ON EXTENSIONS OF BILINEAR MAPS

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ABSTRACT. The paper deals with extension of bounded bilinear maps. It gives a necessary and sufficient condition for extending a bounded bilinear map on the Cartesian product of subspaces of Banach spaces. This leads to a full characterization for extension of bounded bilinear maps on the Cartesian product of arbitrary subspaces of Hilbert spaces. Applications concerning projective tensor products are also investigated.

1. INTRODUCTION

The purpose of this paper is to prove an extension result for bilinear maps. This will be presented in Theorem 5.2. It gives a necessary and sufficient condition for a bounded bilinear map to be extended from the Cartesian product $\mathcal{M} \times \mathcal{N}$ of linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ to the Cartesian product $\mathcal{X} \times \mathcal{Y}$ of the Banach spaces. Such conditions are imposed on the linear manifolds only. This leads to a complete unconditional statement for the extension of bounded bilinear maps acting on the Cartesian product of subspaces of Hilbert spaces. Applications related to extensions of bounded linear transformations on projective tensor products are considered as well.

The paper is organized as follows. Notation and terminology are set in Section 2. A brief review on the bilinear extension problem is considered in Section 3. Auxiliary results required in the sequel are brought together in Section 4. The main results are treated in Section 5, followed by an application in Section 6.

2. NOTATION AND TERMINOLOGY

In the present context the terms *forms* and *functionals*, *bounded linear* and *continuous linear*, *bounded bilinear* and *continuous bilinear*, are pairwise synonyms and we use both forms freely; and $\mathbb{F}$ denotes either the real or the complex field.

All linear spaces are over the same field $\mathbb{F}$. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be nonzero linear spaces. A *bilinear map* $\phi \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a function from the Cartesian product $\mathcal{X} \times \mathcal{Y}$ of linear spaces to a linear space $\mathcal{Z}$ whose sections are linear transformations. In other words, let $\phi^y = \phi(\cdot, y) = \phi|_{\mathcal{X} \times \{y\}} \colon \mathcal{X} \to \mathcal{Z}$ be the y-section of the bilinear map ϕ and let $\phi_x = \phi(x, \cdot) = \phi|_{\{x\} \times \mathcal{Y}} \colon \mathcal{Y} \to \mathcal{Z}$ be the x-section of ϕ. These functions ϕ^y and ϕ_x are linear transformations. From now on suppose $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are normed spaces. If two normed spaces $\mathcal{X}$ and $\mathcal{Y}$ are isometrically isomorphic, and if $y \in \mathcal{Y}$ is the isometrically isomorphic image of $x \in \mathcal{X}$, then write $\mathcal{X} \cong \mathcal{Y}$ and $x \cong y$. By a subspace $\mathcal{M}$ of a normed space $\mathcal{X}$ we mean a *closed* linear manifold of $\mathcal{X}$ (equipped with the norm inherited from $\mathcal{X}$). If $\mathcal{M}$ is a linear manifold of $\mathcal{X}$, then $\mathcal{M}^-$ will denote its

closure in $\mathcal{X}$. A bilinear map is bounded if $\sup_{0 \neq x \in \mathcal{X},\, 0 \neq y \in \mathcal{Y}} \frac{\|\phi(x,y)\|}{\|x\|\,\|y\|}$ is finite. In this case set $\|\phi\| = \sup_{0 \neq x \in \mathcal{X},\, 0 \neq y \in \mathcal{Y}} \frac{\|\phi(x,y)\|}{\|x\|\,\|y\|}$ so that $\|\phi(x,y)\| \leq \|\phi\|\,\|x\|\,\|y\|$ for every $(x,y) \in \mathcal{X} \times \mathcal{Y}$. This defines a norm on the linear space of all bounded bilinear maps.

A bilinear map is continuous (regarding the product topology in $\mathcal{X} \times \mathcal{Y}$) if and only if it is bounded. Let $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$ denote the normed space of all bonded bilinear maps $\phi \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, and let $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ denote the normed space of all bounded linear transformations $T \colon \mathcal{X} \to \mathcal{Y}$. The range of $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ (notation: $\mathcal{R}(T) = T(\mathcal{X}) = \mathrm{range}\,(T)$) is a linear manifold of $\mathcal{Y}$, and its kernel (notation: $\mathcal{N}(T) = T^{-1}(\{0\}) = \mathrm{kernel}\,(T)$) is a subspace of $\mathcal{X}$. Let $\mathcal{X}^* = \mathcal{B}[\mathcal{X}, \mathbb{F}]$ be the dual of $\mathcal{X}$. If one of $\mathcal{X}$ or $\mathcal{Y}$ is a Banach space, then ϕ lies in $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$ if and only if ϕ^y lies in $\mathcal{B}[\mathcal{X}, \mathcal{Z}]$ and ϕ_x lies in $\mathcal{B}[\mathcal{Y}, \mathcal{Z}]$. Both $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$ and $\mathcal{B}[\mathcal{X}, \mathcal{Z}]$ are Banach spaces if and only if $\mathcal{Z}$ is. (For properties on bounded bilinear maps see, e.g., [6, Section 1.1, p.8].)

The algebraic tensor product of linear spaces $\mathcal{X}$ and $\mathcal{Y}$ is a linear space $\mathcal{X} \otimes \mathcal{Y}$ associated with a bilinear map $\theta \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y}$ whose range spans $\mathcal{X} \otimes \mathcal{Y}$ with the following property: for every linear map $\phi \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ into any linear space $\mathcal{Z}$ there exists a (unique) linear transformation $\Phi \colon \mathcal{X} \otimes \mathcal{Y} \to \mathcal{Z}$ for which the diagram

$$
\begin{array}{ccc}
\mathcal{X} \times \mathcal{Y} & \xrightarrow{\ \phi\ } & \mathcal{Z} \\[2pt]
{\scriptstyle\theta}\searrow & & \uparrow{\scriptstyle\Phi} \\[2pt]
& \mathcal{X} \otimes \mathcal{Y} &
\end{array}
$$

commutes. Set $x \otimes y = \theta(x,y)$ for each $(x,y) \in \mathcal{X} \times \mathcal{Y}$. These are the *single tensors*. An element F in $\mathcal{X} \otimes \mathcal{Y}$ is a finite sum $\sum_i x_i \otimes y_i$ of single tensors. (For an exposition on algebraic tensor products see, e.g., [15].)

