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ON CONVERGENCE OF NUCLEAR AND CORRELATION OPERATORS IN HILBERT SPACE

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ABSTRACT: *This paper deals with convergence of sequences of nuclear operators on a separable Hilbert space. Emphasis is given to trace-norm convergence, which is a basic property in stochastic systems theory. Obviously trace-norm convergence implies uniform convergence. The central theme of the paper focus the opposite way, by investigating when convergence in a weaker topology turns out to imply convergence in a stronger topology. The analysis carried out here is exhaustive in the following sense. All possible implications within a selected set of asymptotic properties for sequences of nuclear operators are established. The special case of correlation operators is also considered in detail.*

KEY WORDS: Asymptotic stability • infinite-dimensional systems • operator theory

RESUMO: *Este artigo lida com convergência de seqüências de operadores nucleares em um espaço de Hilbert separável. O problema de convergência na norma do traço, que é uma propriedade básica na teoria de sistemas estocásticos, é enfatizado. Obviamente convergência na norma do traço implica em convergência uniforme. O tema central desse artigo focaliza o sentido inverso da implicação acima, investigando quando a convergência em uma topologia mais fraca vem implicar na convergência em uma topologia mais forte. A análise desenvolvida aqui é exaustiva no seguinte sentido. Todas as possíveis implicações, dentro de um conjunto selecionado de propriedades assintóticas para seqüências de operadores nucleares, são estabelecidas. O caso especial de operadores correlação é também considerado em detalhe.*

PALAVRAS-CHAVE: Estabilidade assintótica • sistemas de dimensão finita • teoria de operadores

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1. INTRODUCTION

A nuclear (or trace-class) operator T on a separable Hilbert space H is a bounded linear operator, which is compact and its absolute value $|T|$ has a finite trace. Obviously any finite-dimensional linear operator (*i.e.* one with a finite-dimensional range) is nuclear and, in particular, any linear operator on a finite-dimensional normed linear space is nuclear. A correlation operator is a nonnegative nuclear operator.

Nuclear operators play a fundamental role in infinite-dimensional stochastic system theory (cf. [3] and [6]), since correlation operators naturally arise in the characterization of second-order H-valued random variables (*e.g.* see [15]). Therefore, some of the main problems for stochastic dynamical systems in Hilbert space, such as identification, filtering, stability, and optimal control, usually require nuclear operators at the modelling stage, either implicitly or explicitly (*e.g.* see [2], [4], [10], and [17], respectively).

However, nuclear and (in particular) correlation operators have a much stronger significance in system theory than just being a proper instrument for a rigorous modelling of infinite-dimensional stochastic systems (*e.g.* see [1] and [5]). Actually, convergence of sequences of correlation operators is a basic property in stochastic system analysis. For instance, mean-square stability for discrete dynamical systems deals essentially with convergence preservation between correlation sequences (*e.g.* see [11]-[13]). Moreover, trace-norm convergence for sequences of correlation operators turns out to be also a necessary and sufficient condition for quadratic-mean convergence of random sequences [cf. [12]).

In this paper we investigate the relationship among a selected set of asymptotic properties for sequences of nuclear operators on a separable Hilbert space. A rather complete analysis on trace-norm convergence is given. The paper is organized as follows. Auxiliary results, which will be needed along the text, are presented in Section 2. There we prove Theorem 1, which deals with convergence of sequences of absolutely summable sequences in a Banach space. The main results, which are stated in Theorem 2, appear in Section 3. There we consider a set of asymptotic properties regarding boundedness, dominance and convergence for sequences of nuclear operators. All possible implications among them are established in Theorem 2. In particular, we show that a sequence of nuclear operators converges in the trace-norm if and only if it converges uniformly and its trace-norm sequence converges. It is also shown that a dominated sequence of nuclear operators converges in the trace-norm, whenever it converges in the uniform norm topology. The special case of correlation operators is considered in Section 4, where further results are obtained. For instance, it is shown that a locally one-sided weakly convergent correlation sequence converges in the trace-norm, whenever its limit is a correlation operator. Illustrative examples are given in Section 5.

2. PRELIMINARY RESULTS

Let $(X, \|\ \|)$ be a normed linear space. $(\ell_1(X), \|\ \|_1)$ and $(\ell_\infty(X), \|\ \|_\infty)$ will denote the normed linear space of all X-valued sequences $x=(x(1),x(2),\ldots)$ such that $\|x\|_1=\sum_{k=1}^\infty\|x(k)\|<\infty$ or $\|x\|_\infty=\sup_{k\geq 1}\|x(k)\|<\infty$, respectively, which are Banach spaces whenever $(X, \|\ \|)$ is a Banach space. Clearly $\ell_1(X)\subset\ell_\infty(X)$.

The purpose of this section is to prove Theorem 1 below. There it will be established the relationship among several asymptotic properties for sequences in $\ell_1(X)$, which will play a fundamental role for supporting the main results in Section 3. We begin by supplying an auxiliary inequality.

