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# Erratum/Addendum to "PF property and property ($\beta$) for paranormal operators"

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The proof of Theorem 7.2(ii) – *If* $A \in B(\mathcal{H}) \cap \mathrm{PF}(\delta)$ *is a contraction, then* $S(\delta_{A,V^*}) \neq \varnothing$ *implies* $A$ *is not* $n$*-supercyclic* – of our paper of the title is incomplete (in that it fails to consider the case $\alpha(A_c^* - \overline{\lambda}) = 0$). We provide here additional argument to complete the proof, and prove an analogue of the result for weakly supercyclic operators.

We follow the notation and terminology of the paper of the title. Thus $B(\mathcal{H})$ denotes the algebra of bounded linear operators on an infinite dimensional complex Hilbert space, $\delta_{A,B} \in B(B(\mathcal{H}))$ denotes the generalized derivation $\delta_{A,B}(X) = AX - XB$, and an operator $A \in B(\mathcal{H})$ satisfies the *Putnam–Fuglede property* $\delta$, denoted $A \in \mathrm{PF}(\delta)$, if whenever the equation $AX = XV^*$ holds for some isometry $V$ and operator $X \in B(\mathcal{H})$, then also $A^*X = XV$. An operator $A \in B(\mathcal{H})$ is $n$-*supercyclic* for some $n \in \mathbb{N}$ if $\mathcal{H}$ has an $n$-dimensional subspace $M$ with dense orbit $\mathrm{Orb}_M(A) = \bigcup_{m\in\mathbb{N}} A^m M$; a 1-supercyclic operator is supercyclic, and we say that $A$ is *weakly supercyclic* if there exists a vector $x \in \mathcal{H}$, with $M$ the corresponding one dimensional subspace generated by $x$, such that $\mathrm{Orb}_M(A)$ is weakly dense (i.e., dense in the weak topology) in $\mathcal{H}$. It is clear that if an operator $A \in B(\mathcal{H})$ is $n$-supercyclic or weakly supercyclic, then $\mathcal{H}$ is separable (see [1, 2, 5, 7] for more information).

For an operator $A \in B(\mathcal{H})$, let $S(\delta_{A,V^*}) = \{V \in B(\mathcal{H}) : V \text{ isometric}, \delta^{-1}_{A,V^*}(0) \neq \{0\}\}$, and let $S_D(\delta_{A,V^*})$ denote the set of those isometries $V \in B(\mathcal{H})$ for which there exists an $X \in B(\mathcal{H})$ with dense range such that $\delta_{A,V^*}(X) = 0$. Clearly, if $A \in B(\mathcal{H})$ is a contraction, $S(\delta_{A,V^*}) \neq \varnothing$ and $A \in PF(\delta)$, then $A$ is the direct sum of a unitary with some (possibly trivial) operator, and if also $S_D(\delta_{A,V^*}) \neq \varnothing$, then $A$ is unitary.

**Theorem 7.2** *Let $A$ be a contraction in $B(\mathcal{H}) \cap PF(\delta)$.*

(a) *If $A$ is $n$-supercyclic, then $S(\delta_{A,V^*}) = \varnothing$.*

(b) *If $A$ is weakly supercyclic, then either*

($b_1$) *$S(\delta_{A,V^*}) = \varnothing$, or*

($b_2$) *$S(\delta_{A,V^*}) \neq \varnothing$ and $A$ is a unitary (hence $S_D(\delta_{A,V^*}) \neq \varnothing$).*

*Proof*. If $S(\delta_{A,V^*}) \neq \varnothing$ and $A \in PF(\delta)$, then there exists a decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$ of $\mathcal{H}$ such that $A = A|_{\mathcal{H}_u} \oplus A|_{\mathcal{H}_c} = A_u \oplus A_c$, where $A_u$ is unitary and $A_c$ is completely non-unitary (indeed, a $C_{.0}$ contraction). (The component $A_c$ may be absent if there exists an $X \in B(\mathcal{H})$ with dense range satisfying $\delta_{A,V^*}(X) = 0$.) Evidently, in such a case, $\sigma(A_u)$ is contained in the boundary $\partial\mathbf{D}$ of the unit disc in the complex plane, $||A_c|| \leq 1$ and the spectral radius $r(A_c)$ of $A_c$ satisfies $r(A_c) \leq 1$.

The proof here is similar to that of [3, Theorem 7.2(ii)]. However, as earlier stated, the argument of the proof of [3, Theorem 7.2(ii)] is incomplete: We provide below the missing argument and reproduce here a complete proof for the reader's convenience.