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. Two reasonable crossnorms on $\mathcal{X} \otimes \mathcal{Y}$ are the *injective* $\| \cdot \|_\vee$ and *projective* $\| \cdot \|_\wedge$ norms, which are given by

$$
\|F\|_\vee = \sup_{\|f\| \leq 1,\, \|g\| \leq 1,\, f \in \mathcal{X}^*,\, g \in \mathcal{Y}^*} \Big| \sum_i f(x_i)\, g(y_i) \Big|,
$$

$$
\|F\|_\wedge = \inf_{\{x_i\},\, \{y_i\},\, F = \sum_i x_i \otimes y_i} \sum_i \|x_i\|\,\|y_i\|,
$$

for every $F = \sum_i x_i \otimes y_i \in \mathcal{X} \otimes \mathcal{Y}$. Let $\mathcal{X} \otimes_\vee \mathcal{Y}$ and $\mathcal{X} \otimes_\wedge \mathcal{Y}$ denote a tensor product space $\mathcal{X} \otimes \mathcal{Y}$ equipped with $\| \cdot \|_\vee$ or $\| \cdot \|_\wedge$. Their completions, denoted by $\mathcal{X} \widehat{\otimes}_\vee \mathcal{Y}$ and $\mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y}$, are referred to as the *injective and projective tensor products*. (For expositions on the Banach spaces $\mathcal{X} \widehat{\otimes}_\vee \mathcal{Y}$ and $\mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y}$ see, e.g., [10], [6], [18], [8].)

3. Preliminaries

A very brief review of previous results on bilinear extension under restrictions.

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be normed spaces over the same field and let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Although there is no Hahn–Banach Theorem for bilinear functionals, variants of it may hold if extra assumptions are imposed. Consider the general case of a bilinear map $\phi \colon \mathcal{M} \times \mathcal{N} \to \mathcal{Z}$ taking values in a Banach space $\mathcal{Z}$ rather than in $\mathbb{F}$. Having in mind the extension of ϕ to $\widehat{\phi} \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ acting on $\mathcal{X} \times \mathcal{Y}$, extra assumptions may be placed on any of the four players involved, namely, the normed spaces $\mathcal{X}$ and $\mathcal{Y}$, the linear manifolds $\mathcal{M}$ and $\mathcal{N}$, the bounded

bilinear map ϕ, and the Banach space $\mathcal{Z}$. The first goal in the present paper is to generalize an extension for bounded bilinear forms proposed in [9], which will be discussed in Section 5. Other ways to face the bounded bilinear extension problem was to assume $\mathcal{X} = \mathcal{Y}$ and $\mathcal{Z} = \mathbb{F}$ (i.e., bilinear or multilinear forms rather than maps). Quite often it is assumed $\mathcal{M} = \mathcal{N}$. Along this line, the role played by the embedding of $\mathcal{M}$ into $\mathcal{X}$ was analyzed in [4]; the case of $\mathcal{X} = L^1$ was presented in [5], [7]; properties of the bilinear (multilinear) form itself were investigated in [11]; the restriction operator taking bounded bilinear forms on $\mathcal{X} \times \mathcal{X}$ into bounded bilinear forms into $\mathcal{M} \times \mathcal{M}$ was considered in [3]; a characterization of the sequence space c_0 in terms of extendible bilinear forms was given [2]; and a necessary and sufficient condition for bilinear forms to be extendible was presented in [1] making a connection with an integral representation of bounded bilinear functionals identified with bounded linear functionals on the injective tensor product (see, e.g., [8, Proposition 1.1.21]). All these were done for bilinear (multilinear) forms (rather than for $\mathcal{Z}$-valued bilinear maps), having in mind the quest for conditions (or particularizations) ensuring Hahn-Banach type results for continuous bilinear functionals.

4. Auxiliary Results

A subspace of a normed space is *complemented* if it has a subspace as an algebraic complement. A normed space is *complemented* if every subspace of it is complemented. If a Banach space is complemented, then it is isomorphic (i.e., topologically isomorphic) to a Hilbert space, and so complemented Banach spaces are identified with Hilbert spaces [16] (see also [12]). We will need the following well-known results, one on complemented subspaces and the other on bilinear maps.

Proposition 4.1. *Let $\mathcal{M}$ be a subspace of an normed space $\mathcal{X}$. If there exists a continuous projection $E \colon \mathcal{X} \to \mathcal{X}$ with $\mathcal{R}(E) = \mathcal{M}$, then $\mathcal{M}$ is complemented. The converse holds if $\mathcal{X}$ is a Banach space.*

Proof. See, e.g., [14, Remark A.4] among many others. $\qquad\square$

Proposition 4.2. *For an arbitrary triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of normed spaces,*

$$\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}] \cong \mathcal{B}[\mathcal{X}, \mathcal{B}[\mathcal{Y}, \mathcal{Z}]].$$

Proof. For $\mathcal{Z} = \mathbb{F}$ and with $\cong$ standing for isometric isomorphism we get

$$\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}] \cong \mathcal{B}[\mathcal{X}, \mathcal{Y}^*].$$

See [6, Section 1.4, p.9]. An extension from $\mathbb{F}$ to $\mathcal{Z}$ follows the same argument. $\quad\square$

If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, then $\mathcal{M} \otimes_{\vee} \mathcal{N}$ is a linear manifold of $\mathcal{X} \otimes_{\vee} \mathcal{Y}$, and consequently $\mathcal{M} \widehat{\otimes}_{\vee} \mathcal{N}$ is a subspace of the injective tensor product $\mathcal{X} \widehat{\otimes}_{\vee} \mathcal{Y}$. In general, this fails for the projective norm $\| \cdot \|_{\wedge}$ of the projective tensor product $\mathcal{X} \widehat{\otimes}_{\wedge} \mathcal{Y}$. Indeed, as is readily verified, $\|F\|_{\mathcal{X} \otimes_{\wedge} \mathcal{Y}} \leq \|F\|_{\mathcal{M} \otimes_{\wedge} \mathcal{N}}$ for every F in $\mathcal{M} \otimes_{\wedge} \mathcal{N}$, the inequality may be strict, and $\mathcal{M} \otimes_{\wedge} \mathcal{N}$ is a subspace of $\mathcal{X} \otimes_{\wedge} \mathcal{Y}$ if and only if $\|F\|_{\mathcal{X} \otimes_{\wedge} \mathcal{Y}} = \|F\|_{\mathcal{M} \otimes_{\wedge} \mathcal{N}}$ for every F in $\mathcal{M} \otimes_{\wedge} \mathcal{N}$. We will need the following well-known results on projective tensor products.

