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A Note On The Lyapunov Equation For Discrete Linear Systems In Hilbert Space

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Abstract. It is given a simple and unified new proof for the following well-known stability condition: an infinite-dimensional time-invariant discrete linear system is uniformly asymptotically stable if and only if the associate Lyapunov equation has a unique strictly positive solution. The proof is partially based on an application of Rota's model construction technique.

1. INTRODUCTION

Throughout this paper H and K will denote abstract Hilbert spaces over the same (real or complex) field, with $<;>$ and $\|\ \|$ standing for inner product and norm in H or in K. Set $l_2(H) = \oplus_{k=0}^{\infty} H$, the direct sum of countably infinite copies of H, which is a Hilbert space made up of all sequences $\{x_k \in H; k \geq 0\}$ such that $\sum_{k=0}^{\infty} \| x_k \|^2 < \infty$. Let $B[H,K]$ be the Banach space of all bounded linear transformations of H into K, and set $B[H] = B[H,H]$. We shall use the same symbol $\|\ \|$ to denote the induced uniform norm in $B[H,K]$, and $r_\sigma(T) = lim_{n\to\infty} \| T^n \|^{\frac{1}{n}}$ will stand for the spectral radius of $T \in B[H]$. Let $V^* \in B[K,H]$ be the adjoint of $V \in B[H,K]$, and set $| V | = (V^*V)^{\frac{1}{2}} \in B^+[H]$, where $B^+[H] = \{L \in B[H] : L = L^* \geq 0\}$. Let $G[H,K] = \{W \in B[H,K] : \exists W^{-1} \in B[K,H]\}$ and set: $G[H] = G[H,H]$ and $G^+[H] = G[H] \cap B^+[H] = \{Q \in B[H] : Q^* = Q \succ 0\}$. Here nonegativity and strict positivity are defined as usual: $L \geq 0$ iff $< Lx; x > \geq 0$ for all $x \in H$, and $Q \succ 0$ iff there exists $\gamma > 0$ such that $< Qx; x > \geq \gamma \| x \|^2$ for all $x \in H$ (or equivalently, $Q \geq \gamma I$).

As a background for our further discussion let us have a very brief look at just two stability concepts, which turn out to be equivalent. Consider a discrete time-invariant infinite-dimensional free linear system described by the following autonomous homogeneous difference equation in H.

$$x_{n+1} = Tx_n, \qquad x_0 = x \in H. \tag{1}$$

The model in (1) (or equivalently, the operator $T \in B[H]$) is uniformly asymptotically stable iff the state sequence $\{x_n = T^n x \in H; n \geq 0\}$ converges to zero uniformly for all initial conditions $x \in H$, or equivalently, $sup_{\|x\|\leq 1} \| T^n x \| \to 0$ as $n \to \infty$, which means that $\| T^n \| \to 0$ as $n \to \infty$ (i.e. $T^n \xrightarrow{u} 0$). Now, for a given H-valued sequence $\{u_n \in H; n \geq 0\}$, consider a forced version of the free model (1), which is given by the following autonomous inhomogeneous difference equation in H.

$$x_{n+1} = Tx_n + u_{n+1}, \qquad x_0 = u_0 \in H. \tag{2}$$

The model in (2) is l_2-stable iff the state sequence $\{x_n = \sum_{k=0}^{n} T^{n-k} u_k \in H; n \geq 0\}$ is in $l_2(H)$ whenever the input sequence $\{u_n \in H; n \geq 0\}$ is in $l_2(H)$. The above two concepts of stability are equivalent in the following sense. The forced model (2) is l_2-stable if and only if the free model (1) is uniformly asymptotically stable [6].

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There exist in current literature a large collection of necessary and sufficient conditions for uniform asymptotic stability. In particular, for $T \in B[H]$ the following assertions are equivalent.

$$T^n \xrightarrow{u} 0. \tag{3}$$

$$r_\sigma(T) < 1. \tag{4}$$

For every $S \in G^+[H]$ there exists a unique $P \in G^+[H]$ such that

$$S = P - T^*PT. \tag{5}$$

The equivalence between (3) and (4) is well known (e.g. see [2] and [3]), and also it is the equivalence between (4) and (5). That assertions (4) and (5) are equivalent even in an infinite dimensional setting has been investigated, for instance, in [1], [6], [2] and [4]. The purpose of this note is to present a simple and unified new proof for the equivalence between (4) and (5) for infinite-dimensional systems, which is essenially based on the theory of similarity to contraction in Hilbert space. The advantage of the new proof is that it is less computational than the previous ones, and it uses a geometrical approach which seems to be promising for investigating further aspects (even in a weaker topology) of stability for infinite-dimensional discrete systems.