Lemma 1. If $x,y\in\ell_1(X)$, then for every $j\geq 0$

$$\|y\|_1+\|x\|_1\leq\|y-x\|_1+2j\|y\|_\infty+2\sum_{k=j+1}^\infty\|x(k)\| ,$$

or equivalently,

$$\|y-x\|_1\leq\|y\|_1-\|x\|_1+2j\|y-x\|_\infty+2\sum_{k=j+1}^\infty\|x(k)\| .$$

Proof. For $j=0$ the above result is reduced to the triangle inequality in $\ell_1(X)$. For each $j\geq 1$ we can write

$$\|y\|_1+\|x\|_1-2\left(\sum_{k=1}^{j}\|y(k)\|+\sum_{k=j+1}^\infty\|x(k)\|\right)$$

$$=\sum_{k=1}^{j}\left(\|x(k)\|-\|y(k)\|\right)+\sum_{k=j+1}^\infty\left(\|y(k)\|-\|x(k)\|\right)$$

$$\leq\sum_{k=1}^{j}\|x(k)-y(k)\|+\sum_{k=j+1}^\infty\|y(k)-x(k)\|=\|x-y\|_1 ,$$

by the triangle inequality in X. Recalling that $\sum_{k=1}^{j}\|y(k)\|\leq j\|y\|_\infty$ for every $j\geq 1$, we get the desired result.

Theorem 1. Set $\ell_\infty=(\ell_\infty(X), \|\ \|_\infty)$ and $\ell_1=(\ell_1(X), \|\ \|_1)$ for some Banach space $(X, \|\ \|)$. Let $x \in \ell_\infty$ and $\{x_i \in \ell_1;\ i\geq 0\}$, and consider the following assertions:

(A) $\|x_i-x\|_1 \to 0$ as $i\to\infty$,

(B) $\|x_i\|_1 \to \|x\|_1$ as $i\to\infty$,

(C) $\sup_{i\geq 0} \|x_i\|_1 < \infty$,

(D) $\|x_i-x\|_\infty \to 0$ as $i\to\infty$,

(E) $x \in \ell_1$.

We claim that the diagram below characterizes all possible implications among the above statements.

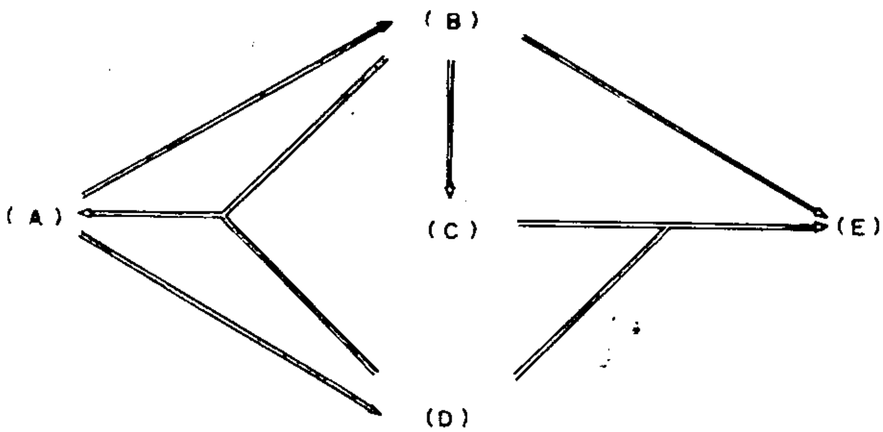

DIAGRAM 1

Now consider a further assertion:

(F) $\sum_{k=1}^{\infty} \sup_{i\geq 0} \|x_i(k)\| < \infty$,

which is equivalent to

(F') $\|x_i(k)\| \leq \|y(k)\|$ $\forall k\geq 1$ $\forall i\geq 0$ for some $y \in \ell_1$,

which means that $\{x_i=(x_i(1),x_i(2),\dots) \in \ell_1;\ i\geq 0\}$ is dominated by some $y \in \ell_1$. We also claim that the implications between (F) and the preceding statements are all characterized by the following diagram.

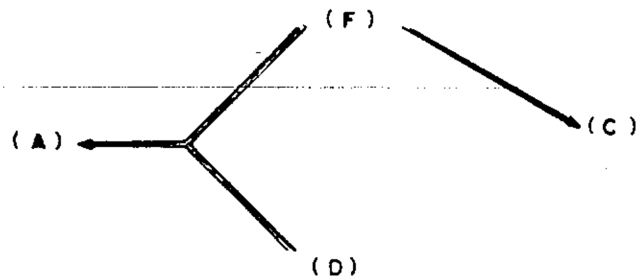

DIAGRAM 2.

Proof. It is trivially verified that (A) $\Longrightarrow$ (B,D), (B) $\Longrightarrow$ (C,E), and (F') $\Longleftrightarrow$ (F) $\Longrightarrow$ (C). By the triangle inequality in X we get

$$\sum_{k=1}^{j} \| x(k)\| \leq \sum_{k=1}^{j} \| x_i(k)-x(k)\| + \sum_{k=1}^{j} \| x_i(k)\|$$

$$\leq j \| x_i-x\|_\infty + \sup_{i\geq 0} \| x_i\|_1$$

for every $j\geq 1$ and $i\geq 0$. Thus (C,D) $\Longrightarrow$ (E). By Lemma 1 we have

$$\| x_i-x\|_1 \leq | \| x_i\|_1 - \| x\|_1 | + 2j\| x_i-x\|_\infty + 2 \sum_{k=j+1}^{\infty} \| x(k)\|$$

for every $j\geq 1$ and $i\geq 0$, whenever x_i, $x \in \ell_1$. Hence, if (B,D) hold true, we get for every $j\geq 1$,

$$\limsup_{i\to\infty} \| x_i-x\|_1 \leq 2 \sum_{k=j+1}^{\infty} \| x(k)\| \to 0 \text{ as } j\to\infty ,$$

since $x \in \ell_1$ and X is a Banach space. Then (B,D) $\Longrightarrow$ (A). Now assume that (D,F) hold true, and notice that $x \in \ell_1$ (since (F) $\Longrightarrow$ (C) and (C,D) $\Longrightarrow$ (E)). Thus, by the triangle inequality in X,

