1}\|A_k\|\left(\sum_{k=n+1}^{\infty}\lambda_k\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

which implies uniform convergence. Note that the limiting map $\mathscr{A}_w$ fails to be a bounded linear one just because its domain $\mathscr{I}_w$ is not a linear subspace of $\mathscr{H}$. Actually, boundedness is straightforward and homogeneity is trivial. The additivity property holds for $i=1$ because $\{\mathscr{A}_w(n);\ n\geq 1\}$ converges (strongly on $\mathscr{I}_w$) to $\mathscr{A}_w$. Since it holds for $i=1$ it is a simple matter to conclude the proof by induction.

Lemma 1. Let $\{w_i\in\mathscr{H};\ i\geq 0\}$ be a correlation stationary sequence, let $\{e_k;\ k\geq 1\}$ be the orthonormal basis for H made up of all eigenvectors of $\mathscr{E}\{w_i\circ w_i\}\in B_1[H]^+$, and let $\{A_k\in B[H];\ k\geq 0\}$ be uniformly bounded. Set, for every $i \geq 0$,

$$\mathscr{A}_{w_i} = A_0 + \sum_{k=1}^{\infty} A_k < w_i; e_k >: \mathscr{I}_{w_i} \to \mathscr{H}$$

according to Proposition 1. Given $x_0 \in \mathscr{H}$ and $\{u_i \in \mathscr{H};\ i \geq 0\}$, assume that $w_0 \in \mathscr{I}_{x_0}$ and $w_j \in \mathscr{I}_{\{x_0, u_0, \cdots, u_{j-1}, w_0, \cdots, w_{j-1}\}}$ for every $j \geq 1$. Then the difference equation in $\mathscr{H}$

$$x_{i+1} = [A_0 + \sum_{k=1}^{\infty} A_k < w_i; e_k >] x_i + u_i$$

has a unique solution, which lies in $\mathscr{I}_{w_i}$ for every $i \geq 0$. The solution is given by

$$x_1 = A_{w_0} x_0 + u_0$$

and

$$x_i = \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_0} x_0 + \sum_{j=1}^{i-1} \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} u_{j-1} + u_{i-1} \text{ for every } i \geq 2.$$

Proof. Set $v_0 = x_0$ and $v_{j+1} = u_j$ for every $j \geq 0$. Under the above independence conditions we get, for every $i \geq 1$,

(a) $$\sum_{j=0}^{i-1} \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} v_j + v_i \in \mathscr{I}_{w_i},$$

(b) $$\mathscr{A}_{w_i} [\sum_{j=0}^{i-1} \mathscr{A}_{w_{i-1}} \cdots \mathscr{A}_{w_j} v_j + v_i] = \sum_{j=0}^{i} \mathscr{A}_{w_i} \cdots \mathscr{A}_{w_j} v_j.$$

This can be readily verified by induction, according to the additivity property for $\mathscr{A}_{w_i}$ in Proposition 1, with the help of Remark 2(b). Now it is a simple matter to verify that the sequence in (a) is the unique solution of $x_{i+1} = \mathscr{A}_{w_i} x_i + v_i$ with $x_0 = v_0$, according to (b).

4. Expectation and correlation evolution

Consider the infinite-dimensional bilinear model defined in Lemma 1. In this section we shall investigate the expectation and correlation evolution for the state sequence $\{x_i \in \mathscr{I}_{w_i};\ i \geq 0\}$ generated by such a model. This will be developed in Lemma 2 below. We begin now with two auxiliary propositions.

Proposition 2. Let $\{e_k;\ k \geq 1\}$ be any orthonormal basis for H, and let $\{A_k \in B[H];\ k \geq 0\}$ be an arbitrary sequence. For a given $w \in \mathscr{H}$ and each $n \geq 1$ set $M_w(n), F_w(n) \in B[H]$, $K_w(n), L_w(n), T_w(n) \in B[B[H]]$, and $P_w^u(n) \in B[H, B_1[H]]$ for any $u \in \mathscr{H}$, as follows;

$$M_w(n) = \sum_{k=1}^{n} A_k < E\{w\};\ e_k >, \qquad F_w(n) = A_0 + M_w(n),$$

$$K_w(n)[Q] = \sum_{k,l=1}^{n} < (E\{w\} \circ E\{w\}) e_l; e_k > A_k Q A_l^*, \qquad \forall Q \in B[H],$$

$$L_w(n)[Q] = \sum_{k,l=1}^{n} < \mathscr{E}\{w \circ w\} e_l; e_k > A_k Q A_l^*, \qquad \forall Q \in B[H],$$

$$T_w(n)[Q] = \sum_{k,l=1}^{n} < (\mathscr{E}\{w \circ w\} - E\{w\} \circ E\{w\}) e_l; e_k > A_k Q A_l^*, \quad \forall Q \in B[H],$$

$$P_w^u(n)[h] = A_0 h \circ E\{u\} + \sum_{k=1}^{n} A_k h \circ \mathscr{E}\{u \circ w\} e_k, \qquad \forall h \in H .$$

Let $\{\mathscr{A}_w(n): \mathscr{I}_w \to \mathscr{H};\ n \geq 1\}$ be given as in Proposition 1. Then, for every $n \geq 1$,

(a) $K_w(n)[Q] = M_w(n) Q M_w(n)^*, \qquad \forall Q \in B[H],$

(b) $T_w(n)[Q] = L_w(n)[Q] - K_w(n)[Q], \qquad \forall Q \in B[H],$

(c) $0 \leq K_w(n)[\,|Q|\,] \leq L_w(n)[\,|Q|\,], \qquad \forall Q \in B[H],$

(d) $\|K_w(n)\| = \|M_w(n)\|^2 \leq \|L_w(n)\|$,

(e) $E\{\mathscr{A}_w(n) v\} = F_w(n) E\{v\}, \qquad \forall v \in \mathscr{I}_w$,

(f) $\mathscr{E}\{\mathscr{A}_w(n) v \circ \mathscr{A}_w(n) v\}$
$= F_w(n) \mathscr{E}\{v \circ v\} F_w(n)^* + T_w(n)[\mathscr{E}\{v \circ v\}], \qquad \forall v \in \mathscr{I}_w$,

(g) $\mathscr{E}\{\mathscr{A}_w(n) v \circ u\} = P_w^u(n)[E\{v\}]. \qquad \forall v \in \mathscr{I}_{\{u,\, w\}}$.