2. PRELIMINARIES

An operator $T \in B[H]$ is a contraction (a strict contraction) iff $\| T \| \leq 1$ ($\| T \| < 1$). $T \in B[H]$ is similar to a contraction (to a strict contraction) iff there exists $W \in G[H,K]$, for some Hilbert space K unitarilly equivalent to H, such that $\| WTW^{-1} \| \leq 1$ ($\| WTW^{-1} \| < 1$). The proposition below gives an alternative definition and a further necessary and sufficient condition for similarity to contraction (or to strict contraction).

Proposition: Recall that H and K are topologically isomorphic if and only if they are unitarily equivalent. In other words,

$$G[H,K] \neq \emptyset \Longleftrightarrow \{U \in G[H,K] : U^{-1} = U^*\} \neq 0. \tag{a}$$

Therefore, for any $T \in B[H]$ and $W \in G[H,K]$,

$$\| WTW^{-1} \| = \|| W | T | W |^{-1}\|, \tag{b}$$

so that (c) $T \in B[H]$ is similar to a contraction (to a strict contraction) if and only if there exists $Q \in G^+[H]$ such that $\| QTQ^{-1} \| \leq 1$ ($\| QTQ^{-1} \| < 1$). Moreover, for any $T \in B[H]$, $Q \in G^+[H]$, and $\alpha \in (0, \infty)$,

$$\| QTQ^{-1} \| \leq \alpha \Longleftrightarrow (\alpha^2 Q^2 - T^*Q^2T) \in B^+[H], \tag{d}$$

$$\| QTQ^{-1} \| < \alpha \Longleftrightarrow (\alpha^2 Q^2 - T^*Q^2T) \in G^+[H]. \tag{e}$$

Proof: If $W \in G[H,K]$, then $U = W \mid W \mid^{-1} \in G[H,K]$ is unitary (i.e. $U^*U = I \in B[H]$ and $UU^* = I \in B[K]$), so that (a) holds true. Hence we get the identity in (b) since $\| WTW^{-1} \| = \| U \mid W \mid T \mid W \mid^{-1} U^* \| = \| \mid W \mid T \mid W \mid^{-1} \|$. Thus the alternative definition in (c) is trivially verified since $W \in G[H,K]$ if and only if $\mid W \mid \in G^+[H]$. Now recall that $\| QTQ^{-1} \| \leq \alpha$ if and only if $\| QTQ^{-1}x \| \leq \alpha \| x \|$ for every $x \in H$, and that $< Q^{-1}T^*Q^2TQ^{-1}x; x > = \| QTQ^{-1}x \|^2$ for every $x \in H$. Hence $\| QTQ^{-1} \| \leq \alpha$ if and only if $< (\alpha^2 Q^2 - T^*Q^2T)Q^{-1}x; Q^{-1}x > \geq 0$ for every $x \in H$, so that (d) holds true. From (d) it follows that, if $\| QTQ^{-1} \| < \alpha$ (or equivalently, if $\| QTQ^{-1} \| \leq \alpha_0$ for some $\alpha_0 \in (0, \alpha)$), then

$(\alpha^2-\alpha_0^2)Q^2 \leq \alpha^2 Q^2 - T^*Q^2T$, so that $0 \prec \alpha^2Q^2 - T^*Q^2T$ (i.e. $(\alpha^2Q^2 - T^*Q^2T) \in G^+[H]$) since $0 < (\alpha^2 - \alpha_0^2)$ and $0 \prec Q^2$. On the other hand, notice that $0 \prec \alpha^2Q^2 - T^*Q^2T$ if and only if there exists $\beta > 0$ such that $0 \leq (\alpha^2Q^2 - \beta I) - T^*Q^2T$. Take $\alpha_0 \in (0, \alpha)$ such that $\alpha_0 \leq \frac{\beta}{\alpha \| Q \|^2}$. Thus $\alpha Q^2 \leq \alpha \| Q \|^2 I \leq (\frac{\beta}{\alpha_0})I$, so that $\alpha^2Q^2 - \beta I \leq \alpha(\alpha - \alpha_0)Q^2$. Hence $0 \leq \alpha(\alpha - \alpha_0)Q^2 - T^*Q^2T$. Therefore $\| QTQ^{-1} \| \leq [\alpha(\alpha - \alpha_0)]^{\frac{1}{2}} < \alpha$, according to ($d$), which completes the proof of (e).