$$| \| x_i\|_1 - \| x\|_1 | = | \sum_{k=1}^{\infty}(\| x_i(k)\| - \| x(k)\|)| \leq \sum_{k=1}^{\infty} \| x_i(k)-x(k)\|$$

$$\leq \sum_{k=1}^{j}\| x_i(k)-x(k)\| + \sum_{k=j+1}^{\infty} (\| x_i(k)\| + \| x(k)\|)$$

$$\leq j\| x_i-x\|_\infty + \sum_{k=j+1}^{\infty}\| y(k)\| + \sum_{k=j+1}^{\infty}\| x(k)\|$$

for every j≥1 and i≥0, for some $y \in \ell_1$. Hence, for every j≥1,

$$\limsup_{i\to\infty} \big| \, \|x_i\|_1 - \|x\|_1 \big| \le \sum_{k=j+1}^{\infty} \|y(k)\| + \sum_{k=j+1}^{\infty} \|x(k)\| \to 0 \qquad j\to\infty$$

since $x,y \in \ell_1$ and X is a Banach space. Therefore (D,F) ⟹ (B), so that (D,F) ⟹ (A), since (B,D) ⟹ (A). Finally we show that the above established results are the only possible implications among assertions (A) to (F). For this we set $(X, \|\ \|)=(\mathbb{C}, |\ |)$, and $e_k=(0,\dots,0,1,0,\dots) \in \ell_1$ for each k≥1, with the nonzero entry at the kth position. To show that (A) ⇏ (F) set x=0 and $x_i=[(i+2)\ell og(i+2)]^{-1}\sum_{k=1}^{i+2}e_k$ for every i≥0. Then (A,B,C,D,E) ⇏ (F). By setting $x=(\sum_{k=1}^{\infty}k^{-2})e_1$ and $x_i=\sum_{k=1}^{i+1} k^{-2}e_k$ for every i≥0 it can be verified that (B,F) ⇏ (D). Thus (B,C,E,F) ⇏ (D). Hence (B,C,E,F) ⇏ (A). Now set x=0 and $x_i(i+1)^{-1}\sum_{k=1}^{i+1}e_k$ for every i≥0 so that (C,D) ⇏ (B). Therefore (C,D,E) ⇏ (B). Hence (C,D,E) ⇏ (A). If we set x=0 and $x_i=[\ell og(i+2)]^{-1}\sum_{k=1}^{i+2}e_k$ for every i≥0 it follows that (D,E) ⇏ (C). With $x=\sum_{k=1}^{\infty}k^{-1}e_k$ and $x_i=\sum_{k=1}^{i+1}k^{-1}e_k$ for every i≥0 it is shown that (D) ⇏ (E). It is obvious that (C,E,F) ⇏ (B) and (C,F) ⇏ (E).

Remark 1. By combining the two diagrams in Theorem 1 there results

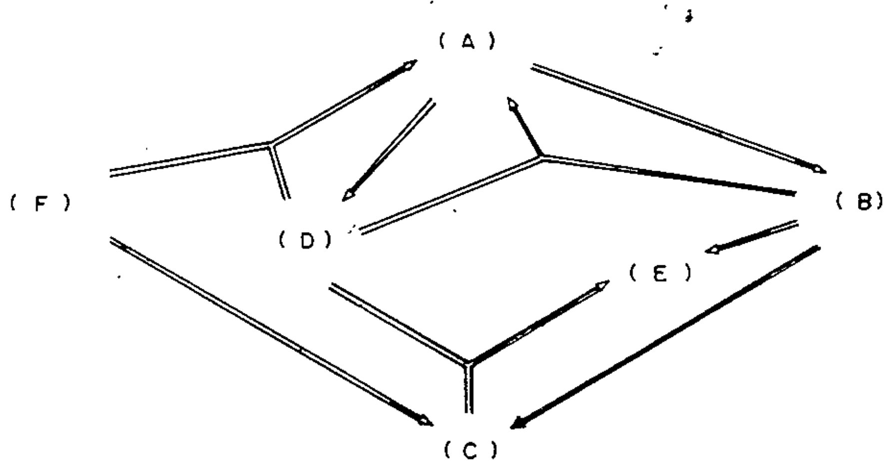

DIAGRAM 3

Notice that the only necessary and sufficient condition in the above diagram is (A) ⟺ (B,D). It is also worth noticing that (D,F) ⟹ (A) is a discrete version of the Dominated Convergence Theorem for p=1 (cf. [7], p.151).

3. MAIN RESULTS

Throughout this section H will denote a separable nontrivial Hilbert space, with < ; > and $\| \;\|$ standing for inner product and norm in H, respectively. We shall use the same symbol $\| \;\|$ to denote the uniform induced norm in B[H], the Banach algebra of all bounded linear operators of H into itself, and $P\sigma(T)$ will denote the point spectrum (*i.e.* the set of all eigenvalues) of $T \in B[H]$. Let $T^* \in B[H]$ be the adjoint of $T \in B[H]$. The standard notation $0 \le T$ will be used if a self-adjoint operator $T=T^* \in B[H]$ is nonnegative (*i.e.* $0 \le \langle Tx;x\rangle \; \forall x \in H$). We set $B[H]^+=\{T \in B[H]: 0 \le T\}$, the closed convex cone of all nonnegative operators on H. Given $T,S \in B[H]$ we shall write $T \le S$ whenever $(S-T) \in B[H]^+$, and $\pm T \le S$ if either $T \le S$ or $-T \le S$. $T^{1/2} \in B[H]^+$ will denote the (unique) square root of $T \in B[H]^+$. Set $|T|=(T^*T)^{1/2} \in B[H]^+$, the absolute value of $T \in B[H]$, so that $B[H]^+=\{T \in B[H]: T=|T|\}$.

