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ASYMPTOTIC LIMITS, BANACH LIMITS, AND CESÀRO MEANS

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Abstract. Every new inner product in a Hilbert space is obtained from the original one by means of a unique positive operator. The first part of the paper is a survey on applications of such a technique, including a characterization of similarity to isometries. The second part focuses on Banach limits for dealing with power bounded operators. It is shown that if a power bounded operator for which the sequence of shifted Cesàro means converges (at least in the weak topology) uniformly in the shift parameter, then it has a Cesàro asymptotic limit coinciding with its φ-asymptotic limit for all Banach limits φ.

1. Introduction

The purpose of this paper is twofold. It is a survey with an expository flavor linking the notions in the title, and also includes original results.

The paper is split into two parts, both dealing with bounded linear operators on a Hilbert space. The first part (Sections 3, 4 and 5) surveys the technique of generating a new inner product from the original one, and its applications to similarity to isometries and asymptotic limit for contractions, emphasizing the common role played by the equation $T^*A\,T = A$. The central results of this part appear in Propositions 4.1, 4.2, giving a comprehensive characterization of similarity to isometries.

The second part (Sections 6 and 7) is a follow-up of the first one, extending it to power bounded operators by means of the φ-asymptotic limit associated with a Banach limit φ and, alternatively, to bounded operators by means of Cesàro asymptotic limit associated with Cesàro means, still focusing on the role played by the equation $T^*A\,T = A$. Theorem 6.1 brings together a large collection of properties of φ-asymptotic limits for power bounded operators, as a generalization of analogous results for contractions. Similarly, Theorem 7.1 brings together a large collection of properties of Cesàro-asymptotic limits for bounded operators. Theorem 7.2 shows that if a power bounded operator is such that its sequence of Cesàro means converges in the weak topology, whose shifted sequences converge uniformly in the shift parameter, then its Cesàro asymptotic limit coincides with its φ-asymptotic limit for all Banach limits φ. This is followed by an application in Corollary 7.1.

2. Notation and Terminology

A linear transformation L on a linear space $\mathcal{X}$ is injective if and only if its kernel $\mathcal{N}(L) = L^{-1}(\{0\})$ is null (i.e., if and only if $\mathcal{N}(L) = \{0\}$). If $\mathcal{X}$ is a normed space, then let $\mathcal{B}[\mathcal{X}]$ stand for the normed algebra of all operators on $\mathcal{X}$ (i.e., of all bounded linear transformations of $\mathcal{X}$ into itself). If $T \in \mathcal{B}[\mathcal{X}]$, then $\mathcal{N}(T)$ is a subspace of $\mathcal{X}$, which means a *closed* linear manifold of $\mathcal{X}$. The range $\mathcal{R}(T) = T(\mathcal{X})$ of $T \in \mathcal{B}[\mathcal{X}]$ is

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a (not necessarily closed) linear manifold of $\mathcal{X}$. An operator T on a normed space $\mathcal{X}$ has a bounded inverse on its range if and only if it is bounded below. An operator T on a Banach space $\mathcal{X}$ is bounded below if and only if it is injective with a closed range (i.e., $\mathcal{N}(T)=\{0\}$ and $\mathcal{R}(T)=\mathcal{R}(T)^-$ where the upper bar denotes closure).

Let $\mathcal{X}$ be a normed space. An operator $T\in\mathcal{B}[\mathcal{X}]$ is an isometry if $\|Tx\|=\|x\|$ for every $x\in\mathcal{X}$ (a unitary operator is an invertible isometry on a Hilbert space). It is a contraction if $\|Tx\|\le\|x\|$ for every $x\in\mathcal{X}$ (i.e., $\|T\|\le 1$), and it is power bounded if $\sup_n\|T^n\|<\infty$. In this case set $\beta=\sup_n\|T^n\|$. Thus T is power bounded if there is a constant $\beta>0$ such that $\|T^nx\|\le\beta\|x\|$ for all integers $n\ge 1$ and every $x\in\mathcal{X}$, which implies $\sup_n\|T^nx\|<\infty$ for every $x\in\mathcal{X}$. The converse holds if $\mathcal{X}$ is a Banach space by the Banach–Steinhaus Theorem. Every isometry is a contraction and every contraction is power bounded. An operator $T\in\mathcal{B}[\mathcal{X}]$ is power bounded below if there is a constant $\alpha>0$ such that $\alpha\|x\|\le\|T^nx\|$ for all integers $n\ge 1$ and every $x\in\mathcal{X}$. A $\mathcal{B}[\mathcal{X}]$-valued sequence $\{S_n\}$ converges uniformly (or in the operator norm topology) to an operator $S\in\mathcal{B}[\mathcal{X}]$ if $\|(S_n-S)\|\to 0$ (notation: $S_n\xrightarrow{u}S$). It converges strongly to S if the $\mathcal{X}$-valued sequence $\{S_nx\}$ converges to Sx in the norm topology (i.e., $\|(S_n-S)x\|\to 0$) for every $x\in\mathcal{X}$ (notation: $S_n\xrightarrow{s}S$). The sequence $\{S_n\}$ converges weakly to $S\in\mathcal{B}[\mathcal{X}]$ if $f((S_n-S)x)\to 0$ for every f in the dual $\mathcal{X}^*$ of $\mathcal{X}$ and every x in $\mathcal{X}$ (notation: $S_n\xrightarrow{w}S$ — if $\mathcal{X}$ is a Hilbert space with inner product $\langle\cdot\,;\cdot\rangle$, weak convergence means $\langle(S_n-S)x\,;y\rangle\to 0$ for every $x,y\in\mathcal{X}$ by the Riesz Representation Theorem, which is equivalent to $\langle(S_n-S)x\,;x\rangle\to 0$ for every $x\in\mathcal{X}$ if the Hilbert space is complex by the Polarization Identity). Uniform convergence clearly implies strong convergence, which in turn implies weak convergence. An operator $T\in\mathcal{B}[\mathcal{X}]$ is of class $C_{0\cdot}$ if the power sequence $\{T^n\}$ converges strongly to the null operator, $T^nx\to 0$ for every $x\in\mathcal{X}$ (i.e., if T is strongly stable — notation: $T^n\xrightarrow{s}O$), and it is of Class $C_{1\cdot}$ if $T^nx\not\to 0$ for every $0\ne x\in\mathcal{X}$.

Suppose $\mathcal{X}$ is an inner product space with inner product $\langle\,\cdot\,;\cdot\,\rangle$. The norm induced by the inner product will be denoted by $\|\cdot\|$. If $\mathcal{X}$ is a Hilbert space and $T\in\mathcal{B}[\mathcal{X}]$, then $T^*\in\mathcal{B}[\mathcal{X}]$ denotes its (Hilbert-space) adjoint. A self-adjoint operator A (i.e., one for which $A^*=A$) is nonnegative or positive if, respectively, $0\le\langle Ax\,;x\rangle$ for every $x\in\mathcal{X}$ or $0<\langle Ax\,;x\rangle$ for every nonzero $x\in\mathcal{X}$ (notation: $A\ge O$ or $A>O$). A (self-adjoint) operator A is positive if and only if it is nonnegative and injective:

$$A>O \iff A\ge O \text{ and } \mathcal{N}(A)=\{0\}.$$

Injective self-adjoint operators have dense range (i.e., $\mathcal{R}(A)^-=\mathcal{X}$ whenever $\mathcal{N}(A)=\{0\}$ if $A^*=A$). Thus positive operators are injective with dense range. Hence a positive operator A on a Hilbert space $\mathcal{X}$ is bounded below if and only if its has a bounded inverse on its closed dense image, which in turn is equivalent to saying that it is injective and surjective, which means invertible (with a bounded inverse). Invertible positive operators are called strictly positive and denoted by $A\succ O$:

$$A\succ 0 \iff A>O \text{ has a bounded inverse on } \mathcal{X}.$$

A nonnegative operator A has a unique nonnegative square root $A^{\frac12}$ which is positive or strictly positive whenever A is (indeed, $\mathcal{N}(A^{\frac12})=\mathcal{N}(A)$ and $\mathcal{R}(A^{\frac12})^-=\mathcal{R}(A)^-$).

3. Generating a New Inner Product

Take a linear space $\mathcal{X}$, let $\langle\,\cdot\,;\cdot\,\rangle$ be an inner product on $\mathcal{X}$, suppose $(\mathcal{X},\langle\,\cdot\,;\cdot\,\rangle)$ is a Hilbert space, and let A be a nonnegative operator (i.e., $A\ge O$) on this Hilbert space. As is readily verified, the nonnegative operator A generates a new semi-inner product $\langle\,\cdot\,;\cdot\,\rangle_A$ on $\mathcal{X}$ defined for every $x, y \in \mathcal{X}$ by

$$\langle x\,;y\rangle_A = \langle Ax\,;y\rangle,$$

which induces a seminorm $\|\cdot\|_A$ on $\mathcal{X}$ given by

$$\|x\|_A = \langle Ax\,;x\rangle^{\frac{1}{2}} = \|A^{\frac{1}{2}}x\|.$$

This seminorm $\|\cdot\|_A$ becomes a norm whenever A is injective (i.e., $\mathcal{N}(A) = \{0\}$ or, equivalently, whenever the nonnegative A is positive). Consider the inner product space $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle_A)$. As defined in Section 2, $\mathcal{B}[\mathcal{X}]$ is the Banach algebra of all linear operators on the Hilbert space $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ which are bounded (i.e., continuous) with respect to the norm $\|\cdot\|$ induced by the inner product $\langle\,\cdot\,;\cdot\,\rangle$. If A is positive, then let $\mathcal{B}[\mathcal{X}]_A$ denote the normed algebra of all linear operators on the inner product space $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle_A)$ which are bounded (i.e., continuous) with respect to the new norm $\|\cdot\|_A$. In this case (i.e., if $A > O$ or, equivalently, if $\|\cdot\|_A$.is a norm), the following elementary result represents an appropriate starting point.