Proof. The results in (a) and (b) are trivial, and (c) is equivalent to

(c′) $0 \leq T_w(n)[\,|Q|\,] \leq L_w(n)[\,|Q|\,], \qquad \forall Q \in B[H],$

according to (b). The first inequality in (c) is immediate from (a). Since

$$\begin{aligned}
&<T_w(n)[\,|Q|\,]h; h> \\
&= \sum_{k,l=1}^{n} \varepsilon\{<e_l; w - E\{w\}> <w - E\{w\};\ e_k>\} <|Q|^{\frac{1}{2}} A_l^* h;\ |Q|^{\frac{1}{2}} A_k^* h> \\
&= \sum_{k,l=1} <|Q|^{\frac{1}{2}} <e_l; w - E\{w\}> A_l^* h;\ |Q|^{\frac{1}{2}} <e_k; w - E\{w\}> A_k^* h>_{\mathscr{H}} \\
&= \|\ |Q|^{\frac{1}{2}} \Big(\sum_{k=1}^{n} <w - E\{w\};\ e_k> A_k \Big)^* h \|_{\mathscr{H}}^2, \quad \forall Q \in B[H], \quad \forall h \in H,
\end{aligned}$$

for every $n \geq 1$, it follows the first inequality in (c′) or, equivalently, the second one in (c). This implies the inequality in (d). Actually, for every $n \geq 1$,

$$\begin{aligned}
\|K_w(n)\| &= \sup_{\|Q\|=1} \|M_w(n) Q M_w(n)^*\| = \|M_w(n)\|^2 \\
&= \|K_w(n)[I]\| \leq \|L_w(n)[I]\| \leq \|L_w(n)\|.
\end{aligned}$$

The results in (e), (f), and (g) are readily verified by Remarks 1(a) and 2 (a, b).

Remark 3: Note that, for every $n \geq 1$,

(a) $\|T_w(n)[\,|Q|\,]\|_1 \leq \|T_w(n)\|\ \|Q\|_1, \qquad \forall Q \in B_1[H],$

(b) $P_w^u(n)[h] = \mathscr{E}\{\mathscr{A}_w(n) h \circ u\}, \qquad \forall h \in H.$

Moreover, if $\{e_k;\ k \geq 1\}$ is made up of all eigenvectors of $\mathscr{E}\{w \circ w\} \in B_1[H]^+$, then

(c) $$\|P_w^u(n)[h]\|_1 \leq (\|A_0\| + \sup_{k\geq 1}\|A_k\| \, \|w\|_{\mathscr{X}}) \, \|u\|_{\mathscr{X}}\|h\|, \qquad \forall h \in H,$$

(d) $$L_w(n)[Q] = \sum_{k=1}^{n} \lambda_k A_k Q A_k^*, \qquad \forall Q \in B[H],$$

with $\lambda_k \geq 0$ standing for the eigenvalue of $\mathscr{E}\{w \circ w\}$ associated with e_k for each $k \geq 1$.

Proposition 3. The sequences defined in Proposition 2 converge uniformly whenever $\sup_{k\geq 0}\|A_k\| < \infty$ and $\{e_k;\ k \geq 1\}$ is made up of all eigenvectors of $\mathscr{E}\{w \circ w\} \in B_1[H]^+$.

Proof. Since $\sum_{k=1}^{\infty}\lambda_k = \|\mathscr{E}\{w \circ w\}\|_1 = \|w\|_{\mathscr{X}}^2 < \infty$, we get by Remark 3(d)

$$\sup_{\nu\geq 1}\|L_w(n+\nu) - L_w(n)\| \leq \sup_{k\geq 1}\|A_k\|^2 \sum_{k=1}^{\infty} \lambda_k \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\{L_w(n);\ n \geq 1\}$ converges uniformly. Now recall that $\|f \circ g\| = \|f\| \, \|g\|$ and $Cf \circ Dg = C(f \circ g)D^*$ for any $f, g \in H$ and $C, D \in B[H]$. Therefore, by Remark 1(c),

$$\begin{aligned}
\|[M_w(n+\nu) - M_w(n)]h\|^2 &= \|[M_w(n+\nu) - M_w(n)](h \circ h)[M_w(n+\nu) - M_w(n)]^*\| \\
&= \Big\|\sum_{k,l=n+1}^{n+\nu} <(E\{w\} \circ E\{w\})e_l; e_k> A_k(h \circ h)A_l^*\Big\| \\
&\leq \sup_{k\geq 1}\|A_k\|^2\|h\|^2 \sum_{k,l=n+1}^{n+\nu} |<\mathscr{E}\{w \circ w\}e_l; e_k>|
\end{aligned}$$

for all $h \in H$ and every $n, \nu \geq 1$, so that

$$\sup_{\nu\geq 1}\|M_w(n+\nu) - M_w(n)\| \leq \sup_{k\geq 1}\|A_k\| \left(\sum_{k=n+1}^{\infty} \lambda_k\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $\{M_w(n);\ n \geq 1\}$ and $\{F_w(n);\ n \geq 1\}$ converge uniformly. Then, by Proposition 2(a, b), $\{K_w(n);\ n \geq 1\}$ and $\{T_w(n);\ n \geq 1\}$ also converge uniformly. By Remarks 1(a, b) and 3(b)

$$\|P_w^u(n+\nu)[h] - P_w^u(n)[h]\|_1 \leq \|\mathscr{A}_w(n+\nu)h - \mathscr{A}_w(n)h\|_{\mathscr{X}}\|u\|_{\mathscr{X}}, \ \forall h \in H$$

for every n, $\nu \geq 1$. Hence $\{P_w{}^u(n);\ n \geq 1\}$ converges uniformly according to Proposition 1.