Remark 2. The following results on absolute value operators will be used later in this section. Consider a sequence $\{T_i \in B[H]: i \ge 0\}$. Then

(a) $\| T_i x\| \to 0 \quad \text{as} \quad i \to \infty \quad \forall x \in H$

if and only if

(b) $\langle |T_i| x \; ; \; x\rangle \to 0 \quad \text{as} \quad i \to \infty \quad \forall x \in H$.

Actually, for any $T \in B[H]$,

$$\langle |T|x \; ; \; x\rangle = \| \, |T|^{1/2}x\|^2 \le \| \, |T|x\| \; \|x\| \; ,$$

$$\| \, |T|x\| = \|Tx\| \le \| \, |T|^{1/2}\| \; \| \, |T|^{1/2}x\| \; ,$$

for all $x \in H$. Thus (a) $\Longrightarrow$ (b) trivially, and (b) $\Longrightarrow$ (a) by the Banach-Steinhaus Theorem (*e.g.* see [16] p.74). Also recall that, for any $T \in B[H]$,

$$\| \, |T| \, \| = \|T\| = \| \, |T|^{1/2}\|^2 \; .$$

According to the above three basic results, which are trivially verified, there follows immediately a useful inequality. For all $x \in H$,

$$\|Tx\|^2 \le \|T\| \; \langle |T|x \; ; \; x\rangle \; .$$

Let $B_\infty[H]$ be the class of all compact operators from $B[H]$. From now on we assume that $T \in B_\infty[H]$, or equivalently, $|T| \in B_\infty[H]$. Thus $P\sigma(|T|) \subset [0, \|T\|]$ is countable, and $0 \neq \mu_T \in P\sigma(|T|)$ has finite multiplicity. Each $\mu_T \in P\sigma(|T|)$ is referred to as a singular value of T. Therefore, for any $T \in B_\infty[H]$ there exists a nonincreasing nonnegative null sequence $m_T=(\mu_T(1),\mu_T(2),\dots) \in \ell_\infty$ made up of all singular values of T, each nonzero one counted according to its multiplicity as an eigenvalue of $|T|$, so that $\|T\| = \|m_T\|_\infty = \mu_T(1)$ and $\mu_T(k+1) \leq \mu_T(k) = \mu_{-T}(k)$ for every $k \geq 1$. Set $\|T\|_1 = \|m_T\|_1 = \sum_{k=1}^\infty \mu_T(k)$ whenever $m_T \in \ell_1$, so that $\|T\| \leq \|T\|_1$, and let $B_1[H]=\{T \in B_\infty[H]: \|T\|_1 < \infty\}$ be the class of all nuclear (or trace-class) operators on H. Actually $\|\ \|_1$ is a norm in $B_1[H]$, the so-called trace-norm, and $(B_1[H], \|\ \|_1)$ is a Banach space. We set $B_1[H]^+ = B_1[H] \cap B[H]^+$, the class of all correlation operators on H, so that $|T| \in B_1[H]^+$ if and only if $T \in B_1[H]$. For any $T \in B_1[H]$ a trace can be defined by $tr(T)=\sum_{k=1}^\infty \langle Te_k;e_k \rangle$ which does not depend on the choice of the orthonormal basis $\{e_k;\ k\geq 1\}$ for H, so that $|tr(T)| \leq tr(|T|) = \|T\|_1$. For a comprehensive discussion on nuclear operators the reader is referred to [8],[9],[14] and [16].

Remark 3. The following properties of singular values will be needed in the sequel. Let j,k,ℓ be arbitrary positive integers. If $T,S \in B_\infty[H]$, then

(a) $\mu_{T+S}(k+\ell-1) \leq \mu_T(k) + \mu_S(\ell)$,

(b) $\sum_{k=1}^{j} \mu_{T+S}(k) \leq \sum_{K=1}^{j} \mu_T(k) + \sum_{k=1}^{j} \mu_S(k)$,

(c) $\sum_{k=1}^{j} |\mu_T(k)-\mu_S(k)| \leq \sum_{k=1}^{j} \mu_{T-S}(k) + 2 \sum_{k=1}^{j} \mu_T(k)$.

Moreover, if $T,S \in B_1[H]$, then

(d) $\sum_{k=2j+1}^{\infty} |\mu_T(k)-\mu_S(k)| \leq 2[\|T-S\|_1 - \|S\|_1 + \sum_{k=1}^{j} \mu_T(k)] + 5 \sum_{k=j+1}^{\infty} \mu_S(k)$.

The results in (a) and (b) are well known (*e.g.* see [9], pp.30,48). To verify the results in (c) and (d) we proceed as follows. For any $j \geq 1$ set $J=\{k=1,\dots, j:\ \mu_S(k) \leq \mu_T(k)\}$. Then, by (b),

$$\sum_{k=1}^{j} |\mu_T(k)-\mu_S(k)| = \sum_{k=1}^{j} [\mu_S(k)-\mu_T(k)] + 2 \sum_{k\in J} [\mu_T(k)-\mu_S(k)]$$

$$\leq \sum_{k=1}^{j} \mu_{S-T}(k) + 2 \sum_{k\in J} \mu_T(k) - 2 \sum_{k\in J} \mu_S(k) .$$