Proposition 3.1. *Let $A > O$ be an arbitrary positive operator on a Hilbert space $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ and consider the norm $\|\cdot\|_A$ induced by the inner product $\langle\,\cdot\,;\cdot\,\rangle_A = \langle A\,\cdot\,;\cdot\,\rangle$. The following assertions are equivalent.*

(a) *A is invertible* (i.e., *has bounded inverse on $\mathcal{X}$; equivalently, $A \succ O$*).

(b) *The norms $\|\cdot\|_A$ and $\|\cdot\|$ on $\mathcal{X}$ are equivalent.*

Proof. Take the Hilbert space $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ and let A be a positive operator on it. So

$$\|Ax\|^2 \le \|A^{\frac{1}{2}}\|^2\|A^{\frac{1}{2}}x\|^2 = \|A\|\,\|x\|_A^2 = \|A\|\,|\langle Ax\,;x\rangle| \le \|A\|\,\|Ax\|\,\|x\| \le \|A\|^2\|x\|^2$$

for every x in $\mathcal{X}$. Since A is positive, it is injective, and so A is bounded below if and only if it has a closed range. Since A is an injective self-adjoint, then $\mathcal{R}(A)^- = \mathcal{X}$. Thus the positive operator A is invertible (i.e., $A > O$ has a bounded inverse on $\mathcal{X}$ or, equivalently, A is bounded below and surjective) if and only if A is bounded below, which means $\alpha^2\|x\|^2 \le \|Ax\|^2$ for every $x \in \mathcal{X}$ and some $\alpha > 0$. Therefore

$$A \succ O \iff \tfrac{\alpha^2}{\|A\|}\|x\|^2 \le \|x\|_A^2 \le \|A\|\|x\|^2 \text{ for every } x \in \mathcal{X}, \text{ for some } \alpha > 0. \qquad \square$$

Perhaps the first result along this line is one ensuring that for every new inner product there is a positive operator generating it.This is a classical result from [33].

Lemma 3.1. *Let $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ be a Hilbert space, and let $[\,\cdot\,;\cdot\,]$ be a semi-inner product in $\mathcal{X}$. Then there exists a unique nonnegative operator $A \in \mathcal{B}[\mathcal{X}]$ for which*

$$[x\,;y] = \langle x\,;y\rangle_A = \langle Ax\,;y\rangle \quad \textit{for every } x, y \in \mathcal{X}.$$

If this unique $A \in \mathcal{B}[\mathcal{X}]$ is positive, then $[\,\cdot\,;\cdot\,]$ becomes an inner product in $\mathcal{X}$.

Proof. This is a particular case of a fundamental result for densely defined bounded sesquilinear forms $[\,\cdot\,;\cdot\,]$ in a Hilbert-space setting [33, Theorem 2.28, p.63]. In particular, if $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ is a Hilbert space, then the result holds for every Hermitian symmetric sesquilinear form inducing either a nonnegative or a positive quadratic form (i.e., it holds for every semi-inner or inner product $[\,\cdot\,;\cdot\,]$ on $\mathcal{X} \times \mathcal{X}$) according to whether A is nonnegative or positive, respectively. $\square$

The inner product space $(\mathcal{X},[\,\cdot\,;\,\cdot\,])$ may not be a Hilbert space if the positive A is not strictly positive. However, if A is invertible (i.e., if $A \succ O$), then the new norm generated by A is equivalent to the original one (by Proposition 3.1), and so $(\mathcal{X},\langle\,\cdot\,;\,\cdot\,\rangle_A)$ becomes a Hilbert space.

4. Similarity to an Isometry and the Equation $T^*AT = A$

Let the seminorm (norm) $\|\cdot\|_A$ be the one induced by the new semi-inner product (inner product) $\langle\cdot\,;\cdot\rangle_A = \langle A\cdot\,;\cdot\rangle$ as discussed in the previous section.

Proposition 4.1. *Let $\mathcal{X}$ be a Hilbert space. Take an arbitrary operator T in $\mathcal{B}[\mathcal{X}]$ and an arbitrary nonnegative operator A in $\mathcal{B}[\mathcal{X}]$.*

(a) *Since $A \ge O$, then $\|Tx\|_A = \|x\|_A$ for every $x \in \mathcal{X}$ if and only if $T^*AT = A$.*

(b) *If $A > O$, then T is an isometry in $\mathcal{B}[\mathcal{X}]_A$ if and only if $T^*AT = A$.*

(c) *If $A \succ O$ and $T^*AT = A$, then T is similar to an isometry.*

(d) *If T is similar to an isometry, then $T^*A'T = A'$ for some $O \prec A' \in \mathcal{B}[\mathcal{X}]$.*

Proof. (a) If $A \ge O$, then $\|\cdot\|_A$ is a semi-norm on $\mathcal{X}$. Since $T^*AT - A$ is self-adjoint, $\langle (T^*AT - A)x, x\rangle = 0$ for every $x \in \mathcal{X}$ if and only if $T^*AT = A$. Therefore, since

$$\|Tx\|_A^2 = \|A^{\frac{1}{2}}Tx\|^2 = \langle ATx\,;Tx\rangle = \langle T^*ATx, x\rangle \qquad \text{and} \qquad \langle Ax, x\rangle = \|A^{\frac{1}{2}}x\|^2 = \|x\|_A^2$$

for every $x \in \mathcal{X}$, we get the result in (a).

(b) If $A > O$, then $\|\cdot\|_A$ is a norm on $\mathcal{X}$ and therefore the identity $\|Tx\|_A = \|x\|_A$ for every $x \in \mathcal{X}$ means the operator T is an isometry in $\mathcal{B}[\mathcal{X}]_A$. Now apply (a).

(c) If $A \ge O$ and $T^*AT = A$, then for every $x \in \mathcal{X}$

$$\|A^{\frac{1}{2}}Tx\|^2 = \langle A^{\frac{1}{2}}Tx\,;A^{\frac{1}{2}}Tx\rangle = \langle T^*ATx\,;x\rangle = \langle Ax\,;x\rangle = \|A^{\frac{1}{2}}x\|^2.$$

If $A \succ O$, then $\|A^{\frac{1}{2}}TA^{-\frac{1}{2}}x\| = \|x\|$ for every $x \in \mathcal{X}$. So T is similar to an isometry.

(d) If $T \in \mathcal{B}[\mathcal{X}]$ is similar to an isometry, then there exists an invertible transformation W in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ (with a bounded inverse W^{-1} in $\mathcal{B}[\mathcal{Y},\mathcal{X}]$ for some Hilbert space $\mathcal{Y}$ unitarily equivalent to $\mathcal{X}$ by the Inverse Mapping Theorem since $\mathcal{X}$ is Banach) for which WTW^{-1} is an isometry in $\mathcal{B}[\mathcal{Y}]$. Since W is invertible, the polar decomposition of W is given by $W = U|W|$ where $U \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ is unitary and $|W| = (W^*W)^{\frac{1}{2}} \in \mathcal{B}[\mathcal{X}]$ is strictly positive. Thus $O \prec |W| = U^*W$. Since WTW^{-1} is an isometry on $\mathcal{Y}$, then $U^*WTW^{-1}U$ is an isometry on $\mathcal{X}$. Thus

$$\begin{aligned}\langle |W|^2x,;x\rangle &= \|\,|W|x\|^2 = \|U^*WTW^{-1}U|W|x\|^2\\ &= \|\,|W|TW^{-1}Wx\|^2 = \|\,|W|Tx\|^2 = \langle T^*|W|^2Tx\,;x\rangle\end{aligned}$$

for every $x \in \mathcal{X}$, and hence $T^*A'T = A'$ with $O \prec A' = |W|^2$. □

Let $T \in \mathcal{B}[\mathcal{X}]$ and $A \in \mathcal{B}[\mathcal{X}]$ be arbitrary operators on a Hilbert space $\mathcal{X}$. In accordance with Proposition 4.1(a), T was called an A-*isometry* in [35] if $T^*AT = A$ for some $A \ge O$. Similarly, T was called an A-*contraction* in [35] if $T^*AT \le A$ for some $A \ge O$ (see also [36, 37]). Actually, T is similar to a contraction if and only if $T^*AT \le A$ some $A \succ O$ (see e.g., [22, Corollary 1.8]). Still along these lines, if an operator A (not necessarily nonnegative) is such that $T^*AT = r(T)^2A$ for some

$T \in \mathcal{B}[\mathcal{X}]$, where $r(T)$ stands for the spectral radius of T, then A was called T-*Toeplitz* in [20] (recall: if T is power bounded and $\|T^n\| \not\to 0$, then $r(T) = 1$). For further applications of the new semi-inner product space $(\mathcal{X}, \langle \cdot\,;\cdot \rangle_A)$ along different lines from those discussed here see, e.g., [1] and the references therein.

Similarity to an isometry is equivalent to the equation $T^*A\,T = A$ for some $A \succ O$ (Proposition 4.1 (c,d)). This is still equivalent to some forms of power boundedness and power boundedness below (including Cesàro means forms). These are brought together in the next proposition.