Lemma 2. Under the assumptions of Lemma 1 the state expectation $\{E\{x_i\} \in H;\ i \geq 0\}$ and correlation $\{\mathscr{E}\{x_i \circ x_i\} \in B_1[H]^+;\ i \geq 0\}$ sequences evolve as follows;

(a) $$E\{x_{i+1}\} = F_{w_i}E\{x_i\} + E\{u_i\}$$

(b) $$\begin{aligned}\mathscr{E}\{x_{i+1} \circ x_{i+1}\} &= F_{w_i}\mathscr{E}\{x_i \circ x_i\}F_{w_i}^* + T_{w_i}[\mathscr{E}\{x_i \circ x_i\}] \\ &\quad + \mathscr{E}\{\mathscr{A}_{w_i}x_i \circ u_i\} + \mathscr{E}\{\mathscr{A}_{w_i}x_i \circ u_i\}^* + \mathscr{E}\{u_i \circ u_i\},\end{aligned}$$

where, for each $i \geq 0$, $F_{w_i} \in B[H]$ and $T_{w_i} \in B[B[H]]$ are the uniform limits of $\{F_{w_i}(n);\ n \geq 1\}$ and $\{T_{w_i}(n);\ n \geq 1\}$, according to Proposition 3. Moreover, if

$x_0 \in \mathscr{I}_{\{u_0,\, w_0\}}$ and

$$\{u_j,\ w_j\} \text{ is independent of } \{x_0,\ u_0,\ \cdots,\ u_{j-1},\ w_0,\ \cdots,\ w_{j-1}\}$$

for every $j \geq 1$, which implies the former independence assumption, then

(c) $$\mathscr{E}\{x_{i+1} \circ x_{i+1}\} = F_{w_i} \mathscr{E}\{x_i \circ x_i\} F_{w_i}^{*} + T_{w_i}[\mathscr{E}\{x_i \circ x_i\}] + P_{w_i}^{u_i}[E\{x_i\}] + P_{w_i}^{u_i}[E\{x_i\}]^{*} + \mathscr{E}\{u_i \circ u_i\}$$

with $P_{w_i}^{u_i} \in B[H, B_1[H]]$ denoting the uniform limit of $\{P_{w_i}^{u_i}(n);\ n \geq 1\}$ for each $i \geq 0$, according to Proposition 3.

Proof. Let $\mathscr{A}_w$, F_w, T_w, P_w^u be the limits of $\{\mathscr{A}_w(n);\ n \geq 1\}$, $\{F_w(n);\ n \geq 1\}$, $\{T_w(n);\ n \geq 1\}$, $\{P_w^u(n);\ n \geq 1\}$, according to Propositions 1 and 3. The following properties are readily verified by Proposition 2(e, f, g) and Remark 1(d): $E\{\mathscr{A}_w v\} = F_w E\{v\}$ and $\mathscr{E}\{\mathscr{A}_w v \circ \mathscr{A}_w v\} = F_w \mathscr{E}\{v \circ v\} F_w^{*} + T_w[\mathscr{E}\{v \circ v\}]$ $\forall v \in I_w$, and $\mathscr{E}\{A_w v \circ u\} = P_w^u[E\{v\}]$ $\forall v \in \mathscr{I}_{\{u,\, w\}}$. Hence we get (a) and (b) by Remark 1(a), since $x_i \in \mathscr{I}_{w_i}$ $\forall i \geq 0$. Moreover, if the above independence assumption holds then, by Lemma 1 and Remark 2(b), $x_i \in \mathscr{I}_{\{u_i,\, w_i\}}$ $\forall i \geq 0$. Thus we get (c).

5. Mean-square stability

Consider an infinite-dimensional discrete bilinear model operating in a stochastic environment as defined in Lemma 1. In this section we shall be interested in the asymptotic behaviour of the state expectation and correlation sequences, whose evolution was given in Lemma 2. In particular, we shall investigate sufficient conditions on the maps $\{\mathscr{A}_{w_i}\}$ to ensure that those sequences converge for any admissible initial condition x_0 and input disturbance $\{u_i\}$, and their limits do not depend on x_0. This will be carried out in Theorems 1 and 2 below. First we consider the following auxiliary result, which involves the concept of Cauchy summable sequence: a Cauchy sequence $\{y_i \in Y;\ i \geq 0\}$ in a normed linear space Y is *Cauchy summable* if and only if

$$\sum_{i=0}^{\infty} \sup_{i \geq 0} \|y_{i+v} - y_i\| < \infty.$$

Proposition 4. Let $\Lambda \in B[X]$ be uniformly asymptotically stable and $\theta: X \to X$ be a proper contraction on a Banach space X, such that

$$\|\Lambda^i\| \leq \sigma \alpha^i \quad \forall i \geq 0, \quad \|\theta x - \theta y\| \leq \varrho \|x - y\| \quad \forall x, y \in X, \quad \alpha + \sigma\varrho < 1,$$

for some real constants $\sigma \geq 1$, $0 < \alpha < 1$, $0 \leq \varrho < 1$. If $\{v_i \in X;\ i \geq 0\}$ is a Cauchy summable sequence then $\{z_i \in X;\ i \geq 0\}$, given by

$$z_{i+1} = \Lambda z_i + \theta z_i + v_i \quad z_0 \in X \text{ arbitrary},$$

is also Cauchy summable, and its limit $z \in X$ does not depend on $z_0 \in X$.

Proof. See Lemma (L–1) in Kubrusly (1986).