Thus the result in (c) follows by symmetry. Now let $T,S \in B_1[H]$, and recall from (a) that

$$\sum_{k=j}^{\infty} \mu_T(2k) \leq \sum_{k=j}^{\infty} \mu_T(2k-1) \leq \sum_{k=j}^{\infty} \mu_{T-S}(k) + \sum_{k=j}^{\infty} \mu_S(k)$$

for every $j \geq 1$. Thus

$$\sum_{k=2j-1}^{\infty} \mu_T(k) \leq 2[\sum_{k=j}^{\infty} \mu_{T-S}(k) + \sum_{k=j}^{\infty} \mu_S(k)]$$

for every $j \geq 1$, so that

$$\sum_{k=2j+1}^{\infty} |\mu_T(k)-\mu_S(k)| \leq 2 \sum_{k=j+1}^{\infty} \mu_{T-S}(k) + 3 \sum_{k=j+1}^{\infty} \mu_S(k)$$

for every $j \geq 1$. But, according to (b),

$$\sum_{k=j+1}^{\infty} \mu_{T-S}(k) - \sum_{k=j+1}^{\infty} \mu_S(k) = \| T-S\|_1 - \| S\|_1 + \sum_{k=1}^{j} [\mu_S(k)-\mu_{S-T}(k)]$$

$$\leq \| T-S\|_1 - \| S\|_1 + \sum_{k=1}^{j} \mu_T(k)$$

for every $j \geq 1$, which completes the proof of (d).

To investigate the relatioship among some asymptotic properties for sequences of nuclear operators is the purpose of this paper. This will be developed in Theorem 2 below. As before, we begin by presenting an auxiliary inequaltiy that will suffice our needs.

Lemma 2. If $T,S \in B_1[H]$, then for every $j \geq 0$

$$\| T-S\|_1 \leq 3(\| T\|_1 - \| S\|_1) + 8j\| T-S\| + 7 \sum_{k=j+1}^{\infty} \mu_S(k) .$$

Proof. The result is trivial for $j=0$ by the triangle inequality in $(B_1[H], \| \|_1)$. Recalling that $\sum_{k=1}^{j}\mu_T(k) \leq j\| T\|$, it follows by Remarks 3(c,d) that

$$\| m_T-m_S\|_1 = \sum_{k=1}^{2j} |\mu_T(k)-\mu_S(k)| + \sum_{k=2j+1}^{\infty} |\mu_T(k)-\mu_S(k)|$$

$$\leq 3\| T-S\|_1 - 2\|S\|_1 + 6j\|T\| + 5 \sum_{k=j+1}^{\infty} \mu_S(k)$$

for every $j \geq 1$. Therefore, by Lemma 1

$$\|T\|_1 \leq \|m_T - m_S\|_1 - \|S\|_1 + 2j\|T\| + 2\sum_{k=j+1}^{\infty} \mu_S(k)$$

$$\leq 3(\|T-S\|_1 - \|S\|_1) + 8j\|T\| + 7\sum_{k=j+1}^{\infty} \mu_S(k)$$

for every $j \geq 1$ and for all $T, S \in B_1[H]$. By replacing T by S-T we get the desired result.

<u>Theorem 2</u>. Let $T \in B_\infty[H]$, $\{T_i \in B_1[H];\ i \geq 0\}$, and consider the following assertions:

(A) $\|T_i - T\|_1 \to 0$ as $i \to \infty$,

(B) $\|T_i\|_1 \to \|T\|_1$ as $i \to \infty$,

(C) $\sup_{i \geq 0} \|T_i\|_1 < \infty$,

(D) $\|T_i - T\| \to 0$ as $i \to \infty$,

(E) $T \in B_1[H]$.

Now consider three further assertions stating that $\{T_i \in B_1[H]:\ i \geq 0\}$ is dominated by some $S \in B_1[H]$.

(F) $\sum_{k=1}^{\infty} \sup_{i \geq 0} \mu_{T_i}(k) < \infty$,

(F') $\mu_{T_i}(k) \leq \mu_S(k)$ $\forall k \geq 1$ $\forall i \geq 0$ for some $S \in B_1[H]$,

(G) $|T_i| \leq S \in B_1[H]^+$ $\forall i \geq 0$,

which are related as follows:

(G) $\Longrightarrow$ (F) $\Longleftrightarrow$ (F') .

The above relationship together with Diagrams 1 and 2 (or Diagram 3) characterize all possible implications among the preceding statements. Finally consider the following additional assertions:

(H) $0 \le \pm(T_i - T) \le S \in B_1[H]^+$ for each $i \ge 0$,

(I) $|\langle (T_i - T)x \, ; \, x\rangle| \to 0$ as $i \to \infty$ $\forall x \in H$.

We also claim that (H,I) $\Longrightarrow$ (A). Actually, by considering the auxiliary statements

(J) $|T_i - T| \le S \in B_1[H]^+$ $\forall i \ge 0$,

(K) $\|(T_i - T)x\| \to 0$ as $i \to \infty$ $\forall x \in H$,

the implications in Diagram 4 hold true.

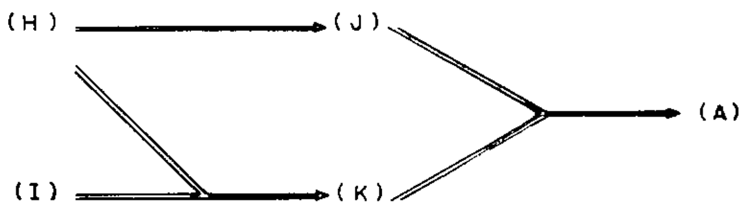

DIAGRAM 4.