Proposition 4.2. *Let T be an operator on a Hilbert space $\mathcal{X}$. The following assertions are pairwise equivalent.*

(a) *T is similar to an isometry.*

(b) *T is power bounded and power bounded below: there exist $\alpha, \beta > 0$ such that*

$$\alpha\|x\| \le \|T^k x\| \le \beta\|x\| \quad \textit{for all}\, k \ge 0 \textit{ and every}\, x \in \mathcal{X}.$$

(c) *There exist $\alpha, \beta > 0$ and an invertible $R \in \mathcal{B}[\mathcal{X}]$ for which*

$$\alpha\|x\|^2 \le \tfrac{1}{n}\sum_{k=0}^{n-1}\|R\,T^k x\|^2 \le \beta\|x\|^2 \quad \textit{for all}\, n \ge 1 \textit{ and every}\, x \in \mathcal{X}.$$

(d) *There exist $\alpha, \beta > 0$ for which*

$$\alpha\|x\|^2 \le \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^k x\|^2 \le \beta\|x\|^2 \quad \textit{for all}\, n \ge 1 \textit{ and every}\, x \in \mathcal{X}.$$

(e) *There exist $\alpha, \beta > 0$ and an invertible $R \in \mathcal{B}[\mathcal{X}]$ for which*

$$\alpha\|x\| \le \|R\,T^k x\| \le \beta\|x\| \quad \textit{for all}\, k \ge 0 \textit{ and every}\, x \in \mathcal{X}.$$

Proof. Suppose (c) holds. Since $\frac{1}{n+1}\|R\,T^n x\|^2 \le \frac{1}{n+1}\sum_{k=0}^{n}\|R\,T^k x\|^2 \le \beta\|x\|^2$, then

$$\begin{aligned}\tfrac{1}{n}\|R\,T^n x\|^2\sum_{k=0}^{n}\tfrac{1}{k+1} &= \tfrac{1}{n}\sum_{k=0}^{n}\tfrac{1}{k+1}\|R\,T^k T^{n-k}x\|^2 \le \tfrac{1}{n}\sum_{k=0}^{n}\beta\|T^{n-k}x\|^2\\ &= \tfrac{n+1}{n}\beta\tfrac{1}{n+1}\sum_{k=0}^{n}\|R^{-1}R\,T^k x\|^2 \le 2\,\beta^2\|R^{-1}\|^2\|x\|^2,\end{aligned}$$

and therefore $\sup_n \big\|\big(\frac{1}{n}\sum_{k=0}^{n}\frac{1}{k+1}\big)^{\frac{1}{2}}R\,T^n x\big\|^2 < \infty$ for every $x \in \mathcal{X}$. By the Banach–Steinhaus Theorem $\sup_n \frac{1}{n}\sum_{k=0}^{n}\frac{1}{k+1}\big\|R\,T^n\big\|^2 < \infty$ which implies $\frac{1}{n}\|R\,T^n\|^2 \to 0$ (as $\sum_{k=0}^{n}\frac{1}{1+k} \to \infty$). Thus since $\|T^{*n}R^*R\,T^n\| = \|R\,T^n\|^2$,

$$\tfrac{1}{n}\|T^{*n}R^*R\,T^n\| \to 0.$$

Now for each $n \ge 1$ consider the Cesàro mean

$$Q_n = \tfrac{1}{n}\sum_{k=0}^{n-1}T^{*k}R^*R\,T^k.$$

Since $\|Q_n^{\frac{1}{2}}x\|^2 = \langle Q_n x\,;x\rangle = \frac{1}{n}\sum_{k=0}^{n-1}\|R\,T^k x\|^2$, then $\alpha\|x\|^2 \le \|Q_n^{\frac{1}{2}}x\|^2 \le \beta\|x\|^2$, and hence $\{Q_n^{\frac{1}{2}}\}$ is a bounded sequence of strictly positive operators (and so is $\{Q_n\}$).

(i) First suppose the Hilbert space $\mathcal{X}$ is separable. In this case the bounded sequence $\{Q_n\}$ has a weakly convergent subsequence (see, e.g., [24, Theorem 5.70]). But the cone of nonnegative operators is weakly closed in $\mathcal{B}[\mathcal{X}]$ and so the weak limit of any weakly convergent subsequence of $\{Q_n\}$ is again a nonnegative operator in $\mathcal{B}[\mathcal{X}]$.

Let the nonnegative $Q \in \mathcal{B}[\mathcal{X}]$ be the weak limit of a weakly convergent subsequence of $\{Q_n\}$. Actually, Q is strictly positive because $\{Q_n\}$ is bounded below. Since

$$T^*Q_nT = Q_n + \tfrac{1}{n}(R^*R - T^{*n}R^*RT^n)$$

for each $n \geq 1$, and since $\frac{1}{n}\|T^{*n}R^*RT^n\| \to 0$, we get (again) the equation

$$T^*QT = Q.$$

So T is similar to an isometry (i.e., (a) holds) by Proposition 4.1(c). In other words, with $O \prec A = Q$ the equation $T^*AT = A$ implies T is similar to an isometry.

(ii) Next suppose the Hilbert space $\mathcal{X}$ is not separable. Since (c) holds, T is nonzero. Take an arbitrary $0 \neq x \in \mathcal{X}$ and set $\mathcal{M}_x = \operatorname{span}\big(\bigcup_{m,n\geq 0}\{T^mT^{*n}x\} \cup \{T^nT^{*m}x\}\big)^-$ which is a separable nontrivial (closed) subspace of the (nonseparable) Hilbert space $\mathcal{X}$ including x and reducing T. Then both $T|_{\mathcal{M}_x}$ and $(T|_{\mathcal{M}_x})^* = T^*|_{\mathcal{M}_x}$ act on the separable Hilbert space $\mathcal{M}_x$. Thus since $T|_{\mathcal{M}_x}$ satisfies (c) — because T does — then according to (i) $T|_{\mathcal{M}_x}$ is similar to an isometry. Consider the collection $\Im = \{\bigoplus \mathcal{M}_x : x \in \mathcal{X}\}$ of all orthogonal direct sums of these subspaces, which is partially ordered (in the inclusion ordering) and is not empty (if $0 \neq y \in {\mathcal{M}_x}^\perp$, then $\mathcal{M}_y \subseteq {\mathcal{M}_x}^\perp$ because ${\mathcal{M}_x}^\perp$ reduces T and so $\mathcal{M}_x \perp \mathcal{M}_y$). Moreover, every chain in $\Im$ has an upper bound in $\Im$ (the union of all orthogonal direct sums in a chain of orthogonal direct sums in $\Im$ is again an orthogonal direct sum in $\Im$). Thus Zorn's Lemma ensures that $\Im$ has a maximal element, say $\mathcal{M} = \bigoplus \mathcal{M}_x$, which coincides with $\mathcal{X}$ (otherwise it would not be maximal since $\mathcal{M} \oplus \mathcal{M}^\perp = \mathcal{X}$). As $T|_{\mathcal{M}_x}x = Tx$, then $T = \bigoplus T|_{\mathcal{M}_x}$ on $\mathcal{X} = \bigoplus \mathcal{M}_x$ is similar to an isometry since each $T|_{\mathcal{M}_x}$ on $\mathcal{M}_x$ is similar to an isometry according to item (i) above. Thus again (a) holds. Hence

$$\text{(c)} \implies \text{(a)}.$$

Now if (a) holds, then (as in the proof of (d) in Proposition 4.1) there exists an invertible transformation $W \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ for which $\|WT^kW^{-1}y\| = \|y\|$ for every $y \in \mathcal{Y}$, equivalently, $\|WT^kx\| = \|Wx\|$ for every $x \in \mathcal{X}$, for all $k \geq 0$. Therefore $\|T^kx\| = \|W^{-1}WT^kx\| \leq \|W^{-1}\|\,\|WT^kx\| = \|W^{-1}\|\,\|Wx\| \leq \|W^{-1}\|\,\|W\|\,\|x\|$ and $\|x\| = \|W^{-1}Wx\| \leq \|W^{-1}\|\,\|Wx\| = \|W^{-1}\|\,\|WT^kx\| \leq \|W^{-1}\|\,\|W\|\,\|T^kx\|$ so that $\|W^{-1}\|^{-1}\|W\|^{-1}\|x\| \leq \|T^kx\| \leq \|W^{-1}\|\,\|W\|\,\|x\|$ for all k for every x. Hence

$$\text{(a)} \implies \text{(b)}.$$

This concludes the proof since (b) $\Longrightarrow$ (d) $\Longrightarrow$ (c) $\Longleftarrow$ (e) $\Longleftarrow$ (b) holds trivially. □

The positive numbers α and β are constant with respect to x but of course they may depend on T. Parts of Proposition 4.2 have appeared in [21, Theorem 2], [31, Proposition 6.3], [15, Theorem 2], [22, Proposition 1.15], [2, Theorem 1], and [27, Corollary 4.2] (and parts of it — e.g., (a) $\Longleftrightarrow$ (b) — may survive in a normed-space setting). For further conditions on similarity to isometries along these lines see, e.g., [4, Proposition 2.6], [38, Theorem 2.1].

5. Contractions and the Equation $T^*AT = A$

The next proposition is a classical result on Hilbert-space contractions dating back to the early 1950s (see, e.g., [22, Chapter 3] and the references therein). It is based on a well-known result which says: *every monotone bounded sequence of Hilbert-space self-adjoint operators converges strongly.* If T is a Hilbert-space contraction, then $\{T^{*n}T^n\}$ is a bounded monotone sequence of self-adjoint operators (in fact a nonincreasing sequence of nonnegative operators) and so it converges strongly to a nonnegative contraction A:

$$\lim T^{*n}T^n = A \quad \text{strongly}$$

(i.e., $\lim_n \|(T^{*n}T^n - A)x\| = 0$ for every x). Such a nonnegative contraction A is usually refereed to as the *asymptotic limit* of the contraction T. So if T is a contraction, then the strong limit $A \geq O$ of $\{T^{*n}T^n\}$ (i.e., the asymptotic limit of T) defines a new semi-inner product $\langle\,\cdot\,;\,\cdot\,\rangle_A$ on $\mathcal{X}$ which becomes an inner product if $A > O$, and in this case T acts as an isometry on $(\mathcal{X}, \langle\,\cdot\,;\,\cdot\,\rangle_A)$ by Proposition 4.1(b).