Proposition 4.3. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are complemented subspaces of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\mathcal{M} \otimes_{\wedge} \mathcal{N}$ is a linear manifold of the normed space $\mathcal{X} \otimes_{\wedge} \mathcal{Y}$, then it is a subspace. If $\mathcal{M} = \mathcal{R}(E)$ and $\mathcal{N} = \mathcal{R}(P)$ for projections E in $\mathcal{B}[\mathcal{X}, \mathcal{X}]$ and P in $\mathcal{B}[\mathcal{Y}, \mathcal{Y}]$ with $\|E\| = \|P\| = 1$, then $\mathcal{M} \otimes_{\wedge} \mathcal{N}$ is a linear manifold of $\mathcal{X} \otimes_{\wedge} \mathcal{Y}$.*

Proof. See, e.g., [18, Proposition 2.4].   □

Proposition 4.4. *For an arbitrary triple* $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ *of Banach spaces,*

$$\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}] \cong \mathcal{B}[\mathcal{X} \widehat{\otimes}_{\wedge} \mathcal{Y}, \mathcal{Z}].$$

Proof. This is the *universal mapping principle*. See, e.g., [8, Theorem 1.1.8].   □

Proposition 4.5. *An alternate expression for the projective norm*:

$$\|F\|_{\wedge} = \sup_{\|\phi\| \leq 1, \, \phi \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \, \mathbb{F}]} \left| \sum_i \phi(x_i, y_i) \right| \quad \text{for every} \quad F = \sum_i x_i \otimes y_i \in \mathcal{X} \otimes_{\wedge} \mathcal{Y}.$$

Proof. See, e.g., [18, p.23]. This follows in part by Proposition 4.4 and the fact that, if $\widehat{\mathcal{X}}$ is a completion of $\mathcal{X}$ and $\mathcal{Z}$ is a Banach space, then $\mathcal{B}[\widehat{\mathcal{X}}, \mathcal{Z}] \cong \mathcal{B}[\mathcal{X}, \mathcal{Z}]$. Hence

$$\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}] \cong (\mathcal{X} \widehat{\otimes}_{\wedge} \mathcal{Y})^* \cong (\mathcal{X} \otimes_{\wedge} \mathcal{Y})^*.$$ □

5. AN EXTENSION FOR BOUNDED BILINEAR MAPS

If one imposes appropriate restrictions on the linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{X}$ and $\mathcal{Y}$, then we get (i) the next extension result for bounded bilinear forms due to [9], and (ii) an extension of it for bilinear maps as in the subsequent theorem.

Theorem 5.1. [9, Corollary 2]. *If* $\mathcal{M}$ *and* $\mathcal{N}$ *are subspaces of Banach spaces* $\mathcal{X}$ *and* $\mathcal{Y}$, *respectively, and if there exists a projection of norm one of* $\mathcal{X}$ *onto* $\mathcal{M}$ *and a projection of norm one of* $\mathcal{Y}$ *onto* $\mathcal{N}$, *then every bounded bilinear functional on* $\mathcal{M} \times \mathcal{N}$ *can be extended to* $\mathcal{X} \times \mathcal{Y}$ *with the same norm.*

It is convenient to summarize Hayden's original proof.

A Sketch of Proof of Theorem 5.1 [9].

Part 1. Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ such that for every $\phi \in \boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathbb{F}]$ there is $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}]$ for which $\widehat{\phi}|_{\mathcal{M} \times \mathcal{N}} = \phi$ and $\|\widehat{\phi}\| = \|\phi\|$. By Proposition 4.2, $\boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathbb{F}] \cong \mathcal{B}[\mathcal{M}, \mathcal{N}^*]$ and $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}] \cong \mathcal{B}[\mathcal{X}, \mathcal{Y}^*]$. This ensures the existence of $\widehat{T} \in \mathcal{B}[\mathcal{X}, \mathcal{Y}^*]$ for every $T \in \mathcal{B}[\mathcal{M}, \mathcal{N}^*]$ with $\widehat{T}|_{\mathcal{M}} = T$ and $\|\widehat{T}\| = \|T\|$.

Part 2. Suppose Part 1 holds for $\mathcal{M} = \mathcal{N}^*$. If $T = I$, then take $\widehat{T} = E$, the continuous projection with $\mathcal{R}(E) = \mathcal{M}$ and $\|E\| = 1$.

Part 3. If there is a continuous projection $E \colon \mathcal{X} \to \mathcal{X}$ with $\mathcal{R}(E) = \mathcal{M}$ and $\|E\| = 1$, and if $\phi \in \boldsymbol{b}[\mathcal{M} \times \mathcal{Y}, \mathbb{F}]$ for some Banach space $\mathcal{Y}$, then $\widehat{\phi}(x, y) = \phi(Ex, y)$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ defines $\widehat{\phi}$ in $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}]$ such that $\widehat{\phi}|_{\mathcal{M} \times \mathcal{N}} = \phi$ and $\|\widehat{\phi}\| = \|\phi\|$.

Part 4. It can be verified that Parts 2 and 3 ensure the following statement. If $\mathcal{M}$ is a subspace of a Banach space $\mathcal{X}$, and $\mathcal{M} = \mathcal{N}^*$ for some Banach space $\mathcal{N}$, and if for every $\phi \in \boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathbb{F}]$ there exists $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X} \times \mathcal{N}, \mathbb{F}]$ such that $\widehat{\phi}|_{\mathcal{M} \times \mathcal{N}} = \phi$ and $\|\widehat{\phi}\| = \|\phi\|$, then for every Banach space $\mathcal{Y}$ and every $\phi \in \boldsymbol{b}[\mathcal{M} \times \mathcal{Y}, \mathbb{F}]$ there exists $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}]$ such that $\widehat{\phi}|_{\mathcal{M} \times \mathcal{Y}} = \phi$ and $\|\widehat{\phi}\| = \|\phi\|$,

Part 5. The statement of Theorem 5.1 can be shown to be a corollary of Part 4.   □

Theorem 5.2 below extends the result from [9] restated in Theorem 5.1 by offering a necessary and sufficient condition, and showing that the norm-one condition is required for the norm inequality only (not for the extension) and, moreover, Theorem 5.2 holds for bilinear maps in general (rather than for bilinear functionals).

Theorem 5.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$.*

(a) *Every bounded bilinear map $\phi\colon \mathcal{M}\times\mathcal{N} \to \mathcal{Z}$ into an arbitrary Banach space $\mathcal{Z}$ has a bounded bilinear extension $\widehat{\phi}\colon \mathcal{X}\times\mathcal{Y} \to \mathcal{Z}$ over $\mathcal{X}\times\mathcal{Y}$ if and only if the closures $\mathcal{M}^-$ and $\mathcal{N}^-$ of $\mathcal{M}$ and $\mathcal{N}$ are complemented subspaces of $\mathcal{X}$ and $\mathcal{Y}$.*

(b) *Moreover, if $\mathcal{M}^- = \mathcal{R}(E)$ and $\mathcal{N}^- = \mathcal{R}(P)$ for some projections $E \in \mathcal{B}[\mathcal{X},\mathcal{X}]$ and $P \in \mathcal{B}[\mathcal{Y},\mathcal{Y}]$ with $\|E\| = \|P\| = 1$, then $\|\widehat{\phi}\| = \|\phi\|$.*

Proof. (a) Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero linear manifolds of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. Let $\mathcal{M}^-$ and $\mathcal{N}^-$ be their closures. Take an arbitrary Banach space $\mathcal{Z}$ and an arbitrary $\phi \in \boldsymbol{b}[\mathcal{M}\times\mathcal{N}, \mathcal{Z}]$. The following assertions are equivalent.

