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ERRATUM/ADDENDUM TO *"POWERS OF POSINORMAL OPERATORS", OPERATORS AND MATRICES* **10** *(2016), 15–27*

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Abstract. Erratum/Addendum to the paper *Powers of posinormal operators*, Operators and Matrices **10** (2016), 15–27.

The statement of Lemma 2 in the above paper is incomplete (in that we overlooked the necessary assumption of closed range to prove the second part of it — we referred to [2, Proof of Lemma 5.31] overlooking that that result is stated for Fredholm operators where ranges are closed). The corrected statement and proof go as follows (notation and terminology as in [4]).

Lemma 2. *Take any operator $A \in \mathcal{B}[\mathcal{H}]$ and an arbitrary integer $k \geq 1$. If*

$$asc(A) \leq k \;\; and \;\; dsc(A) < \infty \quad or \quad asc(A) < \infty \;\; and \;\; dsc(A) \leq k,$$

then

$$dsc(A) = asc(A) \leq k,$$

and so

$$\mathcal{R}(A^n) = \mathcal{R}(A^k) \quad and \quad \mathcal{N}(A^n) = \mathcal{N}(A^k) \quad for\ each\ integer\ n \geq k.$$

If, in addition, $\mathcal{R}(A^n)$ is closed for every n, then

$$dsc(A^*) = asc(A^*) \leq k,$$

and so

$$\mathcal{R}(A^{*n}) = \mathcal{R}(A^{*k}) \quad and \quad \mathcal{N}(A^{*n}) = \mathcal{N}(A^{*k}) \quad for\ each\ integer\ n \geq k.$$

Proof. Take an arbitrary $A \in \mathcal{B}[\mathcal{H}]$. Consider the following auxiliary results.

Claim (i). $\operatorname{asc}(A) < \infty$ and $\operatorname{dsc}(A) < \infty \implies \operatorname{asc}(A) = \operatorname{dsc}(A)$.

Proof of Claim (i). This is a well-known result, see e.g., [6, Theorem 6.2]. □

Claim (ii).

(a) $\operatorname{dsc}(A^*) < \infty \implies \operatorname{asc}(A) < \infty$,

(b) $\operatorname{asc}(A) < \infty \implies \operatorname{dsc}(A^*) < \infty$ if $\mathcal{R}(A^n)$ is closed for every integer $n \geq 1$,

(c) $\operatorname{asc}(A) < \infty \not\Longrightarrow \operatorname{dsc}(A^*) < \infty$ if $\mathcal{R}(A^n)$ is not closed for some integer $n \geq 1$.

Proof of Claim (ii). Take an arbitrary positive integer n.

(a) If $\operatorname{asc}(A) = \infty$, then $\mathcal{N}(A^n) \subset \mathcal{N}(A^{n+1})$ so that $\mathcal{N}(A^{n+1})^\perp \subset \mathcal{N}(A^n)^\perp$ (since $\mathcal{N}(\,\cdot\,)$ is closed — indeed, $\mathcal{M} \subset \mathcal{N} \Longrightarrow \mathcal{N}^\perp \subseteq \mathcal{M}^\perp$ and $\mathcal{N}^\perp = \mathcal{M}^\perp \Longrightarrow \mathcal{M}^- = \mathcal{N}^-$).

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Equivalently, $\mathcal{R}(A^{*(n+1)})^- \subset \mathcal{R}(A^{*n})^-$. As $\mathcal{R}(A^{*n+1}) \subseteq \mathcal{R}(A^{*n})$, the above proper inclusion ensures the proper inclusion $\mathcal{R}(A^{*(n+1)}) \subset \mathcal{R}(A^{*n})$. So $\operatorname{dsc}(A^*) = \infty$, and

$$\operatorname{asc}(A) = \infty \implies \operatorname{dsc}(A^*) = \infty.$$

(b) If $\operatorname{dsc}(A) = \infty$, then $\mathcal{R}(A^{n+1}) \subset \mathcal{R}(A^n)$. Suppose $\mathcal{R}(A^n)$ is closed so that $\mathcal{R}(A^{n+1}) \subset \mathcal{R}(A^n)$ implies $\mathcal{R}(A^n)^\perp \subset \mathcal{R}(A^{n+1})^\perp$. That is, $\mathcal{N}(A^{*n}) \subset \mathcal{N}(A^{*(n+1)})$. Hence $\operatorname{asc}(A^*) = \infty$. Therefore

$$\operatorname{dsc}(A) = \infty \implies \operatorname{asc}(A^*) = \infty \quad \text{if } \mathcal{R}(A^n) \text{ is closed for every integer } n \geq 1.$$

Dually (as $A^{**} = A$ and $\mathcal{R}(A^n)$ closed $\iff$ $\mathcal{R}(A^{*n})$ closed),

$$\operatorname{dsc}(A^*) = \infty \implies \operatorname{asc}(A) = \infty \quad \text{if } \mathcal{R}(A^n) \text{ is closed for every integer } n \geq 1,$$

(c) To verify (c) consider the following example. Take A such that $\mathcal{N}(A^*) = \{0\}$ and $\mathcal{R}(A^*) \neq \mathcal{R}(A^*)^- = \mathcal{H}$. Then $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp = \{0\}$, and hence $\operatorname{asc}(A) = 0$. We show that $\operatorname{dsc}(A^*) = \infty$.