Proposition 5.1. *For every contraction T on a Hilbert space $\mathcal{X}$ there exists a unique nonnegative operator A on $\mathcal{X}$ for which*

(a) $$T^{*n}T^n \xrightarrow{s} A,$$

and so

$$\langle x\,;y\rangle_A = \lim_n \langle T^{*n}T^n x\,;y\rangle = \lim_n \langle T^n x\,;T^n y\rangle = \langle Ax\,;y\rangle$$

for every $x, y \in \mathcal{X}$ or, equivalently,

$$\|T^n x\| \to \|A^{\frac{1}{2}}x\| \quad \text{for every } x \in \mathcal{X},$$

where $\|A^{\frac{1}{2}}x\| = \|x\|_A = \|T^j x\|_A$ for every $x \in \mathcal{X}$ and every $j \geq 0$.

Moreover:

(b) $O \leq A \leq I$ (i.e., *A is a nonnegative contraction on $\mathcal{X}$*).

(c) $T^*AT = A$. *Equivalently,*
$\|A^{\frac{1}{2}}T^n x\| = \|A^{\frac{1}{2}}x\|$ *for every $x \in \mathcal{X}$ and every $n \geq 1$.*

(d) $A \neq O \implies \|A\| = \|T\| = 1$.

(e) $AT = O \iff TA = O \iff A = O$.

(f) $AT = TA \iff A = A^2$.

(g) $(I - A)T^n \xrightarrow{s} O$ *and so* $(I - A^{\frac{1}{2}})T^n \xrightarrow{s} O$.

(h) $\|AT^n x\| \to \|A^{\frac{1}{2}}x\|$ *for every $x \in \mathcal{X}$.*

(i) $\mathcal{N}(A) = \{x \in \mathcal{X} \colon T^n x \to 0\}$ (*so* $T^n \xrightarrow{s} O \iff A = O$).

(j) $\mathcal{N}(I - A) = \{x \in \mathcal{X} \colon \|T^n x\| = \|x\| \ \forall n \geq 1\}$ (*so T is an isometry* $\iff A = I$).

Furthermore,

$$A \text{ is invertible} \iff T \text{ is similar to an isometry.}$$

Proof. See, e.g., [22, Propositions 3.1, 3.2 and 3.8] — see also [8], [30], [6], [28]. □

Remark 5.1. According to Proposition 5.1(i) the strong limit A of $\{T^{*n}T^n\}$ for a Hilbert-space contraction T is positive if and only if T is a contraction of class $C_{1\cdot}$:

(a) $$T \text{ is a } C_{1\cdot}\text{-contraction} \iff \mathcal{N}(A) = \{0\} \iff A > O.$$

Since $A^2 = A > O$ implies $A = I$, the above equivalence and Proposition 5.1(j) ensures

(b) $$T \text{ is a } C_{1\cdot}\text{-contraction with } A = A^2 \iff T \text{ is an isometry.}$$

For a collection of properties of asymptotic limits for Hilbert-space contractions see, for instance, [40, Section I.10], [9, Section 6], [22, Chapter 3], [10], [25]. For the new inner product $\langle\cdot\,;\cdot\rangle_A$ generated by the asymptotic limit A of a $C_{1\cdot}$-contraction T (i.e., for a positive A or, in particular, for a strictly positive A as in Propositions 4.1(c) and 4.2) see, for instance, [16], [22, Remark 3.9] and [23].

6. Power Bounded Operators and the Equation $T^*AT = A$

The existence of Banach limits was established by Banach himself [3, p.21] as a consequence of the Hahn–Banach Theorem. Let ℓ_+^∞ denote the Banach space of all complex-valued bounded sequences equipped with its usual sup-norm. A Banach limit is any *bounded linear* functional $\varphi\colon \ell_+^\infty \to \mathbb{C}$ (i.e., $\varphi \in \ell_+^{\infty *}$, where $\ell_+^{\infty *}$ is the dual of ℓ_+^∞) assigning a complex number to each complex-valued bounded sequence which satisfies the following properties. Take $\{\xi_n\} \in \ell_+^\infty$.

(o) φ is linear (i.e., additive and homogenous),

(i) φ is real (i.e., $\varphi(\{\xi_n\}) \in \mathbb{R}$ whenever $\{\xi_n\}$ is real-valued),

(ii) φ is positive (i.e., $0 \le \varphi(\{\xi_n\})$ whenever $0 \le \xi_n$ for every n),

(iii) φ is order-preserving (i.e., $\varphi(\{\xi_n\}) \le \varphi(\{\upsilon_n\})$ if $\xi_n \le \upsilon_n$ in $\mathbb{R}$ for every n),

(vi) φ is backward-shift-invariant (i.e., $\varphi(\{\xi_{n+1}\}) = \varphi(\{\xi_n\})$),

(v) $\liminf_n \xi_n \le \varphi(\{\xi_n\}) \le \limsup_n \xi_n$ for every real-valued sequence $\{\xi_n\}$,

(vi) φ assigns to a convergent sequence its limit (i.e., $\xi_n \to \xi \Longrightarrow \varphi(\{\xi_n\}) = \xi$) (in particular, $\varphi(\{1,1,1,\dots\}) = 1$),

(vii) $\|\varphi\| = 1$.

For existence of Banach limits see, e.g., [5, Section III.7] or [24, Problem 4.66]). Moreover, for every Banach limit φ there exist Banach limits φ_+ and φ_- such that

$$\lim_n \inf_j \tfrac{1}{n}\sum_{k=0}^{n-1}\xi_{k+j} = \varphi_-(\{\xi_n\}) \le \varphi(\{\xi_n\}) \le \varphi_+(\{\xi_n\}) = \lim_n \sup_j \tfrac{1}{n}\sum_{k=0}^{n-1}\xi_{k+j}$$

for an arbitrary real-valued sequence $\{\xi_n\} \in \ell_+^\infty$, where $\varphi_-(\{\xi_n\})$ and $\varphi_+(\{\xi_n\})$ are the minimum and maximum values of Banach limits at $\{\xi_n\}$, respectively [34, Theorem (β,γ)]; and for every $\xi \in [\varphi_-(\{\xi_n\}), \varphi_+(\{\xi_n\})]$ there exists a Banach limit φ' for which $\varphi'(\{\xi_n\}) = \xi$ (see also [32, (1.1)]). Actually, all Banach limits coincide on a real-valued sequence if and only if their shifted Cesàro means converge uniformly in the shifted parameter. In other words, if $\{\xi_n\}$ is a real-valued sequence, then

$$\varphi(\{\xi_n\}) = \xi \;\text{ for all Banach limits }\; \varphi \iff \lim_n \tfrac{1}{n}\sum_{k=0}^{n-1}\xi_{k+j} = \xi \;\text{ uniformly in } j.$$

[29, Theorem 1] (see also [34, Theorem (δ)]). We will refer to the above displayed results as Lorentz characterizations. Also, and consequently, since φ_- and φ_+ are Banach limits, then for every real-valued sequence $\{\xi_n\}$,

$$\liminf_n \xi_n \le \lim_n \inf_j \tfrac{1}{n}\sum_{k=0}^{n-1}\xi_{k+j} \le \lim_n \sup_j \tfrac{1}{n}\sum_{k=0}^{n-1}\xi_{k+j} \le \limsup_n \xi_n.$$

The Banach limit technique for power bounded operators discussed here is well-known and has been applied quite often (see, e.g., [17, 18, 19] for applications along the lines considered here).

Suppose$T\in\mathcal{B}[\mathcal{X}]$ is a power bounded operator (i.e., $\sup_n\|T^n\|<\infty$) acting on a Hilbert space $(\mathcal{X},\langle\,\cdot\,;\cdot\,\rangle)$. Let $\|\cdot\|\colon\mathcal{X}\to\mathbb{R}$ be the norm induced by the inner product $\langle\,\cdot\,;\cdot\,\rangle\colon\mathcal{X}\times\mathcal{X}\to\mathbb{C}$. Let $\varphi\colon\ell^\infty_+\to\mathbb{C}$ be an arbitrary Banach limit. Since T is power bounded, $\{\langle T^nx\,;T^ny\rangle\}\in\ell^\infty_+$ for each x,y in $\mathcal{X}$. Thus set

$$\langle x\,;y\rangle_\varphi=\varphi(\{\langle T^nx\,;T^ny\rangle\})=\varphi(\{\langle T^{*n}T^nx\,;y\rangle\})$$

for x,y in $\mathcal{X}$. Since every Banach limit is linear (which implies $\varphi(\overline{\{\xi_n\}})=\overline{\varphi}(\{\xi_n\})$) and positive (i.e., $0\le\varphi(\{\xi_n\})$ whenever $0\le\xi_n$ for every n), and since T is linear, then it is readily verified that $\langle\cdot\,;\cdot\rangle_\varphi\colon\mathcal{X}\times\mathcal{X}\to\mathbb{C}$ is a semi-inner product on $\mathcal{X}$. Hence

$$\|x\|^2_\varphi=\varphi(\{\|T^nx\|^2\})$$

for every $x\in\mathcal{X}$ defines the seminorm $\|\cdot\|_\varphi\colon\mathcal{X}\to\mathbb{R}$ induced by the semi-inner product $\langle\cdot\,;\cdot\rangle_\varphi$. (Even in this case of a sequence of norms of powers of a power bounded operator, the squares in the above identity cannot be omitted due to the nonmultiplicativity of Banach limits). Since a Banach limit is order-preserving for real-valued bounded sequences (i.e., $\varphi(\{\xi_n\})\le\varphi(\{v_n\})$ if $\xi_n\le v_n$ in $\mathbb{R}$ for every n), and since $\varphi(\{1,1,1,\dots\})=1$, then as T is power bounded,

$$\|x\|_\varphi\le\sup_n\|T^n\|\|x\|$$

for every $x\in\mathcal{X}$. Since Banach limits are backward-shift-invariant,

$$\|Tx\|^2_\varphi=\varphi(\{\|(T^{n+1}x)\|^2\})=\varphi(\{\|(T^nx)\|^2\})=\|x\|^2_\varphi.$$

for every $x\in\mathcal{X}$. The above setup leads to a generalization of Proposition 5.1 from contractions to power bounded operators as in the forthcoming Theorem 6.1.