(a$_1$) There exists $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X}\times\mathcal{Y}, \mathcal{Z}]$ such that $\widehat{\phi}|_{\mathcal{M}\times\mathcal{N}} = \phi$.

(a$_2$) $\mathcal{M}^-$ and $\mathcal{N}^-$ are complemented subspaces of $\mathcal{X}$ and $\mathcal{Y}$.

(a$_1$)$\Rightarrow$(a$_2$). Consider the linear manifolds $\mathcal{M}$ and $\mathcal{N}$. Take an arbitrary nonzero bounded linear functional $g\colon \mathcal{N} \to \mathbb{F}$ and an arbitrary nonzero bounded linear transformation $T\colon \mathcal{M} \to \mathcal{Z}$ (whose existences are ensured by the Hahn–Banach Theorem), and consider the map $\phi\colon \mathcal{M}\times\mathcal{N} \to \mathcal{Z}$ defined by

$$\phi(u,v) = g(v)Tu \quad \text{for every} \quad (u,v) \in \mathcal{M}\times\mathcal{N}.$$

Since g and T are both linear and bounded, it is readily verified that ϕ is bilinear and bounded (with $\|\phi(u,v)\| \le \|g\|\|T\|\|v\|\|u\|$ for every $u \in \mathcal{M}$ and $v \in \mathcal{N}$). Thus $\phi \in \boldsymbol{b}[\mathcal{M}\times\mathcal{N}, \mathcal{Z}]$. If (a$_1$) holds, then consider its extension $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X}\times\mathcal{Y}, \mathcal{Z}]$, and let $\phi^v \in \mathcal{B}[\mathcal{M}, \mathcal{Z}]$ and $\widehat{\phi}^v \in \mathcal{B}[\mathcal{X}, \mathcal{Z}]$ be their v-sections for each $v \in \mathcal{N} \subseteq \mathcal{Y}$. Thus for every $u \in \mathcal{M}$ and each $v \in \mathcal{N}$,

$$\widehat{\phi}^v|_{\mathcal{M}}(u) = \widehat{\phi}(u,v) = \widehat{\phi}|_{\mathcal{M}\times\mathcal{N}}(u,v) = \phi(u,v) = \phi^v(u).$$

Fix an arbitrary $v \in \mathcal{N}$ for which $g(v) \ne 0$ and set $0 \ne \alpha = g(v) \in \mathbb{F}$. Hence

$$\widehat{\phi}^v|_{\mathcal{M}}(\cdot) = \phi^v(\cdot) = \alpha T(\cdot) \in \mathcal{B}[\mathcal{M}, \mathcal{Z}].$$

Thus $\alpha T = \phi^v \in \mathcal{B}[\mathcal{M}, \mathcal{Z}]$ has an extension $\widehat{\alpha T} = \widehat{\phi}^v \in \mathcal{B}[\mathcal{X}, \mathcal{Z}]$. Then there exists $\widehat{T} \in \mathcal{B}[\mathcal{X}, \mathcal{Z}]$ for which $\widehat{T}|_{\mathcal{M}} = T$. Since $\mathcal{X}$ is a Banach space and since this holds for every nonzero $T \in \mathcal{B}[\mathcal{M}, \mathcal{Z}]$ and every Banach space $\mathcal{Z}$, then $\mathcal{M}^-$ is complemented (see, e.g., [17, Theorem 3.2.17]). Similarly, by taking an arbitrary nonzero bounded linear functional $f\colon \mathcal{M} \to \mathbb{F}$ and an arbitrary nonzero bounded linear transformation $S\colon \mathcal{N} \to \mathcal{Z}$, and defining $\psi \in \boldsymbol{b}[\mathcal{M}\times\mathcal{N}, \mathcal{Z}]$ by

$$\psi(u,v) = f(u)Sv \quad \text{for every} \quad (u,v) \in \mathcal{M}\times\mathcal{N},$$

the same argument ensures that $\mathcal{N}^-$ is complemented. Hence (a$_2$) holds.

(a$_2$)$\Rightarrow$(a$_1$). Consider the normed spaces $\mathcal{M}$ and $\mathcal{N}$. The normed spaces $\boldsymbol{b}[\mathcal{M}\times\mathcal{N}, \mathcal{Z}]$ and $\mathcal{B}[\mathcal{M}, \mathcal{B}[\mathcal{N}, \mathcal{Z}]]$ are isometrically isomorphic for every normed space $\mathcal{Z}$,

$$\boldsymbol{b}[\mathcal{M}\times\mathcal{N}, \mathcal{Z}] \cong \mathcal{B}[\mathcal{M}, \mathcal{B}[\mathcal{N}, \mathcal{Z}]],$$

by Proposition 4.2, where the natural isometric isomorphism

$$\Im\colon \boldsymbol{b}[\mathcal{M}\times\mathcal{N}, \mathcal{Z}] \to \mathcal{B}[\mathcal{M}, \mathcal{B}[\mathcal{N}, \mathcal{Z}]]$$

that sends an arbitrary $\phi \in \boldsymbol{b}[\mathcal{M}\times\mathcal{N}, \mathcal{Z}]$ to $T = \Im(\phi)$ in $\mathcal{B}[\mathcal{M}, \mathcal{B}[\mathcal{N}, \mathcal{Z}]]$ is given by $T(u) = \phi_u \in \mathcal{B}[\mathcal{N}, \mathcal{Z}]$ for each $u \in \mathcal{M}$. Therefore

$$\phi(u,v) = T(u)(v) \quad \text{for every} \quad (u,v) \in \mathcal{M}\times\mathcal{N}.$$

Suppose $\mathcal{Z}$ is a Banach space (which implies that $\mathcal{B}[\mathcal{N}, \mathcal{Z}]$ is a Banach space) in order to allow extension by continuity of uniformly continuous functions on dense sets. Consider the extension by continuity $\widetilde{T} \in \mathcal{B}[\mathcal{M}^-, \mathcal{B}[\mathcal{N}, \mathcal{Z}]]$ of $T \in \mathcal{B}[\mathcal{M}, \mathcal{B}[\mathcal{N}, \mathcal{Z}]]$. For each vector $\widetilde{u} \in \mathcal{M}^-$ consider the extension by continuity $\widetilde{T}^\sim(\widetilde{u}) \in \mathcal{B}[\mathcal{N}^-, \mathcal{Z}]$ of $\widetilde{T}(\widetilde{u}) \in \mathcal{B}[\mathcal{N}, \mathcal{Z}]$, defining a transformation $\widetilde{T}^\sim \in \mathcal{B}[\mathcal{M}^-, \mathcal{B}[\mathcal{N}^-, \mathcal{Z}]]$ such that

$$\widetilde{T}^\sim(u)(v) = T(u)(v) \quad \text{for every} \quad (u, v) \in \mathcal{M} \times \mathcal{N}.$$