Since $\mathcal{R}(A^*) \neq \mathcal{R}(A^*)^- = \mathcal{H}$, take $v \in \mathcal{H} \backslash \mathcal{R}(A^*)$. Suppose $\operatorname{dsc}(A^*) < \infty$, say, suppose $\operatorname{dsc}(A^*) = n$. Then $\mathcal{R}(A^{*n}) = \mathcal{R}(A^{*n+1})$, and so there exists $w \in \mathcal{H}$ such that $A^{*n+1}w = A^{*n}v$. Thus $A^{*n}(A^{*n}w - v) = 0$ so that $A^*w = v$ (since $\operatorname{asc}(A^*) = 0 \implies \mathcal{N}(A^{*n}) = \{0\}$). Hence $v \in \mathcal{R}(A^*)$, which is a contradiction. Thus $\operatorname{dsc}(A^*) = \infty$. □

Claim (iii). $\operatorname{dsc}(A) < \infty \implies \operatorname{asc}(A^*) \leq \operatorname{dsc}(A)$.

Proof of Claim (iii). Consider the argument in the proof of Claim (ii-a). So $\operatorname{dsc}(A) = n_0$ implies $\mathcal{R}(A^n) = \mathcal{R}(A^{n_0})$ for every $n \geq n_0$. Thus $\mathcal{R}(A^n)^- = \mathcal{R}(A^{n_0})^-$. Equivalently, $\mathcal{N}(A^{*n}) = \mathcal{N}(A^{*n_0})$ (as $\mathcal{R}(\,\cdot\,)^\perp = \mathcal{N}(\,\cdot\,^*)$), which implies $\operatorname{asc}(A^n) \leq n_0$. □

If $\operatorname{asc}(A) \leq k$ and $\operatorname{dsc}(A) < \infty$ (or if $\operatorname{asc}(A) < \infty$ and $\operatorname{dsc}(A) \leq k$), then

$$\operatorname{dsc}(A) = \operatorname{asc}(A) \leq k$$

by Claim (i). Moreover, this implies that $\operatorname{asc}(A^*) \leq \operatorname{dsc}(A) \leq k$ by Claim (iii). Now suppose $\mathcal{R}(A^n)$ is closed for every n. Since $\operatorname{asc}(A) \leq k$, we get $\operatorname{dsc}(A^*) < \infty$ by Claim (ii-b). Then, since $\operatorname{asc}(A^*) \leq k$, Claim (i) ensures that

$$\operatorname{dsc}(A^*) = \operatorname{asc}(A^*) \leq k.$$

The range and kernel identities follow from the definition of ascent and descent. □

Consequently, Theorem 1 and Corollary 1 are to be modified, whose proofs follow the same argument as before, now applying the correct version of Lemma 2.

Note. Posinormal operators were introduced in [5] (see also [3]) — an operator is posinormal if its range is included in the range of its adjoint.

Theorem 1. *Take $T \in \mathcal{B}[\mathcal{H}]$. Suppose $\mathcal{R}(T^n)$ is closed for every $n \geq 1$.*

(a) *If T^k is posinormal for some $k \geq 1$ and dsc$(T^m) < \infty$ for some $m \geq 1$, then T^n is posinormal for every $n \geq k$.*

(b) *If T^k is posinormal for some $k \geq 1$ and T^{*m} is posinormal for some $m \geq k$, then T^n is posinormal for every $n \geq k$ and coposinormal for every $n \geq m$.*

Corollary 1. *Take $T \in \mathcal{B}[\mathcal{H}]$. Suppose $\mathcal{R}(T^n)$ is closed for every $n \geq 1$.*

(a) *If T is posinormal and dsc$(T) < \infty$, then T^n is posinormal for every $n \geq 1$.*

(b) *If T is posinormal and coposinormal, then T^n is posinormal and coposinormal for every $n \geq 1$.*

In fact, the assumption "$\mathrm{dsc}(T) < \infty$" in Corollary 1(a) above can be dismissed, yielding a corrected version of Corollary 3:

Corollary 3. *If T is posinormal and $\mathcal{R}(T^n)$ is closed for every $n \geq 1$, then T^n is posinormal.*

In a subsequent paper [1], the above assumption "$\mathcal{R}(T^n)$ is closed for every $n \geq 1$" has been weakened, yielding a sharper result as follows.

Theorem [1]**.** *If T is posinormal and has closed range, then T^n is posinormal and has closed range for every $n \geq 1$.*

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