Remark 6.1. An important particular case. If a power bounded operator T is of class $C_{1\cdot}$ (i.e., $T^nx\not\to0$ for every $0\ne x\in\mathcal{X}$), then $0<\liminf_n\|T^nx\|$ for every $x\ne0$ (see, e.g., proof of Theorem 6.1(i) below). Any Banach limit φ is such that $\liminf_n\xi_n\le\varphi(\{\xi_n\})\le\limsup_n\xi_n$ for a real-valued bounded sequence $\{\xi_n\}$. Then $0<\varphi(\{\|T^nx\|\})=\|x\|_\varphi$ whenever $x\ne0$. So the seminorm $\|\cdot\|_\varphi$ becomes a norm and consequently the semi-inner product $\langle\,\cdot\,;\cdot\,\rangle_\varphi$ becomes an inner product. Conversely, if T is not of class $C_{1\cdot}$, then there is a nonzero $x\in\mathcal{X}$ for which $T^nx\to0$ and so $\limsup_n\|T^nx\|\to0$ which implies $\varphi(\{\|T^nx\|^2|\})=0$ so that $\|x\|_\varphi=0$, and the seminorm $\|\cdot\|_\varphi$ is not a norm. Thus if $\|\cdot\|_\varphi$ is a norm, then T is of class $C_{1\cdot}$. Hence

$$\langle\,\cdot\,;\cdot\,\rangle_\varphi\text{ is an inner product }\iff T\text{ is a power bounded of class }C_{1\cdot}.$$

Thus if T is a power bounded operator of class $C_{1\cdot}$ (with respect to the original norm $\|\cdot\|$), then (since $\|Tx\|_\varphi=\|x\|_\varphi$ for every $x\in\mathcal{X}$ as we saw above) the norm $\|\cdot\|_\varphi$ makes T into an isometry when acting on the inner product space $(\mathcal{X},\langle\,\cdot\,;\cdot\,\rangle_\varphi)$.

Proposition 5.1 can be extended from contractions to power bounded operators (in particular, to power bounded operators of class $C_{1\cdot}$). Given a power bounded operator T and a Banach limit φ, there is a unique nonnegative operator A_φ, referred to as the *φ-asymptotic limit* of T, such that $\langle\cdot\,;\cdot\rangle_\varphi=\langle A\cdot\,;\cdot\rangle$ by Lemma 3.1. So

$$\varphi(\{T^{*n}T^n\})=A_\varphi\quad\text{weakly}$$

(i.e., $\varphi(\{\langle T^{*n}T^nx\,;y\rangle\})=\langle A_\varphi x\,;y\rangle$ for every x,y). The next theorem rounds up a collection of properties (either well-known or not) of the φ-asymptotic limit A_φ for a power bounded operator T into a unified statement. Each assertion in Theorem 6.1 is written so as to establish an injection from the items in Proposition 5.1 into homonymous items in Theorem 6.1.

Theorem 6.1. *Let $T \neq O$ be a power bounded operator on a Hilbert space $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ and let $\varphi\colon \ell_+^\infty \to \mathbb{C}$ be an arbitrary Banach limit. Consider the semi-inner product $\langle\,\cdot\,;\cdot\,\rangle_\varphi = \varphi(\{\langle T^n\cdot\,;T^n\cdot\,\rangle\})$ in $\mathcal{X}$ generated by T and φ. Then there exists a unique nonnegative operator A_φ on $\mathcal{X}$ (which depends on T and φ) such that $\langle\,\cdot\,;\cdot\,\rangle_\varphi = \langle A_\varphi\,\cdot\,;\,\cdot\,\rangle$ and so*

(a) $$\langle x\,;y\rangle_\varphi = \varphi(\{\langle T^n x\,;T^n y\rangle\}) = \varphi(\{\langle T^{*n}T^n x\,;y\rangle\}) = \langle A_\varphi x\,;y\rangle$$

for every $x, y \in \mathcal{X}$ or, equivalently,

$$\varphi(\{\|T^n x\|^2\}) = \|A_\varphi^{\frac{1}{2}} x\|^2 \quad \text{for every } x \in \mathcal{X},$$

where $\|A_\varphi^{\frac{1}{2}} x\| = \|x\|_\varphi = \|T^j x\|_\varphi$ for every $x \in \mathcal{X}$ and every $j \geq 0$.

Moreover:

(b) $O \leq A_\varphi \leq \beta^2 I$ *with* $\beta = \sup_n \|T^n\| \neq 0$.
Thus $\|A_\varphi\| \leq \beta^2$ and the identity $\|A_\varphi\| = \beta^2$ may hold.

(c) $T^* A_\varphi T = A_\varphi$. *Equivalently,*
$\|A_\varphi^{\frac{1}{2}} T^n x\| = \|A_\varphi^{\frac{1}{2}} x\|$ *for every $x \in \mathcal{X}$ and every $n \geq 0$.*

(d) $A_\varphi \neq O \implies 1 \leq \|A_\varphi\|$ *and* $1 \leq \|T\|$.

(e) $A_\varphi T = O \iff T A_\varphi = O \iff A_\varphi = O$.

(f) $A_\varphi T = T A_\varphi \implies A_\varphi = A_\varphi{}^2$.

Conversely, if $A_\varphi = A_\varphi{}^2$, then

(f$_1$) $\|(A_\varphi T^n - T^n A_\varphi)x\|^2 \leq (\beta^2 - 1)\|A_\varphi x\|^2$ *for all n and every $x \in \mathcal{X}$,*
in particular, $\|(A_\varphi T - T A_\varphi)x\|^2 \leq (\|T\|^2 - 1)\|A_\varphi x\|^2$ *for every $x \in \mathcal{X}$,*

(f$_2$) $\varphi(\{\|(A_\varphi T^n - T^n A_\varphi)x\|^2\}) = 0$ *for every $x \in \mathcal{X}$,*

(f$_3$) $\|A_\varphi\| = 1$ *whenever* $O \neq A_\varphi$.

(g) $0 \leq \varphi(\{\|(I - A_\varphi)T^n x\|^2\}) \leq (\|A_\varphi\|^2 - 1)\,\|A_\varphi^{\frac{1}{2}} x\|^2$ *and*
$0 \leq \varphi(\{\|(I - A_\varphi^{\frac{1}{2}})T^n x\|^2\}) \leq \|(I + A_\varphi^{\frac{1}{2}})^{-1}\|^2(\|A_\varphi\|^2 - 1)\|A_\varphi^{\frac{1}{2}} x\|^2$ *for any $x \in \mathcal{X}$,*
which are both null if $\|A_\varphi\| = 1$ (in particular if $O \neq A_\varphi = A_\varphi{}^2$) or $A_\varphi = O$.

(h) $\varphi(\{\|A_\varphi T^n x\|^2\}) = \|A_\varphi^{\frac{1}{2}} x\|^2 + \varphi(\{\|(I - A_\varphi)T^n x\|^2\})$ *for every $x \in \mathcal{X}$.*

(i) $\mathcal{N}(A_\varphi) = \big\{x \in \mathcal{X}\colon\, T^n x \to 0\big\} = \big\{x \in \mathcal{X}\colon\, \varphi(\{\|T^n x\|^2\}) = 0\big\}$.
Hence $\varphi(\{\|T^n x\|^2\}) = 0$ *for every* $x \in \mathcal{X} \iff T^n \xrightarrow{s} O \iff A_\varphi = O$.

(j) $$\begin{aligned}\big\{x \in \mathcal{X}\colon \lim_n \|T^n x\| = \beta\|x\|\big\} &\subseteq \mathcal{N}(\beta^2 I - A_\varphi) = \big\{x \in \mathcal{X}\colon \varphi(\{\|T^n x\|^2\}) = \beta^2\|x\|^2\big\} \\ &\subseteq \big\{x \in \mathcal{X}\colon \|x\| \leq \liminf_n \|T^n x\| \leq \limsup_n \|T^n x\| = \beta\|x\|\big\}.\end{aligned}$$

Hence $\varphi(\{\|T^n x\|^2\}) = \beta^2\|x\|^2$ for every $x \in \mathcal{X} \iff \lim_n \|T^n x\| = \beta\|x\|$ for every $x \in \mathcal{X} \iff A_\varphi = \beta^2 I \iff A_\varphi = I \iff T$ is an isometry on $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$.

Also,

$$\langle\,\cdot\,;\cdot\,\rangle_\varphi \text{ is an inner product} \iff T \text{ is of class } C_{1\cdot} \iff A_\varphi \text{ is positive.}$$

In this case (i.e., *if* $A_\varphi > O$),

(k) $A_\varphi T = T A_\varphi \iff A_\varphi = {A_\varphi}^2 \iff A_\varphi = I \iff T$ *is an isometry on* $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$.

Furthermore, the following assertions are pairwise equivalent.

(1) A_φ *is invertible* (i.e., $A_\varphi \succ O$).

(2) *The norms* $\|\cdot\|_\varphi$ *and* $\|\cdot\|$ *on* $\mathcal{X}$ *are equivalent.*

(3) T *on* $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ *is similar to an isometry.*

(4) T *on* $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ *is power bounded below.*

Proof. Let T be a power bounded operator on a Hilbert space $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$. Thus the sequence $\{\langle T^n x\,; T^n y\rangle\}$ is bounded for every $x, y \in \mathcal{X}$. Then consider the semi-inner product $\langle\,\cdot\,;\cdot\,\rangle_\varphi = \varphi(\{\langle T^n\,\cdot\,; T^n\,\cdot\rangle\})$ in $\mathcal{X}$ generated by T and a Banach limit φ.