Assumption 1. In order to reach a proper balance between generality of results and simplicity of analysis we make the following assumptions. Let $\{w_i \in \mathscr{H};\ i \geq 0\}$ be an expectation and correlation stationary sequence, and set

$$s = E\{w_i\} \in H, \qquad S = \mathscr{E}\{w_i \circ w_i\} \in B_1[H]^+,$$

for every $i \geq 0$. Let $\{e_i;\ k \geq 1\}$ be the orthonormal basis for H made up of all eigenvectors of $S \in B_1[H]^+$, and let $\{A_k \in B[H];\ k \geq 0\}$ be an arbitrary uniformly bounded sequence. Given $x_0 \in \mathscr{H}$ and $\{u_i \in \mathscr{H};\ i \geq 0\}$, assume that $x_0 \in \mathscr{I}_{\{u_0, w_0\}}$ and

$$\{u_j,\ w_j\} \text{ is independent of } \{x_0,\ u_0,\ \cdots,\ u_{j-1},\ w_0,\ \cdots,\ w_{j-1}\}$$

for every $j \geq 1$. Set, for every $i \geq 0$,

$$r_i = E\{u_i\} \in H, \qquad R_i = \mathscr{E}\{u_i \circ u_i\} \in B_i[H]^+,$$
$$G_i = \mathscr{E}\{u_i \circ w_i\} \in B_1[H],$$

and suppose $G_i = \gamma_i G_0$ for some scalar sequence $\{\gamma_i \in \mathbb{C};\ i \geq 0\}$. Assume further that $\{\gamma_i \in \mathbb{C};\ i \geq 0\}$, $\{r_i \in H;\ i \geq 0\}$, and $\{R_i \in B_1[H]^+;\ i \geq 0\}$ are Cauchy summable sequences in $\mathbb{C}$, H, and $B[H]$, respectively; and $\sup_{i \geq 0} \|R_i\|_1 < \infty$.

Now, under Assumption 1 and according to Lemma 1, consider the state sequence $\{x_i \in \mathscr{I}_{w_i};\ i \geq 0\}$ generated by the difference equation

$$x_{i+1} = [A_0 + \sum_{k=1}^{\infty} A_k \langle w_i; e_k \rangle] x_i + u_i, \tag{1}$$

and set for every $i \geq 0$

$$q_i = E\{x_i\} \in H, \qquad Q_i = \mathscr{E}\{x_i \circ x_i\} \in B_i[H]^+.$$

By Lemma 2 it follows that, for every $i \geq 0$,

$$q_{i+1} = F q_i + r_i \tag{2}$$

$$Q_{i+1} = F Q_i F^* + T[Q_i] + V_i[q_i] \tag{3}$$

with $F, M \in B[H]$, $T \in B[B[H]]$, $\{P_i \in B[H, B_1[H]];\ i \geq 0\}$ and $\{V_i : H \to B_1[H];\ i \geq 0\}$ given by

$$F = A_0 + M, \qquad M = \sum_{k=1}^{\infty} A_k \langle s; e_k \rangle,$$

$$T[Q] = \sum_{k,l=1}^{\infty} \langle (S - s \circ s) e_l;\ e_k \rangle A_k Q A_l^*, \qquad \forall Q \in B[H],$$

$$P_i[h] = A_0 h \circ r_i + \sum_{k=1}^{\infty} A_k h \circ G_i e_k, \qquad \forall h \in H,$$

$$V_i[h] = P_i[h] + P_i[h]^* + R_i, \qquad \forall h \in H,$$

according to Propositions 2 and 3. Note that, if $A_0 = 0$ or $M = 0$, then

$$F Q F^* + T[Q] = \sum_{k=0}^{\infty} \lambda_k A_k Q A_k^*, \qquad \forall Q \in B[H], \tag{4}$$

where $\lambda_0=1$ and $\lambda_k\geq 0$ is the eigenvalue of $S\in B_1[H]^+$ associated with the eigenvector e_k for each $k\geq 1$ (cf. Proposition 2(a, b) and Remark 3(d)).

Definition 1. The model (1) is *mean square stable* if, for any $x_0\in\mathscr{H}$ and $\{u_i\in\mathscr{H};\ i\geq 0\}$ satisfying Assumption 1, there exist $q\in H$ and $Q\in B_1[H]^+$ independent of $x_0\in\mathscr{H}$, such that

(a) $\|q_i-q\|\to 0$ as $i\to\infty$, (b) $\|Q_i-Q\|\to 0$ as $i\to\infty$.

Theorem 1. If there exist real constants $\sigma\geq 1$ and $0<\alpha<1$ such that

$$\|F^i\|\leq\sigma\alpha^i\quad \forall i\geq 0\quad\text{and}\quad \alpha^2+\sigma^2\|T\|<1,$$

then the model (1) is mean square stable.

Proof. Consider Eq. (2). Since $\{r_i\in H;\ i\geq 0\}$ is Cauchy summable, it follows from Proposition 4 that $\{q_i\in H;\ i\geq 0\}$ is also Cauchy summable, whose limit $q\in H$ does not depend on the initial condition. Hence Definition 1(a) is satisfied. Now consider Eq. (3). Set $\theta=T\in B[B[H]]$ and $\Lambda[Q]=FQF^*$ for all $Q\in B[H]$, where $\Lambda\in B[B[H]]$ is such that

$$\|\Lambda^i\| = \sup_{\|Q\|=1}\|F^iQF^{*i}\|=\|F^i\|^2\leq\sigma^2\alpha^{2i},\qquad \forall i\geq 0.$$

Thus, from Proposition 4, $\{Q_i\in B_1[H]^+;\ i\geq 0\}$ is a Cauchy summable sequence in $B[H]$, whose limit $Q\in B[H]$ does not depend on the initial condition, if $\{V_i[q_i]\in B_1[H];\ i\geq 0\}$ is Cauchy summable in $B[H]$. Moreover, $Q\in B[H]^+$ since $B[H]^+$ is closed in $B[H]$. Hence Definition 1(b) is satisfied, provided the following hypothesis are verified:

(H_1) $\{V_i[q_i]\in B_1[H];\ i\geq 0\}$ is Cauchy summable in $B[H]$.