Applying the above isometric isomorphic argument (i.e., Proposition 4.2) again,

$$\boldsymbol{b}[\mathcal{M}^- \times \mathcal{N}^-, \mathcal{Z}] \cong \mathcal{B}[\mathcal{M}^-, \mathcal{B}[\mathcal{N}^-, \mathcal{Z}]].$$

Let $\widetilde{\mathfrak{I}} \colon \boldsymbol{b}[\mathcal{M}^- \times \mathcal{N}^-, \mathcal{Z}] \to \mathcal{B}[\mathcal{M}^-, \mathcal{B}[\mathcal{N}^-, \mathcal{Z}]]$ be the natural isometric isomorphism. Set $\widetilde{\phi} = \widetilde{\mathfrak{I}}^{-1}(\widetilde{T}^\sim)$ in $\boldsymbol{b}[\mathcal{M}^- \times \mathcal{N}^-, \mathcal{Z}]$ for each $\widetilde{T}^\sim$ in $\mathcal{B}[\mathcal{M}^-, \mathcal{B}[\mathcal{N}^-, \mathcal{Z}]]$ so that $\widetilde{\phi}_{\widetilde{u}} = \widetilde{T}^\sim(\widetilde{u}) \in \mathcal{B}[\mathcal{N}^-, \mathcal{Z}]$ for each $\widetilde{u} \in \mathcal{M}^-$, and so

$$\widetilde{\phi}(\widetilde{u}, \widetilde{v}) = \widetilde{T}^\sim(\widetilde{u})(\widetilde{v}) \quad \text{for every} \quad (\widetilde{u}, \widetilde{v}) \in \mathcal{M}^- \times \mathcal{N}^-.$$

Thus $\widetilde{\phi}(u, v) = \widetilde{T}^\sim(u)(v) = T(u)(v) = \phi(u, v)$ for every $(u, v) \in \mathcal{M} \times \mathcal{N}$. Therefore

$$\widetilde{\phi}\big|_{\mathcal{M} \times \mathcal{N}} = \phi.$$

Now suppose (a$_2$) holds. By Proposition 4.1 this means there are projections E in $\mathcal{B}[\mathcal{X}, \mathcal{X}]$ and P in $\mathcal{B}[\mathcal{Y}, \mathcal{Y}]$ with $\mathcal{R}(E) = \mathcal{M}^-$ and $\mathcal{R}(E) = \mathcal{N}^-$. Set

$$\widehat{\phi}(x, y) = \widetilde{\phi}(Ex, Py) \quad \text{for every} \quad (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

As $E \colon \mathcal{X} \to \mathcal{X}$ and $P \colon \mathcal{Y} \to \mathcal{Y}$ are linear and bounded, and $\widetilde{\phi} \colon \mathcal{M}^- \times \mathcal{N}^- \to \mathcal{Z}$ is bilinear and bounded, then $\widehat{\phi} \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is bilinear and bounded (i.e., $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$). Also, as E and P act as the identity on $\mathcal{M} \subseteq \mathcal{R}(E)$ and $\mathcal{N} \subseteq \mathcal{R}(P)$, we get (a$_1$):

$$\widehat{\phi}\big|_{\mathcal{M} \times \mathcal{N}} = \widetilde{\phi}(E\big|_{\mathcal{M}}, P\big|_{\mathcal{N}}) = \widetilde{\phi}\big|_{\mathcal{M} \times \mathcal{N}} = \phi.$$

(b) Hence $\|\phi\| = \|\widehat{\phi}\big|_{\mathcal{M} \times \mathcal{N}}\| \leq \|\widehat{\phi}\| = \|\widetilde{\phi}(E(\cdot), P(\cdot))\| \leq \|\widetilde{\phi}\| \|E\| \|P\|$. Moreover, although there is no extension by continuity for bilinear maps,

$$\|\widetilde{\phi}\| = \sup_{0 \neq \widetilde{u} \in \mathcal{M}^-,\, 0 \neq \widetilde{v} \in \mathcal{N}^-} \frac{\|\phi(\widetilde{u}, \widetilde{v})\|}{\|\widetilde{u}\| \|\widetilde{v}\|} = \sup_{0 \neq \widetilde{u} \in \mathcal{M}^-,\, 0 \neq \widetilde{v} \in \mathcal{N}^-} \frac{\|\widetilde{T}^\sim(\widetilde{u})(\widetilde{v})\|}{\|\widetilde{u}\| \|\widetilde{v}\|} = \sup_{0 \neq u \in \mathcal{M},\, 0 \neq v \in \mathcal{N}} \frac{\|\widetilde{T}^\sim(u)(v)\|}{\|u\| \|v\|}$$

$$= \sup_{0 \neq u \in \mathcal{M},\, 0 \neq v \in \mathcal{N}} \frac{\|T(u)(v)\|}{\|u\| \|v\|} = \sup_{0 \neq u \in \mathcal{M},\, 0 \neq v \in \mathcal{N}} \frac{\|\phi(u, v)\|}{\|u\| \|v\|} = \|\phi\|.$$

Thus $\|\phi\| \leq \|\widehat{\phi}\| \leq \|\phi\| \|E\| \|P\|$. Then $\|E\| = \|P\| = 1$ implies $\|\widehat{\phi}\| = \|\phi\|$. $\qquad\square$

The converse of Theorem 5.2(b) fails: $\|\widehat{\phi}\| = \|\phi\|$ does not imply $\|E\| = \|P\| = 1$. For instance, $E \in \mathcal{B}[\mathbb{R}^2, \mathbb{R}^2]$ given by $E(x_1, x_2) = (0, x_1 + x_2)$ defines a projection on $\mathbb{R}^2$ with $\|E\| = \sqrt{2}$. Take $f \in \mathcal{B}[\mathbb{R}^2, \mathbb{R}]$ given by $f((x_1, x_2)) = x_2$ for $(x_1, x_2) \in \mathbb{R}^2$ so that $\|f\| = 1$. Take $\widehat{\phi} \in \boldsymbol{b}[\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}]$ given by $\widehat{\phi}(x, y) = f(x) f(y)$ for $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ so that $\|\widehat{\phi}\| = \|f\|^2$. Set $\mathcal{M} = \mathcal{R}(E)$. The restriction $\phi = \widehat{\phi}\big|_{\mathcal{M} \times \mathcal{M}} \in \boldsymbol{b}[\mathcal{M} \times \mathcal{M}, \mathbb{R}]$ is such that $\phi((0, \alpha), (0, \beta)) = \alpha\beta$ for $(0, \alpha), (0, \beta) \in \mathcal{M}$ and so $\|\phi\| = 1$.