(a) As $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ is a Hilbert space, an application of Lemma 3.1 ensures the existence of a unique nonnegative operator A_φ on $\mathcal{X}$ for each power bounded operator T and each Banach limit φ, the φ-asymptotic limit of T, such that

$$\varphi(\{\langle T^{*n}T^n x\ \ y\rangle\}) = \langle x\,; y\rangle_\varphi = \langle x\,; y\rangle_{A_\varphi} = \langle A_\varphi x\,; y\rangle \quad \text{for every} \ \ x, y \in \mathcal{X}.$$

The nontrivial part of the next equivalence follows by the polarization identity. The last identity is a consequence of the shift invariance property for Banach limits:

$$\|x\|_\varphi^2 = \|{A_\varphi}^{\frac{1}{2}} x\|^2 = \varphi(\{\|T^n x\|^2\}) = \varphi(\{\|T^n T^j x\|^2\}) = \|{A_\varphi}^{\frac{1}{2}} T^j x\|^2 = \|T^j x\|_\varphi^2$$

for every $x \in \mathcal{X}$ and every $j \geq 0$.

(b) Now set $\beta = \sup_n \|T^n\|$. Since $\|{A_\varphi}^{\frac{1}{2}} x\|^2 = \varphi(\{\|T^n x\|^2\}) \leq \sup_n \|T^n x\|^2 \leq \beta^2 \|x\|^2$ for every $x \in \mathcal{X}$ (because $\|\varphi\| = 1$) we get

$$\|A_\varphi\| = \|{A_\varphi}^{\frac{1}{2}}\|^2 \leq \beta^2.$$

Also $\langle (A_\varphi - \beta^2 I)x\,; x\rangle = \|{A_\varphi}^{\frac{1}{2}} x\|^2 - \beta^2\|x\|^2 \leq (\|A_\varphi\| - \beta^2)\|x\|^2 \leq 0$ for every $x \in \mathcal{X}$ by the above inequality. Thus the inequalities in (b) hold (since A_φ is self-adjoint):

$$O \leq A_\varphi \leq \beta^2 I.$$

(If $T = \text{shift}\{\beta, 1, 1, 1, \dots\}$, then $A_\varphi = \text{diag}\{\beta^2, 1, 1, 1, \dots\} = T^{*n}T^n$ for all $n \geq 1$.)

(c) By definition, $\|x\|_{A_\varphi} = \|x\|_\varphi$ and so $\|Tx\|_{A_\varphi} = \|Tx\|_\varphi$. Since $\|Tx\|_\varphi = \|x\|_\varphi$ according to (a), then by (b) and Proposition 4.1(a)

$$T^* A_\varphi T = A_\varphi.$$

Equivalently, $T^{*n} A_\varphi T^n = A_\varphi$ for every $n \geq 1$ by induction, which means

$$\|{A_\varphi}^{\frac{1}{2}} T^n x\|^2 = \langle T^{*n} A_\varphi T^n x\,; x\rangle = \langle A_\varphi x\,; x\rangle = \|{A_\varphi}^{\frac{1}{2}} x\|^2$$

for every $x \in \mathcal{X}$ since A_φ is nonnegative.

(d) Then $\|{A_\varphi}^{\frac{1}{2}} x\|^2 = \|{A_\varphi}^{\frac{1}{2}} T^n x\|^2 \leq \|{A_\varphi}^{\frac{1}{2}}\|^2 \|T^n x\|^2 = \|A_\varphi\|\,\|T^n x\|^2$. So (for a constant sequence) $\|{A_\varphi}^{\frac{1}{2}} x\|^2 = \varphi(\{\|{A_\varphi}^{\frac{1}{2}} x\|^2\}) \leq \|A_\varphi\|\,\varphi(\{\|T^n x\|^2\}) = \|A_\varphi\|\,\|{A_\varphi}^{\frac{1}{2}} x\|^2$ for every $x \in \mathcal{X}$ by (a). Hence if there is $x_0 \in \mathcal{X}$ for which ${A_\varphi}^{\frac{1}{2}} x_0 \neq 0$, then $1 \leq \|A_\varphi\|$:

$$1 \leq \|A_\varphi\| \quad \text{whenever} \quad A_\varphi \neq O.$$

Since $A_\varphi = T^* A_\varphi T$, then $\|A_\varphi\| \leq \|A_\varphi\|\,\|T\|^2$. Hence $A_\varphi \neq O$ implies $1 \leq \|T\|$.

(e) If $A_\varphi = O$, then $A_\varphi T = T A_\varphi = O$ trivially. By (c), $A_\varphi T = O$ implies $A_\varphi = O$. Finally, if $T A_\varphi = O$, then $\varphi(\{\|T^n A_\varphi x\|\}) = 0$, and so $\|{A_\varphi}^{\frac{3}{2}} x\| = 0$, for every $x \in \mathcal{X}$, by (a). Thus $A_\varphi = O$ (by the Spectral Theorem since by (b) A_φ is nonnegative).

(f) Take $x, y \in \mathcal{X}$. Since $T^{*n}A_\varphi T^n = A_\varphi$, then $\varphi(\{\langle T^{*n}A_\varphi T^n x\,;y\rangle\}) = \varphi(\{\langle A_\varphi x\,;y\rangle\})$ $= \langle A_\varphi x\,;y\rangle$ (constant sequence). Also by (a) $\varphi(\{\langle T^{*n}T^n A_\varphi x\,;y\rangle\}) = \langle {A_\varphi}^2 x\,;y\rangle$. So

$$A_\varphi T = TA_\varphi \implies A_\varphi = {A_\varphi}^2.$$

Conversely, $\langle A_\varphi T^n x\,;T^n A_\varphi x\rangle = \|{A_\varphi}^{\frac{1}{2}} T^n x\|^2 = \|{A_\varphi}^{\frac{1}{2}} x\|^2$ by (c). If $A_\varphi = A_\varphi^2$, then ${A_\varphi}^{\frac{1}{2}} = A_\varphi$ by uniqueness of the nonnegative square root, and hence

$$\begin{aligned}\|(A_\varphi T^n - T^n A_\varphi)x\|^2 &= \|A_\varphi T^n x\|^2 + \|T^n A_\varphi x\|^2 - 2\|{A_\varphi}^{\frac{1}{2}} x\|^2\\ &= \|T^n A_\varphi x\|^2 - \|A_\varphi x\|^2 \le (\beta^2 - 1)\|A_\varphi x\|^2\end{aligned}$$

for all n and every $x \in \mathcal{X}$. In particular, for $n = 1$ we get for every $x \in \mathcal{X}$

$$\|(A_\varphi T - TA_\varphi)x\|^2 \le (\|T\|^2 - 1)\|A_\varphi x\|^2.$$

Asymptotically, if $A_\varphi = A_\varphi^2$, then ${A_\varphi}^{\frac{1}{2}} = A_\varphi$. So we get by (a) and the above identity

$$\varphi(\{\|(A_\varphi T^n - T^n A_\varphi)x\|^2\}) = \varphi(\{\|T^n A_\varphi x\|^2\}) - \|A_\varphi x\|^2 = \|{A_\varphi}^{\frac{3}{2}} x\|^2 - \|A_\varphi x\|^2 = 0$$

for every $x \in \mathcal{X}$. Moreover, if $A_\varphi = {A_\varphi}^2$, then A_φ is an orthogonal projection (since it is self-adjoint) and so $\|A_\varphi\| = 1$ whenever $A_\varphi \ne O$.

(g) Take an arbitrary $x \in \mathcal{X}$. Again, by (c) we get $\langle T^n x\,;A_\varphi\, T^n x\rangle = \|{A_\varphi}^{\frac{1}{2}} T^n x\|^2 = \|{A_\varphi}^{\frac{1}{2}} x\|^2$ for all $n \ge 0$, and $\varphi(\{\|T^n x\|^2\}) = \|{A_\varphi}^{\frac{1}{2}} x\|^2$ according to (a). Thus

$$\begin{aligned}0 \le \varphi(\{\|(I - A_\varphi)T^n x\|^2\}) &= \varphi(\{\|T^n x\|^2\}) + \varphi(\{\|A_\varphi T^n x\|^2\}) - 2\varphi(\{\|{A_\varphi}^{\frac{1}{2}} x\|^2\})\\ &= \varphi(\{\|A_\varphi T^n x\|^2\}) - \|{A_\varphi}^{\frac{1}{2}} x\|^2 \le (\|A_\varphi\|^2 - 1)\|{A_\varphi}^{\frac{1}{2}} x\|^2.\end{aligned}$$

Since $I - A_\varphi = (I + {A_\varphi}^{\frac{1}{2}})\,(I - {A_\varphi}^{\frac{1}{2}})$, and since $I + {A_\varphi}^{\frac{1}{2}}$ is invertible with a bounded inverse (because $A_\varphi \ge O$), then $I - {A_\varphi}^{\frac{1}{2}} = (I + {A_\varphi}^{\frac{1}{2}})^{-1}(I - A_\varphi)$ and so

$$0 \le \varphi(\{\|(I - {A_\varphi}^{\frac{1}{2}})T^n x\|^2\}) \le \|(I + {A_\varphi}^{\frac{1}{2}})^{-1}\|^2 \varphi(\{\|(I - A_\varphi)T^n x\|^2\}).$$

(h) This was proved above: $\varphi(\{\|(I - A_\varphi)T^n x\|^2\}) = \varphi(\{\|A_\varphi T^n x\|^2\}) - \|{A_\varphi}^{\frac{1}{2}} x\|^2$.