(H_2) The uniform limit $Q\in B[H]^+$ is nuclear (i.e. $Q\in B_1[H]^+$).

Proof of (H_1). Recall that $e\circ f-g\circ h=(e-g)\circ f+g\circ(f-h)$ and $\|f\circ g\|=\|f\|\,\|g\|$ for every $e,f,g,h\in H$ and, from the definition of P_i and Remarks 1(b) and 3(c), that

$$\Big\|\sum_{k=1}^{\infty}A_kh\circ G_ie_k\Big\|\leq(2\|A_0\|+\sup_{k\geq 1}\|A_k\|\ \|S\|_1^{\frac{1}{2}})\|R_i\|_1^{\frac{1}{2}}\|h\|$$

for all $h\in H$ and every $i\geq 0$. Therefore, for every $i,v\geq 0$,

$$\begin{aligned}\|P_{i+v}[q_{i+v}]-P_i[q_i]\| &\leq \|A_0(q_{i+v}-q_i)\circ r_{i+v}+A_0q_i\circ(r_{i+v}-r_i)\|\\ &\quad+\Big\|\sum_{k=1}^{\infty}A_k(q_{i+v}-q_i)G_{i+v}e_k\\ &\quad+\sum_{k=1}^{\infty}A_kq_i\circ(G_{i+v}-G_i)e_k\Big\|\\ &\leq\|A_0\|\ (\|q_{i+v}-q_i\|\,\|r_{i+v}\|+\|q_i\|\,\|r_{i+v}-r_i\|)\\ &\quad+(2\|A_0\|+\sup_{k\geq 1}\|A_k\|\,\|S\|_1^{\frac{1}{2}})\\ &\quad\times\sup_{i\geq 0}\|R_i\|_1^{\frac{1}{2}}(\|q_{i+v}-q_i\|+|\gamma_{i+v}-\gamma_i|\,\|q_i\|),\end{aligned}$$

since $\sup_{i\geq 0}\|R_i\|_1<\infty$ and $G_i=\gamma_i G_0$ for every $i\geq 0$. Now

$$\|V_{i+v}[q_{i+v}]-V_i[q_i]\| \leq 2\|P_{i+v}[q_{i+v}]-P_{i+v}[q_i]\|+\|R_{i+v}-R_i\|$$

for every $i, v\geq 0$, so that $\{V_i[q_i]\in B_1[H];\ i\geq 0\}$ is Cauchy summable in $B[H]$, by the Cauchy summability of $\{\gamma_i\in \mathbb{C};\ i\geq 0\}$, $\{q_i\in H;\ i\geq 0\}$, $\{r_i\in H;\ i\geq 0\}$, and $\{R_i\in B_1[H]^+;\ i\geq 0\}$.

Proof of (H_2). From Eq. (3) we get by induction

$$Q_i = \Lambda^i Q_0 + \sum_{j=0}^{i-1}\Lambda^{i-j-1}(\theta[Q_j]+V_j[q_j])$$

for every $i\geq 1$. Moreover, according to Remark 3(a),

$$\|\theta[Q]\|_1 = \|T[Q]\|_1 \leq \|T\|\,\|Q\|_1 \qquad \forall Q \in B_1[H]^+.$$

Hence, since $\|\Lambda^i\|\leq\sigma^2\alpha^{2i}$ for every $i\geq 0$,

$$\|Q_i\|_1 \leq \sigma^2[\alpha^{2i}\|Q_0\|_1 + \sum_{j=0}^{i-1}\alpha^{2(i-j-1)}(\|T\|\,\|Q_j\|_1+\|V_j[q_j]\|_1)]$$

for every $i\geq 0$, so that (cf. Proposition (P–1) in Kubrusly, 1986)

$$\|Q_i\|_1 \leq \sigma^2[(\alpha^2+\sigma^2\|T\|)^i\|Q_0\|_1 + \sum_{j=0}^{i-1}(\alpha^2+\sigma^2\|T\|)^{i-j-1}\|V_j[q_j]\|_1]$$

for every $i\geq 0$. However, by the definition of V_i and Remark 3(c),

$$\|V_i[q_i]\|_1 \leq 2\,(\|A_0\|+\sup_{k\geq 1}\|A_k\|\,\|S\|_1^{\frac{1}{2}})\|R_i\|_1^{\frac{1}{2}}\|q_i\| + \|R_i\|_1$$

for every $i\geq 0$, so that $\sup_{i\geq 0}\|R_i\|_1<\infty \Longrightarrow \sup_{i\geq 0}\|V_i[q_i]\|_1 < \infty$. Then

$$\sup_{i\geq 0}\|Q_i\|_1 \leq \sigma^2[\|Q_0\|_1+ \sup_{i\geq 0}\|V_i[q_i]\|_1(1-\alpha^2-\sigma^2\|T\|)^{-1}]$$

since $(\alpha^2+\sigma^2\|T\|)<1$. By hypothesis (H_1) $\{Q_i\in B_1[H]^+;\ i\geq 0\}$ is Cauchy summable in $B[H]$, so that it converges uniformly. Thus $Q\in B_1[H]$, since $\sup_{i\geq 0}\|Q_i\|_1<\infty$ (cf. Weidmann, 1980, p.179).