If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then one gets a full extension with no restrictions on the linear manifolds $\mathcal{M}$ and $\mathcal{N}$, as expected.

Corollary 5.3. *Every bounded bilinear map $\phi \colon \mathcal{M} \times \mathcal{N} \to \mathcal{Z}$ defined on the Cartesian product of arbitrary linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of arbitrary Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ into an arbitrary Banach space $\mathcal{Z}$ has a bounded bilinear extension $\widehat{\phi} \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ over $\mathcal{X} \times \mathcal{Y}$ such that $\|\widehat{\phi}\| = \|\phi\|$.*

Proof. (a) Hilbert spaces are complemented. Thus every subspace of a Hilbert space is complemented, and so Theorem 5.2(a) applies to every linear manifold $\mathcal{M}$ of a Hilbert space $\mathcal{X}$ and to every linear manifold $\mathcal{N}$ of a Hilbert space $\mathcal{Y}$.

(b) If $\mathcal{X}$ ad $\mathcal{Y}$ are Hilbert spaces, then the orthogonal projections $E \in \mathcal{B}[\mathcal{X}, \mathcal{X}]$ with $\mathcal{R}(E) = \mathcal{M}^-$ and $P \in \mathcal{B}[\mathcal{Y}, \mathcal{Y}]$ with $\mathcal{R}(P) = \mathcal{N}^-$ are such that $\|E\| = \|P\| = 1$, and therefore $\|\widehat{\phi}\| = \|\phi\|$ by Theorem 5.2(b). $\qquad\square$

Extensions of linear transformations (or forms) are not unique, and so extensions of bilinear maps are not unique (since a product of linear forms is a bilinear form).

Remark 5.4. Consider the following classes of operators on a Banach space $\mathcal{X}$.

$$\Gamma_R[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}, \mathcal{X}] \colon \mathcal{R}(T)^- \text{ is a complemented subspace of } \mathcal{X}\big\},$$
$$\Gamma_N[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}, \mathcal{X}] \colon \mathcal{N}(T) \text{ is a complemented subspace of } \mathcal{X}\big\},$$
$$\Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}], \qquad \mathcal{S}\Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cup \Gamma_N[\mathcal{X}].$$

The class $\Gamma[\mathcal{X}]$ is large enough. It includes, for instance, the class of all compact perturbations of semi-Fredholm operators (see, e.g., [13, Remark 5.1(b)]). A straightforward consequence of Theorem 5.2 leads to the following result.

Let $\mathcal{M}$ and $\mathcal{N}$ be linear manifolds of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ for which there are operators $T \in \mathcal{S}\Gamma[\mathcal{X}] \subseteq \mathcal{B}[\mathcal{X}, \mathcal{X}]$ and $S \in \mathcal{S}\Gamma[\mathcal{Y}] \subseteq \mathcal{B}[\mathcal{Y}, \mathcal{Y}]$ such that

$$\mathcal{M}^- = \mathcal{R}(T)^- \ \text{ or } \ \mathcal{M}^- = \mathcal{N}(T) \quad \text{and} \quad \mathcal{N}^- = \mathcal{R}(S)^- \ \text{ or } \ \mathcal{N}^- = \mathcal{N}(S),$$

according to whether T lies in $\Gamma_R[\mathcal{X}]$ or $\Gamma_N[\mathcal{X}]$ and S lies in $\Gamma_R[\mathcal{Y}]$ or $\Gamma_N[\mathcal{Y}]$. If $\phi \colon \mathcal{M} \times \mathcal{N} \to \mathcal{Z}$ is a bounded bilinear map into an arbitrary Banach space $\mathcal{Z}$, then there exists a bounded bilinear extension $\widehat{\phi} \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ of ϕ over $\mathcal{X} \times \mathcal{Y}$.

6. An Application to Projective Tensor Products

If $\mathcal{M}$ and $\mathcal{N}$ are linear manifolds of linear spaces $\mathcal{X}$ and $\mathcal{Y}$, then $\mathcal{M} \otimes \mathcal{N}$ is a regular linear manifold of the linear space $\mathcal{X} \otimes \mathcal{Y}$. If $\mathcal{X} \otimes \mathcal{Y}$ is equipped with the injective norm, then $\mathcal{M} \otimes_\vee \mathcal{N}$ is a linear manifold of $\mathcal{X} \otimes_\vee \mathcal{Y}$. However, this is not always the case if $\mathcal{X} \otimes \mathcal{Y}$ is equipped with the projective norm. The next corollaries give necessary and sufficient conditions for $\mathcal{M} \otimes_\wedge \mathcal{N}$ to be a linear manifold of $\mathcal{X} \otimes_\wedge \mathcal{Y}$. Moreover, extensions of bounded linear transformations on regular linear manifolds of projective tensor products also comes as another consequence of Theorem 5.2.

Corollary 6.1. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be arbitrary Banach spaces, let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and consider the following assertions.*

(o) *$\mathcal{M}$ and $\mathcal{N}$ are complemented in $\mathcal{X}$ and $\mathcal{Y}$ with $\mathcal{M} = \mathcal{R}(E)$ and $\mathcal{N} = \mathcal{R}(P)$ for projections $E \in \mathcal{B}[\mathcal{X}, \mathcal{X}]$ and $P \in \mathcal{B}[\mathcal{Y}, \mathcal{Y}]$ such that $\|E\| = \|P\| = 1$.*

(a) *$\mathcal{M} \otimes_\wedge \mathcal{N}$ is a linear manifold of $\mathcal{X} \otimes_\wedge \mathcal{Y}$.*

(b) *Every bounded bilinear map $\phi \colon \mathcal{M} \times \mathcal{N} \to \mathcal{Z}$ has a bounded bilinear extension $\widehat{\phi} \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ with $\|\widehat{\phi}\| = \|\phi\|$.*

(c) *Every bounded linear transformation $T \colon \mathcal{M} \otimes_\wedge \mathcal{N} \to \mathcal{Z}$ has a bounded linear extension $\widetilde{T} \colon \mathcal{X} \otimes_\wedge \mathcal{Y} \to \mathcal{Z}$ with $\|\widetilde{T}\| = \|T\|$.*

(d) *Every bounded linear transformation $\overline{T} \colon \mathcal{M} \widehat{\otimes}_\wedge \mathcal{N} \to \mathcal{Z}$ has a bounded linear extension $\widehat{T} \colon \mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y} \to \mathcal{Z}$ with $\|\widehat{T}\| = \|\overline{T}\|$.*

The above assertions are related as follows:

$$(\text{o}) \Longrightarrow (\text{a,b}), \qquad (\text{a,b}) \Longleftrightarrow (\text{c}) \Longleftrightarrow (\text{d}).$$

Proof. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be arbitrary Banach spaces.