3. Conclusion

In this section we conclude the anounced proof for the equivalence between assertions (4) and (5) stated in section 1. Such a proof, which is the purpose of this note, uses the previous Proposition in one way and, in the other way, it applies the model construction technique provided by Rota in [5] to show that $T \in B[H]$ is similar to a part of the backward shift on $l_2(H)$ whenever $r_\sigma(T) < 1$.

Theorem. *Let $T \in B[H]$. The following assertions are equivalent.*

(1) $r_\sigma(T) < 1$.
(2) *T is similar to a strict contraction.*
(3) *There exists $P \in G^+[H]$ such that $P - T^*PT \in G^+[H]$.*
(4) *For every $S \in G^+[H]$ there exists $P \in G^+[H]$ such that $S = P - T^*PT$.*

Morover, if the above holds, then the solution $P \in G^+[H]$ of the Lyapunov equation $S = P - T^*PT$, for any $S \in G^+[H]$, is unique and given by

$$P = \sum_{k=0}^{\infty} T^{*k} S T^k,$$

where the above convergence is in the uniform topology.

Proof: (d) $\Rightarrow$ (c) trivially. (c) $\Rightarrow$ (b) according to the previous Proposition. (b) $\Rightarrow$ (a) since similarity preserves the spectrum, so that $r_\sigma(T) = r_\sigma(WTW^{-1}) \leq \| WTW^{-1} \|$ for every $W \in G[H, K]$. The proof of (a) $\Rightarrow$ (d) goes as follows. Suppose (a) holds true (so that $\sum_{k=0}^{\infty} \| T^k \|^2 < \infty$), take an arbitrary $S \in G^+[H]$, and set $R = S^{\frac{1}{2}} \in G^+[H]$. Then, for any $x \in H$,

$$\| R^{-1} \|^{-2} \| x \|^2 \leq \| Rx \|^2 \leq \sum_{k=0}^{n} \| RT^k x \|^2 \leq \| R \|^2 \left(\sum_{k=0}^{\infty} \| T^k \|^2\right) \| x \|^2$$

for every $n \geq 0$. Hence we may define a map $W : H \to l_2(H)$ given by $Wx = (Rx, RTx, RT^2x, \ldots)$ for all $x \in H$. Such a map is clearly linear and bounded with $\| Wx \|^2 = \sum_{k=0}^{\infty} \| RT^k x \|^2$ for every $x \in H$, so that it has a bounded inverse on its (closed) range. Thus $W \in G[H, K]$, where K is the range of W, which is a subspace (i.e. a closed linear manifold) of $l_2(H)$. Therefore $| W |^2 \in G^+[H]$. Note that

$$< (R^2 - | W |^2 + T^* | W |^2 T)x; x > = \| Rx \|^2 - \| Wx \|^2 + \| WTx \|^2 = 0$$

for every $x \in H$. Thus $R^2 = | W |^2 - T^* | W |^2 T$, so that ($a$) $\Rightarrow$ (d) with $P = | W |^2 \in G^+[H]$. Moreover, such an operator is unique. For, if $R^2 = Q^2 - T^*Q^2T$ for some $Q^2 \in G^+[H]$, then

$$\| RT^k x \|^2 = < T^{*k} R^2 T^k x; x > = \| QT^k x \|^2 - \| QT^{k+1} x \|^2$$

for every $x \in H$ and for each $k \geq 0$. Hence

$$\| Wx \|^2 = \lim_{n \to \infty} \sum_{k=0}^{n} \| RT^k x \|^2 = \lim_{n \to \infty} (\| Qx \|^2 - \| QT^{n+1} x \|^2) = \| Qx \|^2$$

for every $x \in H$, since $r_\sigma(T) < 1$ (i.e. $T^n \xrightarrow{u} 0$). Therefore,

$$< (|W|^2 - Q^2)x; x > = \| Wx \|^2 - \| Qx \|^2 = 0$$

for every $x \in H$, so that $Q^2 = |W|^2$, which proves uniqueness. Finally set $P_n = \sum_{k=0}^{n} T^{*k} S T^k = P - T^{*n+1} P T^{n+1} \in B^+[H]$ for each $n \geq 0$, where $P \in G^+[H]$ is the solution of $S = P - T^* PT$. Thus $\| P_n - P \| = \| T^{*n+1} P T^{n+1} \| \leq \| P \| \| T^{n+1} \|^2$ for every $n \geq 0$, so that $\| P_n - P \| \to 0$ as $n \to \infty$ since $r_\sigma(T) < 1$ (i.e. $T^n \xrightarrow{u} 0$).

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