(i) Part of assertion (i) follows at once from (a) since $\mathcal{N}(A_\varphi) = \mathcal{N}({A_\varphi}^{\frac{1}{2}})$. Indeed,

$$T^n x \to 0 \implies \varphi(\{\|T^n x\|\}) = 0 \iff \|{A_\varphi}^{\frac{1}{2}} x\|^2 = 0 \iff x \in \mathcal{N}({A_\varphi}^{\frac{1}{2}}) \iff x \in \mathcal{N}(A_\varphi).$$

Conversely, suppose $\beta \ge 1$. (Otherwise $T^n x \to 0$ for every $x \in \mathcal{X}$ since $\sup_n \|T^n\| = \beta < 1$ implies $\|T^n\| \le \|T\|^n \le \beta^n \to 0$.) If $\varphi(\{\|T^n x\|\}) = 0$ for some Banach limit φ, then $\liminf_n \|T^n x\| = 0$ (recall: $0 \le \liminf_n \xi_n \le \varphi(\{\xi_n\}) \le \limsup_n \xi_n$ for $\xi_n \ge 0$). However, if $\liminf_n \|T^n x\| = 0$, then for every $\varepsilon > 0$ there is an integer n_ε such that $\|T^{n_\varepsilon} x\| < \varepsilon$, which implies $\|T^n x\| \le \beta\|T^{n_\varepsilon} x\| < \beta\varepsilon$ for all $n \ge n_\varepsilon$. Thus $\|T^n x\| \to 0$.

(j) Take any $0 \ne x \in \mathcal{X}$ (nonzero to avoid trivialities). Since φ is a Banach limit,

$$\|T^n x\| \to \beta\|x\| \implies \varphi(\{\|T^n x\|^2\}) = \beta^2\|x\|^2.$$

According to (a),

$$\varphi(\{\|T^n x\|^2\}) = \beta^2\|x\|^2 \iff \|{A_\varphi}^{\frac{1}{2}} x\|^2 = \beta^2\|x\|^2 \iff \langle(\beta^2 I - A_\varphi)x\,;x\rangle = 0,$$

and according to (b), since $\mathcal{N}((\beta^2 I - A_\varphi)^{\frac{1}{2}}) = \mathcal{N}(\beta^2 I - A_\varphi)$,

$$\langle(\beta^2 I - A_\varphi)x\,;x\rangle = 0 \iff \|(\beta^2 I - A_\varphi)^{\frac{1}{2}} x\| = 0 \iff x \in \mathcal{N}(\beta^2 I - A_\varphi).$$

Conversely, since $\varphi(\{\|T^n x\|^2\}) = \beta^2\|x\|^2$ means $\varphi(\{\beta^2\|x\|^2 - \|T^n x\|^2\}) = 0$, and since $0 \le \beta^2\|x\|^2 - \|T^n x\|^2$ for every n because $\sup_n \|T^n x\| \le \beta\|x\|$, then we get

$$\varphi(\{\|T^n x\|^2\}) = \beta^2\|x\|^2 \implies \liminf_n\{\beta^2\|x\|^2 - \|T^n x\|^2\} = 0.$$

However, recalling again that $0 \le \|T^n x\| \le \beta\|x\|$ for every n,

$$\liminf_n\{\beta^2\|x\|^2 - \|T^n x\|^2\} = 0 \iff \limsup_n\|T^n x\| = \beta\|x\|.$$

Moreover, for $x \neq 0$ and since $\beta > 0$ (as $T \neq O$),

$$\limsup_n\|T^n x\| = \beta\|x\| \implies \beta \ge 1$$

(indeed, if $\beta < 1$, then $\beta = \limsup_n \frac{\|T^n x\|}{\|x\|} \le \limsup_n \sup_{x\neq 0} \frac{\|T^n x\|}{\|x\|} = \limsup_n\|T^n\| \le \limsup_n\|T\|^n \le \limsup_n \beta^n = \lim_n \beta^n = 0$). Also, since $\|T^{n+m}x\| \le \beta\|T^n x\|$ for each $m, n \ge 0$, then $\limsup_n\|T^n x\| = \limsup_n\|T^{n+m}x\| \le \beta \liminf_m\|T^m x\|$, and so

$$\limsup_n\|T^n x\| = \beta\|x\| \implies \|x\| \le \liminf_m\|T^m x\|.$$

Finally, by the above implications and equivalences, if $\lim_n \|T^n x\| = \beta\|x\|$ for every $x \in \mathcal{X}$, then $\varphi(\{\|T^n x\|^2\}) = \beta^2\|x\|^2$ for every $x \in \mathcal{X}$, which means $A_\varphi = \beta^2 I$. But this implies $T^*T = I$ by (c) (since $\beta \neq 0$ whenever $T \neq O$), which in turn implies $A_\varphi = I$ by (a) (i.e., $\langle A_\varphi x\,; y\rangle = \varphi(\{\langle T^{*n}T^n x\,; y\rangle\})$ for every $x, y \in \mathcal{X}$). However, if $A_\varphi = I$, then $\|Tx\| = \|x\|$ for every $x \in \mathcal{X}$ by (c), which means T is an isometry on $(\mathcal{X}, \langle\cdot\,;\cdot\rangle)$, and we are back to $\lim_n \|T^n x\| = \beta\|x\|$ for every $x \in \mathcal{X}$ with $\beta = 1$.

As we saw in Remark 6.1, $\langle\cdot\,;\cdot\rangle_\varphi$ is an inner product if and only if T is a power bounded of class $C_{1\cdot}$, and the semi-inner product $\langle\cdot\,;\cdot\rangle_\varphi = \langle A_\varphi\cdot\,;\cdot\rangle$ is an inner product (i.e., the seminorm $\|\cdot\|_\varphi = \|{A_\varphi}^{\frac{1}{2}}\cdot\|$ is a norm) if and only if $\mathcal{N}(A_\varphi) = \{0\}$, which means the nonnegative A_φ is positive. Thus from now on suppose $A_\varphi > O$.

(k) If $A_\varphi T = T A_\varphi$ then $A_\varphi = {A_\varphi}^2$ by (f) and so A_φ is an orthogonal projection (since it is a seif-adjoint idempotent) which implies $A_\varphi = I$ (because $A_\varphi > O$), and hence $A_\varphi T = T A_\varphi$ trivially. Therefore if $A_\varphi > O$,

$$A_\varphi T = T A_\varphi \implies A_\varphi = {A_\varphi}^2 \implies A_\varphi = I \implies A_\varphi T = T A_\varphi.$$

But $A_\varphi = I$ if and only if T is an isometry, as we saw in the proof of item (j).

Since T is power bounded, to prove assertions (1) to (4) proceed as follows.

$$(1) \iff (2), \quad (3) \iff (4), \quad \text{and} \quad (1) \implies (3)$$

by Proposition 3.1, Proposition 4.2(a,b), and Proposition 4.1(c), respectively. Conversely, as T is power bounded, if (4) holds, then $\alpha\|x\| \le \|T^n x\| \le \beta\|x\|$ for all $n \ge 0$, and so $\alpha\|x\| \le \varphi(\{\|T^n x\|\}) \le \beta\|x\|$, for every $x \in \mathcal{X}$. Since $\varphi(\{\|T^n x\|^2\}) = \|x\|_\varphi^2$ by (a), then $\alpha^2\|x\|^2 \le \|x\|_\varphi^2 \le \beta^2\|x\|^2$ for every $x \in \mathcal{X}$ and so (2) holds. Thus

$$(4) \implies (2). \qquad \square$$

Remark 6.2. If T is a contraction (equivalently, if $\beta \le 1$), then Theorem 6.1 is reduced to Proposition 5.1, and $\{T^{*n}T^n\}$ converges strongly (thus weakly) to $A_\varphi = A$ for every Banach limit φ, with $\|A\| = 1$ or $\|A\| = 0$. For a $C_{1\cdot}$-contraction, $\|A\| = 1$.

Such a combined procedure (of using Lemma 3.1 together with an inner product generated by a power bounded operator and a Banach limit) seems to have been originated in the celebrated Nagy's 1947 paper [39] (see also [40, Section II.5]). Subsequent applications of it appear, for instance, in [16, 17, 18] and, recently, in [11, 12, 27]. Proposition 4.2(a,b), however, supplies an elementary and straightforward proof of Nagy's result as follows.

Corollary 6.1. [39] *On a Hilbert space, an invertible power bounded operator with a power bounded inverse is similar to a unitary operator — the converse is trivial.*

Proof. Let $T \in \mathcal{B}[\mathcal{X}]$ and $T^{-1} \in \mathcal{B}[\mathcal{X}]$ be power bounded. Thus there exist real constants $0 < \alpha \le 1$ and $1 \le \beta$ for which $\|T^n\| \le \beta$ and $\|T^{-n}\| \le \alpha^{-1}$ for all $n \ge 0$. So $\alpha\|x\| \le \|T^{-n}\|^{-1}\|x\| \le \|T^n x\| \le \beta\|x\|$ for all n and every x. Hence T is similar to an isometry by Proposition 4.2(a,b). Since T is invertible, then so is the isometry similar to it: an invertible Hilbert-space isometry means a unitary operator. □

7. Cesàro Means and the Equation $T^*AT = A$

A word on terminology. An $\mathcal{X}$-valued sequence in an arbitrary normed space $\mathcal{X}$ is called Cesáro convergent if its sequence of arithmetic means (referred to as Cesáro means) converges in $\mathcal{X}$, whose limit is called Cesáro limit.

Banach limits have been related to Cesàro means since the very beginning [29], and Cesàro means are naturally linked to the Ergodic Theorem for power bounded operators. If a sequence $\{Q_n\}$ of Cesàro means $Q_n = \frac{1}{n}\sum_{k=0}^{n-1} T^{*k}T^k$ for an operator T converges (either weakly, strongly, or uniformly), then its limit Q (if it exists) has been refereed to as the *Cesàro asymptotic limit* of T (see [11]). As is well-known, the strong limit Q always exists for contractions and coincides with the asymptotic limit A: for a contraction T the sequence of Cesàro means $\{Q_n\}$ converges strongly to $Q = A$. An elementary quick proof is readily obtained as follows.