(o) $\Longrightarrow$ (a,b). If (o) holds, then according to Proposition 4.3 and Theorem 5.2,

    (a) $\mathcal{M} \otimes_{\wedge} \mathcal{N}$ is a linear manifold of $\mathcal{X} \otimes_{\wedge} \mathcal{Y}$
        (or, equivalently, $\mathcal{M} \otimes_{\wedge} \mathcal{N}$ is a subspace of $\mathcal{X} \otimes_{\wedge} \mathcal{Y}$),   and

    (b) every $\phi \in \boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathcal{Z}]$ into an arbitrary Banach space $\mathcal{Z}$ has an extension
        $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$ with $\|\widehat{\phi}\| = \|\phi\|$.

(a,b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d). Each extension in (c) or (d) only makes sense if (a) holds. According to Proposition 4.4

$$\mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}] \cong \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}].$$

The natural isometric isomorphism between them,

$$\mathcal{J} : \mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}] \to \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$$

such that $\psi = \mathcal{J}(S)$ for $S \in \mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}]$ and $\psi \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}.\mathcal{Z}]$, is given by

$$\psi(x, y) = \mathcal{J}(S)(x, y) = S(x \otimes y) \quad \text{and so} \quad S(x \otimes y) = \mathcal{J}^{-1}(\psi)(x \otimes y) = \psi(x, y)$$

for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$. By Theorem 5.2(a) $\mathcal{M}$ and $\mathcal{N}$ are complemented if and only if every ϕ in $\boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathcal{Z}]$ has an extension $\widehat{\phi}$ in $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$. Thus in this case with T in $\mathcal{B}[\mathcal{M} \otimes_{\wedge} \mathcal{N}, \mathcal{Z}]$ and $\widetilde{T}$ in $\mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}]$ being the isometrically isomorphic images of ϕ in $\boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathcal{Z}]$ and $\widehat{\phi}$ in $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$,

$$T \cong \phi = \widehat{\phi}|_{\mathcal{M} \times \mathcal{N}} \quad \text{and} \quad \widehat{\phi} \cong \widetilde{T}.$$

As the restriction $\widetilde{T}|_{\mathcal{M} \otimes_{\wedge} \mathcal{N}} \in \mathcal{B}[\mathcal{M} \otimes_{\wedge} \mathcal{N}, \mathcal{Z}]$ of $\widetilde{T} \in \mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}]$ to $\mathcal{M} \otimes_{\wedge} \mathcal{N}$ only makes sense if (a) holds, then in this case with $\phi = \mathcal{J}(T)$ and $\widehat{\phi} = \mathcal{J}(\widetilde{T})$,

$$T(u \otimes v) = \mathcal{J}^{-1}(\phi)(u \otimes v) = \phi(u, v) = \widehat{\phi}|_{\mathcal{M} \times \mathcal{N}}(u, v) = \widehat{\phi}(u, v) = \mathcal{J}(\widetilde{T})(u, v)$$
$$= \widetilde{T}(u \otimes v) = \widetilde{T}|_{\mathcal{M} \otimes_{\wedge} \mathcal{N}}(u \otimes v) \quad \text{for every} \quad (u \otimes v) \in \mathcal{M} \otimes_{\wedge} \mathcal{N}.$$

Therefore $T(F) = \widetilde{T}|_{\mathcal{M} \otimes_{\wedge} \mathcal{N}}(F)$ for every $F = \sum_i u_i \otimes v_i \in \mathcal{M} \otimes_{\wedge} \mathcal{N}$. Hence

$$\widehat{\phi}|_{\mathcal{M} \times \mathcal{N}} = \phi \quad \text{implies} \quad \widetilde{T}|_{\mathcal{M} \otimes_{\wedge} \mathcal{N}} = T.$$

    Thus *if every $\phi \in \boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathcal{Z}]$ has an extension $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$ and (a) holds, then every $T \in \mathcal{B}[\mathcal{M} \otimes_{\wedge} \mathcal{N}, \mathcal{Z}]$ has an extension $\widetilde{T} \in \mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}]$.*

A symmetric argument ensures the converse:

    if every $T \in \mathcal{B}[\mathcal{M} \otimes_{\wedge} \mathcal{N}, \mathcal{Z}]$ has an extension $\widetilde{T} \in \mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}]$, then every $\phi \in \boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathcal{Z}]$ has an extension $\widehat{\phi} \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathcal{Z}]$ and (a) holds.

Suppose either (c) or (d) holds, and so (a) holds. Thus the completions $\mathcal{M} \widehat{\otimes}_{\wedge} \mathcal{N}$ of $\mathcal{M} \otimes_{\wedge} \mathcal{N}$ and $\mathcal{X} \widehat{\otimes}_{\wedge} \mathcal{Y}$ of $\mathcal{X} \otimes_{\wedge} \mathcal{Y}$ are such that $\mathcal{M} \widehat{\otimes}_{\wedge} \mathcal{N}$ is a subspace of $\mathcal{X} \widehat{\otimes}_{\wedge} \mathcal{Y}$. Since $\mathcal{Z}$ is a Banach space,

$$\mathcal{B}[\mathcal{M} \widehat{\otimes}_{\wedge} \mathcal{N}, \mathcal{Z}] \cong \mathcal{B}[\mathcal{M} \otimes_{\wedge} \mathcal{N}, \mathcal{Z}] \quad \text{and} \quad \mathcal{B}[\mathcal{X} \widehat{\otimes}_{\wedge} \mathcal{Y}, \mathcal{Z}] \cong \mathcal{B}[\mathcal{X} \otimes_{\wedge} \mathcal{Y}, \mathcal{Z}].$$

Again, let $\overline{T}$ in $\mathcal{B}[\mathcal{M} \widehat{\otimes}_{\wedge} \mathcal{N}, \mathcal{Z}]$ be the extension over completion of T in $\mathcal{B}[\mathcal{M} \otimes_{\wedge} \mathcal{N}, \mathcal{Z}]$ so that $\overline{T} \cong T$, and let $\widehat{T}$ in $\mathcal{B}[\mathcal{X} \widehat{\otimes}_{\wedge} \mathcal{Y}, \mathcal{Z}]$ be the extension over completion of $\widetilde{T}$ in

$\mathcal{B}[\mathcal{X} \otimes_\wedge \mathcal{Y}, \mathcal{Z}]$ so that $\widehat{T} \cong \widetilde{T}$. Extensions over completions are unique up to isometric isomorphism. Then $\widehat{T}$ in $\mathcal{B}[\mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y}, \mathcal{Z}]$ extends $\overline{T}$ in $\mathcal{B}[\mathcal{M} \widehat{\otimes}_\wedge \mathcal{N}, \mathcal{Z}]$ if and only if $\widetilde{T}$ in $\mathcal{B}[\mathcal{X} \otimes_\wedge \mathcal{Y}, \mathcal{Z}]$ extends T in $\mathcal{B}[\mathcal{M} \otimes_\wedge \mathcal{N}, \mathcal{Z}]$:

$$\widetilde{T}|_{\mathcal{M} \otimes_\wedge \mathcal{N}} \cong \widehat{T}|_{\mathcal{M} \widehat{\otimes}_\wedge \mathcal{N}} = \overline{T} \cong T \quad \text{and} \quad \widehat{T}|_{\mathcal{M} \widehat{\otimes}_\wedge \mathcal{N}} \cong \widetilde{T}|_{\mathcal{M} \otimes_\wedge \mathcal{N}} = T \cong \overline{T}.$$