Remark 4: Recall that, for any $F\in B[H]$ there always exist real constants $\alpha>0$ and $\sigma\geq 1$ such that $\|F^i\|\leq\sigma\alpha^i$ for every $i\geq 0$. Thus, for any operator $F\in B[H]$, set

$$D_F = \{(\alpha,\sigma) \in \mathbb{R}^2:\ \ \alpha > 0,\ \ \sigma \geq 1,\ \ \|F^i\| \leq \sigma\alpha^i,\ \ \forall i \geq 0\},$$

which is nonempty. Therefore, Theorem 1 can be restated as follows; if there exists $(\alpha,\ \sigma)\in D_F$ such that

(a) $$\alpha^2+\sigma^2\|T\| < 1,$$

then the model (1) is mean square stable. Note that the above condition can be replaced by

(b) $$\alpha^2\sigma^2 + \|T\| < 1$$

in the following sense: if there exists $(\alpha, \sigma) \in D_F$ such that (b) holds true, then there exists a pair in D_F (not necessarily the same one) for which (a) also holds true (and, consequently, the model (1) is mean square stable). However, (b) is stronger than (a) (i.e. (b)$\rightleftarrows$(a)). Actually, if (b) holds true for some pair in D_F, say (α', σ'), then $\|F\| \leq \sigma'\alpha' < 1$ (i.e. F is a proper contraction, which is not imposed by (a)). Hence (for $F \neq 0$—otherwise the result is trivial) we may choose $\sigma = 1$ and $\alpha = \|F\| < 1$, so taht $(\alpha, \sigma) \in D_F$ (since $\|F^i\| \leq \|F\|^i$ for every $i \geq 0$) and (a) holds true. On the other hand, even if there is no pair in D_F for which (a) is satisfied, the model (1) may still be mean square stable. This will be the subject of the next Theorem.

Theorem 2. Suppose $M=0$ so that (4) holds. For each $l \geq 0$ let $\Lambda_l, \theta_l \in B[B[H]]$ be given by

$$\Lambda_l[Q] = \lambda_l A_l Q A_l^*, \qquad \theta_l[Q] = \sum_{\substack{k=0 \\ k \neq l}}^{\infty} \lambda_k A_k Q A_k^*$$

for all $Q \in B[H]$, and let $\sigma_l \geq 1$ and $\alpha_l \geq 0$ be real constants such that

$$\|\Lambda_l^i\|^{\frac{1}{2}} = \lambda_l^{\frac{i}{2}} \|A_l^i\| \leq \sigma_l \alpha_l^i \qquad \forall i \geq 0.$$

If $\inf_{l \geq 0}(\alpha_l^2 + \sigma_l^2 \|\theta_l\|) < 1$, then the model (1) is mean square stable.

Proof. First note that, if $l \neq m$ then

$$\lambda_l \|A_l^* h\|^2 \leq \sum_{\substack{k=0 \\ k \neq m}}^{\infty} \lambda_k \|A_k^* h\|^2 = \langle \theta_m[I]h; h \rangle$$

for every $h \in H$. Thus, for any $l \neq m$,

$$\lambda_l \|A_l\|^2 \leq \sup_{\|h\|=1} \langle \theta_m[I]h; h \rangle = \|\theta_m[I]\| \leq \|\theta_m\|.$$

If $\inf_{l \geq 0}(\alpha_l^2 + \sigma_l^2 \|\theta_l\|) < 1$, then there exists $m \geq 0$ such that $\alpha_m^2 + \sigma_m^2 \|\theta_m\| < 1$. Hence $\alpha_m < 1$ and $\|\theta_m\| < 1$, since $\sigma_m \geq 1$. Moreover, $M=0$ implies that $F = A_0$, so that

$$\|F^i\| \leq \sigma_0 \alpha_0^i \quad \forall i \geq 0, \qquad \|F\|^2 \leq \|\theta_m\| \quad \text{if } m > 0,$$

since $\lambda_0 = 1$. Therefore $\|F^i\| \leq \sigma\alpha^i$ for every $i \geq 0$, with $\sigma = \sigma_0 \geq 1$ and $0 < \alpha < 1$, where either $\alpha = \alpha_0$ if $m = 0$ or $\|\theta_m\|^{1/2} \leq \alpha < 1$ if $m > 0$. Now consider Eq. (2), and recall that Eq. (3) can be written as

$$Q_{i+1} = \Lambda_m[Q_i] + \theta_m[Q_i] + V_i[q_i],$$

according to Eq. (4). Then the desired result follows from Proposition 4 exactly as in the proof of Theorem 1. Note that the proof of (H_1) remains literally the same, and the proof of (H_2) also applies to the above equation since

$$\|\theta_m[Q]\|_1 \leq \|\theta_m\| \|Q\|_1 \qquad \forall Q \in B_1[H]^+.$$

6. Illustrative examples

In this section we shall present an application of Theorems 1 and 2 through simple illustrative examples. Note that mean-square stability conditions are given in Theorem 1 in terms of the operators F and T. However, F and T are defined as functions of $\{A_k;\ k\geq 0\}$, $S=\mathscr{E}\{w_i\circ w_i\}$ and $s=E\{w_i\}$. Similarly, mean square stability conditions are given in Theorem 2 in terms of the operators Λ_l and θ_l for $l\geq 0$. However, Λ_l and θ_l are defined as functions of $\{A_k;\ k\geq 0\}$ and $\{\lambda_l;\ l\geq 1\}$, which are the eigenvalues of $S=\mathscr{E}\{w_i\circ w_i\}$. Therefore, both Theorem 1 and Theorem 2 supply sufficient conditions for mean-square stability of the model (1) in terms of the model operators $\{A_k;\ k\geq 0\}$ and the disturbance $\{w_i;\ i\geq 0\}$, which define the maps $\{\mathscr{A}_{w_i};\ i\geq 0\}$.