(a) Suppose a contraction $A \in B(\mathcal{H}) \cap PF(\delta)$ is $n$-supercyclic and $S(\delta_{A,V^*}) \neq \varnothing$. Then, as observed above, $A = A_u \oplus A_c$, where $A_u$ is unitary, $\sigma(A_u) \subseteq \{\lambda : |\lambda| = 1\} = \partial\mathbf{D}$ (= the boundary of the unit disc $\mathbf{D}$ in the complex plane). If $r(A_c) < 1$, then there exists a real number $\rho < 1$ such that $\sigma(A_c) \subseteq \{\lambda : |\lambda| \leq \rho\}$. Consequently, $\sigma(A) \subseteq \partial\mathbf{D} \cup \{\lambda : |\lambda| \leq \rho\}$, and this by [5, Proposition 4.5] is a contradiction. Hence $r(A_c) = 1$, which then implies that $A_c$ is normaloid. Letting $\sigma_\pi(A_c) = \{\lambda \in \sigma(A_c) : |\lambda| = r(A_c)\}$ denote the *peripheral spectrum* of $A_c$, [6, Proposition 54.2] implies that $\mathrm{asc}(A_c - \lambda) \leq 1$ and $\beta(A_c - \lambda) = \mathrm{dim}(\mathcal{H}_c/(\mathrm{A_c} - \lambda)(\mathcal{H}_c)) > 0$ for all $\lambda \in \sigma_\pi(A_c)$. Since $\alpha(A_c^* - \overline{\lambda}) = \mathrm{dim}(\mathrm{A_c^*} - \overline{\lambda}) \leq \beta(\mathrm{A_c} - \lambda)$, either $\alpha(A_c^* - \overline{\lambda}) = 0$ or $\alpha(A_c^* - \overline{\lambda}) > 0$. We prove that neither of these alternatives is feasible, leading us to conclude that our hypothesis $r(A_c) = 1$ is not possible. Recall the (easily proved) fact that the eigenvalues $\mu$ of a contraction in $B(\mathcal{H})$ such that $|\mu| = 1$ are normal (i.e., the corresponding eigenspace is reducing). Hence, if $\alpha(A_c^* - \overline{\lambda}) > 0$, then $A_c$ has a unitary direct summand — a contradiction. Consequently, $\alpha(A_c^* - \overline{\lambda}) = 0$. We claim that $\overline{\lambda} \notin \mathrm{iso}\sigma(A_c^*)$. If $\overline{\lambda} \in \mathrm{iso}\sigma(A_c^*)$, then $\mathcal{H}_c = P(\mathcal{H}_c) \oplus (I - P)(\mathcal{H}_c)$, where $P$ is the Riesz projection associated with $\overline{\lambda}$ [6, Theorem 49.1]. Evidently, $\mathrm{dim}(\mathrm{P}(\mathcal{H}_c)) = \infty$ (for if $\mathrm{dim}(\mathrm{P}(\mathcal{H}_c)) < \infty$, then $\overline{\lambda}$ is a pole, hence an eigenvalue). Since the adjoint of an $n$-supercyclic operator can not have an infinite dimensional invariant subspace [2], we have a contradiction. This leaves us with the case $\overline{\lambda} \in \sigma_a(A_c^*)$ and $\overline{\lambda} \notin \mathrm{iso}\sigma(A_c^*)$. Then, given $\epsilon > 0$, every $\epsilon$-neighbourhood of $\overline{\lambda}$ contains an element of the point spectrum of $A_c^*$. Since the adjoint of an $n$-supercyclic operator has at most (counting multiplicity) $n$ eigenvalues [2], again we have a contradiction. Hence $\alpha(A_c^* - \overline{\lambda}) \neq 0$, which leads to yet another contradiction. Therefore, if a contraction $A \in B(\mathcal{H}) \cap PF(\delta)$ is $n$-supercyclic, then $S(\delta_{A,V^*}) = \varnothing$. (Equivalently, if $S(\delta_{A,V^*}) \neq \varnothing$, then a contraction $A \in B(\mathcal{H}) \cap PF(\delta)$ is not $n$-supercyclic.)