Proof. It is trivially verified that (A) $\Longrightarrow$ (B,D), (B) $\Longrightarrow$ (C,E), and (F') $\Longrightarrow$ (F) $\Longrightarrow$ (C). Moreover, it is well known (cf. [9], p.26) that (G) $\Longrightarrow$ (F'). Furthermore, it is a simple matter to show that (F) $\Longrightarrow$ (F') by setting $\mu_S(k) = \sup_{i \ge 0} \mu_{T_i}(k)$ for every $k \ge 1$, and defining $S \in B_1[H]^+$ by the formula

$$Sx = \sum_{k=1}^{\infty} \mu_S(k) \langle x \, ; \, e_k \rangle \, e_k \qquad \forall x \in H \quad ,$$

for some orthonormal basis $\{e_k ; \, k \ge 1\}$ for H. By Remark 3(a) or 3(b)

$$\sum_{k=1}^{j} \mu_T(k) \le j\mu_{T-T_i}(1) + \sum_{k=1}^{j} \mu_{T_i}(k) \le j\| T - T_i \| + \sup_{i \ge 0} \| T_i \|_1$$

for every $j \ge 1$ and $i \ge 0$. Thus (C,D) $\Longrightarrow$ (E). By Lemma 2 we have

$$\| T_i - T \|_1 \le 3| \; \| T_i \|_1 - \|T\|_1 | + 8j \; \|T_i - T\| + 7 \sum_{k=j+1}^{\infty} \mu_T(k)$$

for every $j\geq 1$ and $i\geq 0$, whenever $T_i, T \in B_1[H]$. Hence, if (B,D) hold true, we get for every $j\geq 1$

$$\limsup_{i\to\infty} \| T_i - T\|_1 \leq 7 \sum_{k=j+1}^{\infty} \mu_T(k) \to 0 \quad \text{as} \quad j\to\infty \quad ,$$

since $T \in B_1[H]$. Then (B,D) $\Longrightarrow$ (A). Now assume that (D,F) hold true, and notice that $T \in B_1[H]$ (because (F) $\Longrightarrow$ (C) and (C,D) $\Longrightarrow$ (E)). Thus, since $\sup_{k\geq 1}|\mu_{T_i}(k)-\mu_T(k)|\leq \| T_i - T\|$ for every $i\geq 0$ according to Remark 3(a),

$$\begin{aligned} | \, \| T_i\|_1 - \| T\|_1 \, | &= \Big| \sum_{k=1}^{\infty} [\mu_{T_i}(k)-\mu_T(k)] \Big| \\ &\leq \sum_{k=1}^{j} |\mu_{T_i}(k)-\mu_T(k)| + \sum_{k=j+1}^{\infty} [\mu_{T_i}(k)+\mu_T(k)] \\ &\leq j \, \| T_i - T\| + \sum_{k=j+1}^{\infty} \mu_S(k) + \sum_{k=j+1}^{\infty} \mu_T(k) \end{aligned}$$

for every $j\geq 1$ and $i\geq 0$, for some $S \in B_1[H]$. Hence, for every $j\geq 1$,

$$\limsup_{i\to\infty} | \, \| T_i\| - \| T\|_1 \, | \leq \sum_{k=j+1}^{\infty} \mu_S(k) + \sum_{k=j+1}^{\infty} \mu_T(k) \to 0 \quad \text{as} \quad j\to\infty \; ,$$

since $T,S \in B_1[H]$. Therefore (D,F) $\Longrightarrow$ (B), so that (D,F) $\Longrightarrow$ (A), since (B,D) $\Longrightarrow$ (A). In order to verify that the results in Diagram 3 plus (G) $\Longrightarrow$ (F) comprise all possible implications among assertions (A) to (G), set $H=\ell_2$. First we take $T=\text{diag}(m_T) \in B_1[\ell_2]^+$ and $T_i=\text{diag}(m_{T_i}) \in B_1[\ell_2]^+$ for every $i\geq 0$, with $m_T=x$ and $m_{T_i}=x_i$ for those sequences $x, x_i \in \ell_1$ for every $i\geq 0$ (which are all nondecreasing and nonnegative) used as counter-examples in the proof of Theorem 1. This establishes that no further implication, involving only assertions (A) to (F), can be added to Diagram 3. Now let $\{e_k;\ k\geq 1\}$ be the standard orthonormal basis for ℓ_2. If we set $T=0$ and $T_i=(i+1)^{-1}\text{diag}(e_{i+1})$ for every $i\geq 0$ it follows that (A,F) $\not\Longrightarrow$ (G). Hence (A,B,C,D,E,F) $\not\Longrightarrow$ (G). By setting $T_i=2^{-1}\text{diag}([1+(-1)^i]e_1+[1+(-1)^{i+1}]e_2)$ for every $i\geq 0$ it can be verified that (B,G) $\not\Longrightarrow$ (D). Thus (B,C,E,F,G) $\not\Longrightarrow$ (D), and so (B,C,E,F,G) $\not\Longrightarrow$ (A). It is obvious that (C,E,F,G) $\not\Longrightarrow$ (B) and (C,F,G) $\not\Longrightarrow$ (E). Therefore, besides (G) $\Longrightarrow$ (F), no other result involving assertions (A) to (G) can be added to Diagram 3. Finally let us consider assertions (H) to (K). It is trivial that (H) $\Longrightarrow$ (J), and (H,I) $\Longrightarrow$ (K) according to Remark 2. Now suppose (J) holds true and take any orthonormal basis $\{e_k;\ k\geq 1\}$ for H, so that

$$\| T_i-T\|_1 = tr(|T_i-T|) = \sum_{k=1}^{\infty} \langle |T_i-T|e_k;\ e_k\rangle$$
$$\leq \sum_{k=1}^{j} \langle |T_i-T|e_k;\ e_k\rangle + \sum_{k=j+1}^{\infty} \langle Se_k;\ e_k\rangle$$

for every $j\geq 1$ and $i\geq 0$. Hence, for every $j\geq 1$,

$$\limsup_{i\to\infty} \| T_i-T\|_1 \leq \sum_{k=j+1}^{\infty} \langle Se_k;\ e_k\rangle \to 0 \quad \text{as} \quad j\to\infty \quad ,$$

whenever (K) also holds true, by Remark 2, since $S\in B_1[H]^+$. Then (J,K) $\Longrightarrow$ (A).