Consider the set-up of the previous section. Set $H=l_2$ (the Hilbert space made up of all square-summable one-sided infinite sequences of complex numbers), and let $A_k\in B[l_2]$ be the sum of a right and a left weighted shift with weighting sequences

$$\delta_k(1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \cdots),$$
$$\varepsilon_k(1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \cdots),$$

respectively, for each $k\geq 0$. Here $\{\delta_k\neq 0;\ k\geq 0\}$ and $\{\varepsilon_k\neq 0;\ k\geq 0\}$ are bounded complex sequences. Thus

$$A_k=\begin{bmatrix} \begin{matrix} 0 & \varepsilon_k \\ \delta_k & 0 \end{matrix} & & \\ & \begin{matrix} 0 & \varepsilon_k \\ \delta_k & 0 \end{matrix} & \\ & & \ddots \end{bmatrix} \qquad \forall k\geq 0.$$

It is a simple matter to verify that

$$\|A_k{}^i\| = \begin{cases} \varrho_k{}^i, & \text{if } i=0,\ 2,\ 4,\ \cdots \\ \sigma_k\varrho_k{}^i, & \text{if } i=1,\ 3,\ 5,\ \cdots \end{cases}$$

with

$$\varrho_k = |\delta_k\varepsilon_k|^{\frac{1}{2}},$$
$$\sigma_k = \max\{|\delta_k|,\ |\varepsilon_k|\}\varrho_k{}^{-1}$$

for each $k\geq 0$. By setting

$$m_k = \min\{|\delta_k|,\ |\varepsilon_k|\},$$
$$M_k = \max\{|\delta_k|,\ |\varepsilon_k|\},$$

we may write

$$\sigma_k^2 = M_k/m_k \geq 1, \qquad \varrho_k = \sigma_k m_k,$$

for each $k\geq 0$. Now set $S=\mathrm{diag}(\lambda_1, \lambda_2, \cdots)\in B_1[l_2]^+$, where $\lambda_k\geq 0$ for every $k\geq 1$ and $\sum_{k=1}^{\infty}\lambda_k<\infty$, and let $s=(\mu_1, \mu_2, \cdots)\in l_2$ be such that $\sum_{k=1}^{\infty}\mu_k\varepsilon_k=\sum_{k=1}^{\infty}\mu_k\delta_k=0$. Thus $M=\sum_{k=1}^{\infty}A_k\mu_k=0$, so that

$$\|F^i\| = \|A_0{}^i\| \leq \sigma\alpha^i, \qquad \forall i \geq 0,$$

where $\sigma^2=\sigma_0{}^2=M_0/m_0$ and $\alpha=\varrho_0=\sigma_0 m_0$. Note that

$$\|\Lambda_1{}^i\| = \lambda_1^{\frac{1}{2}}\|A_1{}^i\| \leq \sigma_1\alpha_1{}^i, \qquad \forall i \geq 0,$$

where $\sigma_1{}^2=M_1/m_1$ and $\alpha_1=\lambda_1{}^{1/2}\varrho_1=\lambda_1{}^{1/2}\sigma_1 m_1$. For simplicity, without loss of generality as far as the purpose of the present section is concerned, suppose

$$|\delta_k| = |\varepsilon_k| \qquad \forall k \geq 2,$$

and set

$$\beta = \sum_{k=2}^{\infty}\lambda_k|\varepsilon_k|^2 = \sum_{k=2}^{\infty}\lambda_k|\delta_k|^2.$$

Under the above assumption it is readily verified that

$$\|T\| = \sup_{\|Q\|=1}\|\sum_{k=1}^{\infty}\lambda_k A_k Q A_k{}^*\| = \lambda_1 M_1{}^2+\beta,$$

$$\|\theta_1\| = \sup_{\|Q\|=1}\|\sum_{\substack{k=0\\k\neq 1}}^{\infty}\lambda_k A_k Q A_k{}^*\| = M_0{}^2+\beta.$$

Hence

$$\alpha^2+\sigma^2\|T\| = \frac{M_0}{m_0}(m_0{}^2+\lambda_1 M_1{}^2+\beta),$$

$$\alpha_1{}^2+\sigma_1{}^2\|\theta_1\| = \frac{M_1}{m_1}(\lambda_1 m_1{}^2+M_0{}^2+\beta).$$

Now let us asign some numerical values for the parameters involved, in order to illustrate an application of Theorems 1 and 2. We shall consider four simple examples.

Example 1. (perturbation on the model operator A_0) Set $\lambda_1=1$, $\beta=3/16$, $m_1=M_1=1/4$, and $M_0=4/5$. Thus

$$\alpha^2+\sigma^2\|T\| = \frac{4m_0{}^2+1}{5m_0}.$$

Therefore, $\alpha^2+\sigma^2\|T\|<1$ whenever $1/4<m_0\leq M_0=4/5$, so that mean-square stability for the model (1) follows by Theorem 1. On the other hand, $\alpha^2+\sigma^2\|T\|\geq 1$ whenever $0<m_0\leq 1/4$, and mean-square stability does not follow by Theorem (1) in this case. However,

$$\alpha_1{}^2+\sigma_1{}^2\|\theta_1\| = 0.89,$$

so that the model (1) is mean square stable for any admissible value of m_0 (i.e. for $0<m_0\leq M_0=4/5$) by Theorem 2 (with $l=1$).