(b) The proof here is similar to that of the first part, and we shall use freely the relevant parts of the argument above in our proof below. Suppose that $A \in B(\mathcal{H})$ is (w-s) (short for "weakly supercyclic"). If $S(\delta_{A,V^*}) \neq \varnothing$, and $A \in PF(\delta)$, then $A = A_u \oplus A_c$. Since the compression of a (w-s) operator to the orthogonal complement of an invariant subspace of the operator is again (w-s), $A_c$ is a (w-s) contraction. If $r(A_c) < 1$, then there exists a real number $0 < \rho < 1$ such that $\sigma(A) \subseteq \partial\mathbf{D} \cup \{\lambda : |\lambda| \leq \rho\}$, and this by [1, Lemma 3.3] is a contradiction. We prove next that $r(A_c) \neq 1$, which then leads us to conclude that either our hypothesis $S(\delta_{A,V^*}) \neq \varnothing$ is false or $A_c$ acts on the trivial space 0 (and hence $A$ is a unitary). Suppose then that $r(A_c) = 1$. Then, see above, $\alpha(A_c^* - \overline{\lambda}) > 0$ for a $\lambda \in \sigma_\pi(A_c)$ implies $A_c$ has a unitary summand, and hence we must have that $\alpha(A_c^* - \overline{\lambda}) = 0$. If $\sigma_\pi(A_c) \subset \text{iso}\sigma(A_c)$, then, upon letting $P$ denote the Riesz projection associated with the spectral set $\sigma_\pi(A_c)$, we have $\mathcal{H}_c = P(\mathcal{H}_c) \oplus (I-P)(\mathcal{H}_c)$. Recall now the fact that an invariant subspace of a (w-s) operator has co-dimension 1 or $\infty$. (This is a simple consequence of the fact that if $T$ is a (w-s) operator on a finite dimensional space, then it is supercyclic, and there are no supercyclic operators on a finite dimensional space of dimension greater than one.) Hence $\dim((\text{I}-\text{P})(\mathcal{H}_c))$ is either 1 or $\infty$. In either case $\sigma(A_c|_{(I-P)(\mathcal{H}_c)}) \neq \varnothing$ and $|\mu| < 1$ for every $\mu \in \sigma(A_c|_{(I-P)(\mathcal{H}_c)})$. Since this contradicts [1, Lemma 3.3] for $A_c$, we must have that there is a $\lambda \in \sigma_\pi(A_c)$ such that $\lambda \notin \text{iso}\sigma(A_c)$. But then $\overline{\lambda} \notin \text{iso}\sigma_a(A_c^*)$, and hence, given $\epsilon > 0$, every $\epsilon$-neighbourhood of $\overline{\lambda}$ contains an element of the point spectrum of $A_c^*$. Hence ($\sigma_\pi(A_c) = \varnothing$ forcing thereby that) $r(A_c) \neq 1$, and $\sigma(A) \subseteq \partial\mathbf{D} \cup \{\lambda : |\lambda| \leq \rho\}$ for some positive number $\rho < 1$. But then (in view of [1, Lemma 3.3]) we must have either that $A$ is (w-s) and $S(\delta_{A,V^*}) = \varnothing$ or $A_c$ acts on the trivial space 0 (so that $\sigma(A_c) = \varnothing$, $A$ is unitary and $\delta_{A,V^*}(I) = 0$, where $V$ is the isometry $V = A^*$). $\square$

Let $\triangle_{A,B} \in B(B(\mathcal{H}))$ denote the *elementary operator* $\triangle_{A,B}(X) = AXB - X$. Then an operator $A \in B(\mathcal{H})$ is said to satisfy property PF($\triangle$) if whenever the equation $AXV^* - X = 0$ holds for some isometry $V$ and operator $X \in B(\mathcal{H})$, then $A^*XV^* - X = 0$. It is known, [4, Theorem 2.4], that $A \in \text{PF}(\delta) \iff \text{A} \in \text{PF}(\triangle)$. If we let $S(\triangle_{A,V^*}) = \{V \in B(\mathcal{H}) : V \text{ isometric}, \triangle_{A,V^*}^{-1}(0) \neq \{0\}\}$, then $S(\triangle_{A,V^*}) \neq \varnothing \iff S(\delta_{A,V^*}) \neq \varnothing$; furthermore, there exists an isometry $V$ and an operator $X$ with dense range satisfying $\delta_{A,V^*}(X) = 0$ if and only if there exists an isometry $W$ and an operator $Y$ with dense range such that $AYW^* = Y$ (see the proof [4, Theorem 2.4]. Hence:

**Corollary 1** *Let $A$ be a contraction in $B(\mathcal{H}) \cap \text{PF}(\triangle)$.*

(a) *If $S(\triangle_{A,V^*}) \neq \varnothing$, then $A$ is not $n$-supercyclic.*

(b) *If $A$ is weakly supercyclic, then either*

  ($b_1$) *$S(\triangle_{A,V^*}) = \varnothing$, or,*

  ($b_2$) *$S(\triangle_{A,V^*}) \neq \varnothing$ and $A$ is unitary.*

Contractions belonging to a subclass $\mathcal{S} \subset B(\mathcal{H})$ may possess property PF($\delta$) (hence also property PF($\triangle$)) without operators (not necessarily contraction operators) in $\mathcal{S}$ possessing the property [4]. However, since an $A \in B(\mathcal{H})$ is $n$-supercyclic (or, weakly supercyclic) implies $rA$ is $n$-supercyclic (resp., weakly supercyclic) for every non-zero scalar $r$, the theorem (above) has an analogue for (general) operators in $B(\mathcal{H})$.

**Corollary 2** *Let $A \in B(\mathcal{H})$ be such that the contraction operator $B = \frac{1}{||A||}A \in \text{PF}(\delta)$.*

(a) *If $S(\delta_{B,V^*}) \neq \varnothing$, then $A$ is not $n$-supercyclic.*

(b) *If $A$ is weakly supercyclic, then either*

(b$_1$) $S(\delta_{B,V^*}) = \varnothing$*, or,*

(b$_2$) $S(\delta_{B,V^*}) \neq \varnothing$ *and $A$ is a scalar multiple of a unitary operator.*

## References


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