4. PARTICULAR RESULTS

As one could expect, some further and sharper results can be drawn from Theorem 2 for the case of correlation operators, where the nuclear sequence involved is supposed to be also nonnegative. So let us take a closer look at this special case.

Correlation operators. If we take a correlation sequence $\{T_i\in B_1[H]^+;\ i\geq 0\}$, the statements (C) and (G) in Theorem 2 become equivalent to

(C^+) $\sup_{i\geq 0} tr(T_i) < \infty$,

(G^+) $T_i \leq S \in B_1[H]^+ \quad \forall i\geq 0$,

respectively, since $|T|=T \iff T\in B[H]^+$. The first further result we get from Theorem 2 is that now

$$(B^+,D) \iff (A) \quad ,$$

where assertion (B) can be replaced by

(B^+) $tr(T_i) \to tr(T) \quad$ as $\quad i\to\infty$,

since $B[H]^+$ is closed in B[H], so that assertion (E) can also be replaced by

(E^+) $T\in B_1[H]^+$.

Thus, trace-norm convergence for sequences of correlation operators means uniform convergence added to a trace convergence condition. Also notice that Diagram 3 plus the result (G) $\Longrightarrow$ (F) still characterizes all possible implications among assertions (A) to (G), since correlation operators have been used as counter-examples in the proof of Theorem 2. Now let us assume further that, for each $i \geq 0$ and for some $S \in B_1[H]^+$,

$$(H^+) \text{ either } 0 \leq T_i \leq T \leq S \text{ or } 0 \leq T \leq T_i \leq S \quad ,$$

which is equivalent to (H, E^+) for correlation sequences. Under assumption (H^+) we get from Theorem 2 that all the preceding convergence properties are equivalent, since (H,I) $\Longrightarrow$ (A) and $tr(\cdot): B_1[H] \to \mathbb{C}$ is a linear functional. That is, if (H^+) holds true, then

$$(A) \Longleftrightarrow (B) \Longleftrightarrow (B^+) \Longleftrightarrow (D) \Longleftrightarrow (I) \Longleftrightarrow (K) \quad .$$

It is worth noticing that the property (H^+), which ensures the above convergence equivalence, is shared by a somewhat wide class of convergent correlation sequences. For instance, consider a sequence $\{T_i \in B[H];\ i \geq 0\}$. It is said to be nondecreasing or nonincreasing if $T_i \leq T_{i+1}$ for every $i \geq 0$ or $T_{i+1} \leq T_i$ for every $i \geq 0$, respectively. It is said to be monotone if it is either nondecreasing nonincreasing. It is said to converge from below or from above if it converges to $T \in B[H]$ and $T_i \leq T$ for every $i \geq 0$ or $T \leq T_i$ for every $i \geq 0$, respectively. It is said to be one-sided convergent if it converges either from below or from above. It is said to be locally one-sided convergent if it converges to $T \in B[H]$ and for each $i \geq 0$ either $T_i \leq T$ or $T \leq T_i$. Obviously monotone convergence implies one-sided convergence, which implies local one-sided convergence. A convergent sequence with the property (H^+) is just a locally one-sided convergent correlation sequence with a correlation limit.

5. ILLUSTRATIVE EXAMPLES

Let $\mathcal{H}$ denote the Hilbert space of all second-order H-valued random variables, and let $E\{w \circ w\} \in B_1[H]^+$ denote the correlation operator of $w \in \mathcal{H}$, as usual (*e.g.* see [12]). Now consider the following discrete autonomous linear system:

$$(1) \quad u_{i+1} = L u_i + v_{i+1} \quad , \qquad u_0 = v_0 \quad ,$$

where $L \in B[H]$ is the system operator, $\{v_i \in H;\ i \geq 0\}$ is the input disturbance, and $\{u_i \in H;\ i \geq 0\}$ is the state sequence. Assume for simplicity that the input disturbance is a white noise sequence (*i.e.* $E\{v_i \circ v_j\}=0$ for every $i \neq j$), and set for every $i \geq 0$

$$R_i = E\{v_i \circ v_i\} \in B_1[H]^+ , \qquad Q_i = E\{u_i \circ u_i\} \in B_1[H]^+ .$$

The purpose of this section is to provide some simple illustrative examples for applications of Theorem 2. We shall focus our attention on two useful results, which play an essential role in the stability theory for infinite-dimensional discrete linear systems operating in a stochastic environment as in (1). Such results will be concerned with the asymptotic behaviour of the state correlation sequence $\{Q_i \in B_1[H]^+;\ i \geq 0\}$. More precisely, we shall be dealing with the question of whether

$$Q_i \to Q \in B_1[H]^+ \qquad \text{as} \qquad i \to \infty$$

in some topology. By convergence in some topology we mean weak, strong, uniform, or trace-norm convergence, which will be denoted by $\xrightarrow{w}$ $\xrightarrow{s}$ $\xrightarrow{u}$, or $\xrightarrow{t}$, respectively. Note that the limit of a correlation sequence is necessarily a nonnegative operator, since $B[H]^+$ is (weakly) closed in $B[H]$. On the other hand, the (nonnegative) limit may not be a correlation operator, since it may fail to be nuclear. However, if we expect that a second-order random sequence has a correlation sequence converging to a correlation operator, such that its limit is a correlation operator for some second-order random variable, then we have to assume further that the limit is nuclear, as we did above. Such a further assumption may be dropped out in the case of trace-norm convergence, when nuclearity necessarily holds.