Example 2. (perturbation on the model operator A_1) Replace the parameters of the operator A_0 by those of the operator A_1, and vice versa, so that we get a situation which is the oposit of that in the preceding example. That is, set $\lambda_1=1$, $\beta=3/16$, $m_0=M_0=1/4$, and $M_1=4/5$. Thus

$$\alpha^2+\sigma^2\|T\| = 0.89,$$

and the model (1) is mean square stable for any admissible value of m_1 (i.e. for $0<m_1\leq M_1=4/5$) according to Theorem 1 (or equivalently, according to Theorem 2 for $l=0$). Note that

$$\alpha_1{}^2+\sigma_1{}^2\|\theta_1\| = \frac{4m_1{}^2+1}{5m_1}\begin{cases}\geq 1 & \text{if } 0<m_1\leq 1/4,\\ <1 & \text{if } 1/4<m_1\leq M_1=4/5.\end{cases}$$

Example 3. (perturbation on the disturbance correlation) Set $\beta=3/16$, $m_0=M_0=1/4$, $m_1=1/2$, $M_1=1$. Thus

$$\alpha^2+\sigma^2\|T\| = \lambda_1+\frac{1}{4}.$$

If $\lambda_1\in[0,\ 3/4)$, then $\alpha^2+\sigma^2\|T\|<1$, so that mean-square stability for the model (1) follows by Theorem 1. On the other hand, for $\lambda_1\in[3/4,\ 1)$ we get $\alpha^2+\sigma^2\|T\|\geq 1$, and mean-square stability for the model (1) does not follow by Theorem (1) in this case. However

$$\alpha_1{}^2+\sigma_1{}^2\|\theta_1\| = \frac{\lambda_1+1}{2} < 1$$

for any $\lambda_1\in[0,\ 1)$, so that mean-square stability for the model (1) still holds in that case, according to Theorem 2 (with $l=1$).

Example 4. (perturbation on the mixed parameter β) Set $m_0=m_1=M_1=1/4$, $M_0=1/2$, and $\lambda_1=1$. Thus

$$\alpha^2+\sigma^2\|T\| = 2\beta+\frac{1}{4} < 1$$

if and only if $\beta\in[0,\ 3/8)$. Hence, for $\beta\in[3/8,\ 11/16)$, Theorem 1 does not indicate mean-square stability for the model (1). However

$$\alpha_1{}^2+\sigma_1{}^2\|\theta_1\| = \beta+\frac{5}{16} < 1,$$

whenever $\beta\in[0,\ 11/16)$, so that the model (1) is actually mean square stable for every $\beta\in[0,\ 11/16)$, according to Theorem 2 (with $l=1$).

7. Concluding remarks

In this paper we have established conditions for mean-square stability of infinite-dimensional discrete bilinear systems driven by H-valued second-order random sequences. The main results were presented in Theorems 1 and 2.

The finite-dimensional stability results proposed in Kubrusly (1986) were extended here to infinite-dimensional models, which are endowed with a much richer topological structure. Such an extension comprised three steps. First, the infinite-dimensional model under consideration required a proper definition, since it is not a mere enlargement of a finite-dimensional one, as Proposition 1 indicates. There it was proven the existence of the bounded linear-like map $\mathscr{A}_w : \mathscr{I}_w \to \mathscr{H}$ by uniform convergence arguments. Such a map plays a fundamental role in bilinear modelling, since it characterizes the multiplicative action of the input over the state. The infinite-dimensional model was then properly posed in Lemma 1. Secondly, the evolution of the state expectation and correlation sequences was considered in Lemma 2. The evolution equations were given in terms of convergent series of operators, whose essential properties were derived in Proposition 2, and uniform (rather than just strong) convergence was ensured in Proposition 3. Last, mean-square stability conditions were supplied in Theorems 1 and 2. As one could expect, the infinite-dimensional case under consideration also demanded a much deeper stability analysis than its finite-dimensional counterpart in Kubrusly (1986), as the proof of hypothesis H_1 and H_2 of Theorem 1 revealed.

Finally let us remark on the independence conditions assumed so far. Let $x_0 \in \mathscr{H}$, $\{u_i \in \mathscr{H};\ i \geq 0\}$ and $\{w_i \in \mathscr{H};\ i \geq 0\}$ be the random disturbances involved in the Lemmas and Theorems of Secs. 3 to 5, and consider the following conditions.

(I) $x_0 \in \mathscr{I}_{\{(u_i,\, w_i);\ i \geq 0\}}$, and $\{(u_i,\, w_i);\ i \geq 0\}$ is an independent sequence in $\mathscr{H}^2$.

(II) $u_i = w_i\ \forall i \geq 0$, and $\{x_0,\ w_i;\ i \geq 0)$ is an independent sequence.

(III) $x_0 \in \mathscr{I}_{\{u_i,\, w_i;\ i \geq 0\}}$, and $\{u_i;\ i \geq 0\}$ and $\{w_i;\ i \geq 0\}$ are independent sequences, which are independent of each other.

(IV) $x_0 \in \mathscr{I}_{\{u_i,\, w_i;\ i \geq 0\}}$, $\{u_i;\ i \geq 0\}$ and $\{w_i,\ i \geq 0\}$ are independent sequences, and $\{u_j,\ w_j\}$ is independent of $\{u_i,\ w_i;\ j \neq i \geq 0\}\ \forall j \geq 0$.

(V) $x_0 \in \mathscr{I}_{\{u_0,\, w_0\}}$, and $\{u_j,\ w_j\}$ is independent of $\{x_0,\ u_0,\ \cdots,\ u_{j-1},\ w_0,\ \cdots,\ w_{j-1}\}\ \forall j \geq 1$.

It is readily verified that

$$\text{(I)} \Longrightarrow \text{(IV)}, \quad \text{(II)} \Longrightarrow \text{(IV)}, \quad \text{(III)} \Longrightarrow \text{(IV)}, \quad \text{(IV)} \Longrightarrow \text{(V)}.$$

Note that condition (V) is certainly stronger than what is actually needed to ensure just the results in Lemma 1 and those in parts (a) and (b) of Lemma 2. On the other hand, it is enough to ensure all the results in Secs. 3 to 5. However, condition (V) may look somewhat artificial, so that the stronger condition in (IV) is sometimes assumed in the related literature (e.g. see Kubrusly, 1986). Indeed, the even stronger conditions (I), (II) and (III), which may be aesthetically more attractive, are very often assumed for modelling stochastic bilinear systems (e.g. see Kubrusly and Costa, 1985; Zabczyk, 1979; Haussmann, 1974, respectively). It is also worth remarking that conditions (II) and (III) represent rather different situations, which turn out to suffice our modelling purposes.

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