Hence *every $\overline{T} \in \mathcal{B}[\mathcal{M} \widehat{\otimes}_\wedge \mathcal{N}, \mathcal{Z}]$ has an extension $\widehat{T} \in \mathcal{B}[\mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y}, \mathcal{Z}]$ if and only if every $T \in \mathcal{B}[\mathcal{M} \otimes_\wedge \mathcal{N}, \mathcal{Z}]$ has an extension $\widetilde{T} \in \mathcal{B}[\mathcal{X} \otimes_\wedge \mathcal{Y}, \mathcal{Z}]$.*

Also, since $\widehat{\phi} \cong \widetilde{T}$, $\phi \cong T$, $\widetilde{T} \cong \widehat{T}$, and $T \cong \overline{T}$, if one of the norms (b), (c), or (d) coincide, then so does the others. $\qquad\square$

A particular case with a rather simplified statement is immediately obtained by fixing $\mathcal{Z} = \mathbb{F}$ in Corollary 6.1 as follows. (This extends [18, Proposition 2.11]).

Corollary 6.2. *If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, then the following assertions are pairwise equivalent.*

(a) *$\mathcal{M} \otimes_\wedge \mathcal{N}$ is a linear manifold of $\mathcal{X} \otimes_\wedge \mathcal{Y}$.*

(b) *Every bounded bilinear form $\phi \colon \mathcal{M} \times \mathcal{N} \to \mathbb{F}$ has a bounded bilinear extension $\widehat{\phi} \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{F}$ with $\|\widehat{\phi}\| = \|\phi\|$.*

(c) *Every bounded linear form $f \colon \mathcal{M} \otimes_\wedge \mathcal{N} \to \mathbb{F}$ has a bounded linear extension $\widetilde{f} \in \mathcal{X} \otimes_\wedge \mathcal{Y}$ with $\|\widetilde{f}\| = \|f\|$.*

(d) *Every bounded linear form $\overline{f} \colon \mathcal{M} \widehat{\otimes}_\wedge \mathcal{N} \to \mathbb{F}$ has a bounded linear extension $\widehat{f} \colon \mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y} \to \mathbb{F}$ with $\|\widehat{f}\| = \|\overline{f}\|$.*

Proof. If (b) holds, then the set of all bilinear forms in $\boldsymbol{b}[\mathcal{M} \times \mathcal{N}, \mathbb{F}]$ with norm less than 1 is included in the set of all restrictions to $\mathcal{M} \times \mathcal{N}$ of all bilinear forms in $\boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}]$ with norm less that 1. Then by Proposition 4.5

$$\|F\|_{\mathcal{M} \otimes_\wedge \mathcal{N}} \leq \sup_{\|\psi\| \leq 1, \, \psi \in \boldsymbol{b}[\mathcal{X} \times \mathcal{Y}, \mathbb{F}]} \left| \sum_i \psi(u_i, v_i) \right| = \|F\|_{\mathcal{X} \otimes_\wedge \mathcal{Y}}.$$

Since $\|F\|_{\mathcal{X} \otimes_\wedge \mathcal{Y}} \leq \|F\|_{\mathcal{M} \otimes_\wedge \mathcal{N}}$, then $\|F\|_{\mathcal{M} \otimes_\wedge \mathcal{N}} = \|F\|_{\mathcal{X} \otimes_\wedge \mathcal{Y}}$ which is equivalent to (a). So (b) implies (a). Conversely, (a) implies (c) by the Hahn–Banach Theorem. Also, (c) implies (b) and (c) is equivalent to (d) by Corollary 6.1. $\qquad\square$

According to Corollary 5.3, in a Hilbert-space setting each assertion in Corollaries 6.1 and 6.2 holds true.

REFERENCES

1. A. Bowers, *Representation of extendable bilinear forms*, Math. Slovaca **60** (2015), 1123–1136.
2. D. Carando and P. Sevilla-Peris, *Extendibility of bilinear forms on Banach sequence spaces*, Israel J. of Math. **199** (2014), 941–954.
3. J.M.F. Castillo, R. García, A. Defant, D.Pérez-García, and J. Suárez, *Local complementation and the extension of bilinear mappings*, Math. Proc. Camb. Phil. Soc. **152** (2012), 132–166.
4. J.M.F. Castillo, R. García, and J.A. Jaramillo, *Extension of bilinear forms on Banach spaces*, Proc. Amer. Math. Soc. **129** (2001), 3647–3656.
5. J.M.F. Castillo, R. García, and J.A. Jaramillo, *Extension of bilinear forms from subspaces of $\mathcal{L}_1$-spaces*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 91–96.
6. A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam, 1993.
7. A. Defant, D. García, M. Maestre, and D. Pérez-García, *Extension of multilinear forms and polynomials from subspaces of $\mathcal{L}_1$-space*, Houston J. Math. **33** (2007), 839–860.

8. J. Diestel, J.H. Fourie, and J. Swart, *The Metric Theory of Tensor Products – Grothendieck's Résumé Revisited*, American Mathematical Society, Providence, 2008.

9. T.L. Hayden, *The extension of bilinear functionals*, Pacific J. Math. **22** (1967), 99–108.

10. H. Jarchow, *Locally Convex Spaces*, B.G. Teubner, Stuttgart, 1981.

11. H. Jarchow, C. Palazuelos, D. Pérez-García, and I. Villanueva, *Hahn–Banach extension of multilinear forms and summability*, J. Math. Anal. Appl. **336** (2007), 1161–1177.

12. N.J. Kalton, *The complemented subspace problem revisited*, Studia Math. **188** (2008), 223–257.

13. C.S. Kubrusly, *Range-kernel complementation*, Studia Sci. Math. Hungar. **55** (2018), 327–344.

14. C.S. Kubrusly, *Spectral Theory of Bounded Linear Operators*, Birkhäuser-Springer-Switzerland, Cham, 2020.

15. C.S. Kubrusly, *Algebraic tensor products revisited: Axiomatic approach*, Bull. Malays. Math. Sci. Soc. **44** (2021), 2335–2355.

16. J. Lindenstrauss and L. Tzafriri, *On the complemented subspaces problem*, Israel J. Math. **9** (1971), 263–269.

17. R.E. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 1998.

18. R.A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer, London, 2002.

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