Example 1. Consider the linear system in (1), and suppose the input disturbance is correlation stationary. Note that the state correlation sequence converges weakly if and only if it converges strongly. We claim further that "it either converges in the trace-norm or it does not converge to a correlation operator at all".

Proof. If $R_i=R \in B_1[H]^+$ for every $i \geq 0$, then it is a simple matter to verify that

$$Q_i = \sum_{j=0}^{i} L^j R L^{*j} \in B_1[H]^+ \qquad \forall i \geq 0 ,$$

so that

$$Q_i \xrightarrow{w} Q \in B[H]^+ \implies 0 \leq Q - Q_i \leq Q \qquad \forall i \geq 0$$

(or equivalently, $0 \leq Q_i \leq Q \quad \forall i \geq 0$), since for every $i \geq 0$

$$<Q_i x \; ; \; x> = \sum_{j=0}^{i} \| R^{1/2} L^{*j} x \|^2 \leq \sum_{j=0}^{\infty} \| R^{1/2} L^{*j} x \|^2 = <Qx \; ; \; x>$$

for all $x \in H$. Hence weak and strong convergence are trivially equivalent, since $0 \leq Q - Q_i$ for every $i \geq 0$, so that

$$Q_i \xrightarrow{w} Q \implies Q_i \xrightarrow{s} Q$$

according to Remark 2. Furthermore, by a straightforward application of Theorem 2 (cf. (H,I) $\implies$ (A)) we get the desired result, since $Q - Q_i \leq Q$ for every $i \geq 0$. That is

$$Q_i \xrightarrow{w} Q \in B_1[H]^+ \implies Q_i \xrightarrow{t} Q \; .$$

Example 2. Consider the following mean-square stability result [12]. If the system operator is uniformly asymptotically stable, then uniform convergence for the correlation sequences is preserved through the linear system (1). That is, if $r_\sigma(L) < 1$ then

$$R_i \xrightarrow{u} R \in B[H]^+ \implies Q_i \xrightarrow{u} Q \in B[H]^+ \; ,$$

where $r_\sigma(L)$ denotes the spectral radius of $L \in B[H]$. Assuming $r_\sigma(L) < 1$ we claim further that "nuclearity for the uniform limits R and Q is also preserved, and trace-norm convergence for the correlation sequences is preserved as well".

Proof. First consider the following auxiliary result from [12]. Suppose $r_\sigma(L) < 1$, so that the above mean-square stability result holds true. Then

$$R_i \xrightarrow{u} R \implies P_i \overset{\text{def.}}{=} \sum_{j=0}^{i} L^j R L^{*j} \xrightarrow{u} Q \; .$$

Now recall that

$$R \in B_1[H]^+ \implies \sup_{i \geq 0} \| P_i \|_1 \leq \| R \|_1 \sum_{j=0}^{\infty} \| L^j \|^2 < \infty \quad ,$$

because the above series converges whenever $r_\sigma(L)<1$. Hence, since $0\leq P_i\leq Q$ for every $i\geq 0$,

$$R_i \xrightarrow{u} R \in B_1[H]^+ \implies P_i \xrightarrow{t} Q \in B_1[H]^+ ,$$

as an immediate consequence of Theorem 2 (cf. (C,D) $\implies$ (E) and (G,D) $\implies$ (A)). Moreover, it is readily verified that

$$Q_i = \sum_{j=0}^{i} L^{i-j} R_j L^{*i-j} \qquad \forall i\geq 0 ,$$

so that

$$\| Q_i-P_i\|_1 \leq \sum_{j=0}^{i} \| L^{i-j}\|^2 \| R_j-R\|_1 \to 0 \qquad \text{as} \qquad i\to\infty$$

whenever $\| R_i-R\|_1 \to 0$ as $i\to\infty$, since $r_\sigma(L)<1$. Thus

$$\| Q_i-Q\|_1 \leq \| Q_i-P_i\|_1 + \| P_i-Q\|_1 \to 0 \qquad \text{as} \qquad i\to\infty ,$$

and we get the final result:

$$R_i \xrightarrow{t} R \in B_1[H]^+ \implies Q_i \xrightarrow{t} Q \in B_1[H]^+ .$$

6. CONCLUDING REMARKS

The results of this paper were heavily based on Lemma 1. Actually, Theorems 1 and 2 were supported by Lemmas 1 and 2, respectively, and Lemma 2 came out of Lemma 1 with the help of Remark 3.

In Theorem 1 we have established all possible implications among some basic asymptotic properties for sequences of absolutely summable sequences in a Banach space X (*i.e.* for sequences in $\ell_1(X)$). The results proved in Theorem 2, which mirror those in Theorem 1, are concerned with asymptotic properties for sequences of nuclear operators on a separable Hilbert space H (*i.e.* for sequences in $B_1[H]$) and, in particular, for sequences of correlation operators (*i.e.* for sequences in $B_1[H]^+$).

Two important applications of Theorem 2 were considered in Examples 1 and 2. There we presented some very simple new proofs, based on a direct application of Theorem 2, for some relevant results on mean-square stability of infinite-dimensional discrete linear systems (*e.g.* see [11] and [12]).

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