Proposition 7.1. *For a Hilbert-space contraction the sequence of Cesàro means converges strongly and the Cesàro asymptotic limit coincides with the asymptotic limit*:

$$\|T\| \le 1 \quad \Longrightarrow \quad Q_n = \frac{1}{n}\sum_{k=0}^{n-1} T^{*k}T^k \xrightarrow{\;s\;} Q = A.$$

Proof. If T is a contraction, then the sequence $\{T^{*n}T^n\}$ converges strongly to A by Proposition 5.1. Since $x_n \to x$ implies $\frac{1}{n}\sum_{k=1}^{n} x_k \to x$ for any normed-space-valued sequence $\{x_n\}$, then $T^{*n}T^n \xrightarrow{\;s\;} A$ implies $Q_n \xrightarrow{\;s\;} Q = A$. □

As before, the next theorem brings together scattered properties (again, either well-known — e.g., [11, Theorems 2.5, 2.6 and Proposition 5.1] — or not) of Cesàro asymptotic limits Q into a unified statement. Some parts in the proof behave similarly to their equivalent in the proof of Theorem 6.1, as expected; some other parts require an independent and different approach. Each assertion in Theorem 7.1 below is written so as to establish a bijection with the items in Theorem 6.1 and so, by transitivity, it establishes an injection from the items in Proposition 5.1 into homonymous items in Theorems 6.1 and 7.1.

Theorem 7.1. *Let $O \ne T \in \mathcal{B}[\mathcal{X}]$ be a Hilbert-space operator. For each positive integer n consider the Cesàro mean*

$$Q_n = \frac{1}{n}\sum_{k=0}^{n-1} T^{*k}T^k$$

in $\mathcal{B}[\mathcal{X}]$. Suppose the sequence $\{Q_n\}$ converges weakly to $Q \in \mathcal{B}[\mathcal{X}]$. That is, suppose

(a) $Q_n \xrightarrow{\;w\;} Q$. *Equivalently,*

$\|Q_n^{\frac{1}{2}}x\| \to \|Q^{\frac{1}{2}}x\|$ *for every $x \in \mathcal{X}$.*

Then in this case:

(b) $O \leq Q$ *and, if* $\sup_n \|T^n\| = \beta$ *(so that* $\beta \neq 0$*), then* $Q \leq \beta^2$.
If T *is power bounded, then* $\|Q\| \leq \beta^2$ *and the identity* $\|Q\| = \beta^2$ *may hold.*

(c) $T^*Q\,T = Q$. *Equivalently,*
$\|Q^{\frac{1}{2}} T^n x\| = \|Q^{\frac{1}{2}} x\|$ *for every* $x \in \mathcal{X}$ *and every* $n \geq 0$. *Therefore*
$\|{Q_n}^{\frac{1}{2}} T^j x\|^2 = \frac{1}{n}\sum_{k=0}^{n-1}\|T^{k+j}x\|^2 \to \|Q^{\frac{1}{2}}x\|^2$ *for every* $x \in \mathcal{X}$ *and every* $j \geq 0$.

(d) $Q \neq O \implies 1 \leq \|Q\|$ *and* $1 \leq \|T\|$.

(e) $QT = O \iff TQ = O \iff Q = O$.

(f) $QT = TQ \implies Q = Q^2$.
Conversely, if $Q = Q^2$*, then*
(f$_1$) $\|(QT^n - T^nQ)x\|^2 \leq (\sup_n \|T^n\|^2 - 1)\|Qx\|^2$ *for all* n *and every* $x \in \mathcal{X}$,
in particular, $\|(QT - TQ)x\|^2 \leq (\|T\|^2 - 1)\|Qx\|^2$ *for every* $x \in \mathcal{X}$,
(f$_2$) $\|(QT^n - T^nQ)x\| \to 0$ *for every* $x \in \mathcal{X}$,
(f$_3$) $\|Q\| = 1$ *whenever* $O \neq Q$.

(g) $\|(I - Q){Q_n}^{\frac{1}{2}}x\|^2 \leq (\|Q\|^2 - 1)\,\|{Q_n}^{\frac{1}{2}}x\|^2 \to (\|Q\|^2 - 1)\,\|Q^{\frac{1}{2}}x\|^2$ *and*
$\|(I - Q^{\frac{1}{2}}){Q_n}^{\frac{1}{2}}x\|^2 \leq \|(I + Q^{\frac{1}{2}})^{-1}\|^2(\|Q\|^2 - 1)\,\|{Q_n}^{\frac{1}{2}}x\|^2$ *for every* $x \in \mathcal{X}$,
which are both null if $\|Q\| = 1$ *or asymptotically null if* $Q = O$.

(h) $\|Q\,{Q_n}^{\frac{1}{2}}x\|^2 = \|(2Q - I)^{\frac{1}{2}}{Q_n}^{\frac{1}{2}}x\|^2 + \|(I - Q){Q_n}^{\frac{1}{2}}x\|^2$ *for every* $x \in \mathcal{X}$,

(i) *If* T *is power bounded, then*
$\mathcal{N}(Q) = \big\{x \in \mathcal{X}\colon\, T^n x \to 0\big\} = \big\{x \in \mathcal{X}\colon\, \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x\| \to 0\big\}$.
Hence $T^n \xrightarrow{\;s\;} O \iff Q = O$ *and* T *is power bounded.*

(j) *If* $\sup_n \|T^n\| = \beta$ *(so that* $\beta \neq 0$*) then*
$\big\{x \in \mathcal{X}\colon \lim_n \|T^n x\| = \beta\|x\|\big\}$
$\subseteq \mathcal{N}(\beta^2 I - Q) = \big\{x \in \mathcal{X}\colon \lim_n \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x\|^2 = \beta^2\|x\|\big\}$
$\subseteq \big\{x \in \mathcal{X}\colon \|x\| \leq \liminf_n \|T^n x\| \leq \limsup_n \|T^n x\| = \beta\|x\|\big\}$.
Hence $\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x\|^2 = \beta^2\|x\|$ *for every* $x \in \mathcal{X} \iff \lim_n \|T^n x\| = \beta\|x\|$ *for every* $x \in \mathcal{X} \iff Q = \beta^2 I \iff Q = I \iff T$ *is an isometry on* $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$.

Also if T *is power bounded, then*

$\langle\,\cdot\,;\cdot\,\rangle_Q$ *is an inner product* $\iff$ T *is of class* $C_{1\cdot}$ $\iff$ Q *is positive.*

In the case of $Q > O$ *we get*

(k) $QT = TQ \iff Q = Q^2 \iff Q = I \iff T$ *is an isometry on* $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$.

Furthermore, the following assertions are pairwise equivalent.

(1) Q *is invertible* (i.e., $Q \succ O$).
(2) *The norms* $\|\cdot\|_Q$ *and* $\|\cdot\|$ *on* $\mathcal{X}$ *are equivalent.*
(3) T *on* $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ *is similar to an isometry.*
(4) T *on* $(\mathcal{X}, \langle\,\cdot\,;\cdot\,\rangle)$ *is power bounded below.*

Proof. Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator acting on a Hilbert space $\mathcal{X}$, take the sequence $\{T^{*n}T^n\}$ of nonnegative operators in $\mathcal{B}[\mathcal{X}]$, and consider the Cesàro mean

$$Q_n = \tfrac{1}{n}\sum_{k=0}^{n-1} T^{*k}T^k \qquad \text{for every } \ n \geq 1$$

associated with $\{T^{*n}T^n\}$. Suppose the $\mathcal{B}[\mathcal{X}]$-valued sequence $\{Q_n\}$ of nonnegative operators converges weakly to the Cesàro asymptotic limit $Q \in \mathcal{B}[\mathcal{X}]$ of T.

(a) Since the class of nonnegative operators is weakly closed in $\mathcal{B}[\mathcal{X}]$, then the weak limit Q is nonnegative, and hence $Q_n \xrightarrow{\;w\;} Q$ is equivalent to

$$\|Q_n{}^{\frac{1}{2}}x\|^2 = \langle Q_n x \;\; x\rangle \to \langle Qx \;\; x\rangle = \|Q^{\frac{1}{2}}x\|^2 \qquad \text{for every } \ x \in \mathcal{X}.$$

(b) By (a), $O \leq Q$. If $\sup_n \|T^n\| = \beta$, then $Q_n \leq \frac{1}{n}I + \frac{1}{n}\sum_{k=1}^{n-1}\beta^2 I \to \beta^2 I$ (and if $T = \text{shift}\{\beta, 1, 1, 1, \dots\}$, then $Q = T^{*n}T^n = \text{diag}\{\beta^2, 1, 1, 1, \dots\}$) for every $n \geq 1$.

(c) Since $Q_n = \frac{1}{n}\sum_{k=0}^{n-1} T^{*k}T^k$, then (compare with the proof of Proposition 4.2)

$$T^*Q_nT = Q_{n+1} + \tfrac{1}{n}(Q_{n+1} - I)$$

for each $n \geq 1$. If $\langle Q_n x\,;x\rangle \to \langle Qx\,;x\rangle$ for every $x \in \mathcal{X}$, then $\{Q_n\}$ is bounded and so

$$T^*Q\,T = Q.$$

By induction, $T^{*n}Q\,T^n = Q$ for all $n \geq 0$. As $O \leq Q$ and $O \leq Q_n$ for $n \geq 1$, then (i)

$$\|Q^{\frac{1}{2}}T^n x\|^2 = \langle QT^n x\,;T^n x\rangle = \langle T^{*n}QT^n x\,;x\rangle = \langle Qx\,;x\rangle = \|Q^{\frac{1}{2}}x\|$$

for every $x \in \mathcal{X}$ and every $n \geq 0$, and (ii) $T^{*j}Q_nT^j \xrightarrow{\;w\;} Q$ for every $j \geq 0$, and so

$$\|Q_n{}^{\frac{1}{2}}T^j x\|^2 = \langle Q_nT^j x\,;T^j x\rangle = \tfrac{1}{n}\sum_{k=0}^{n-1}\langle T^{*k+j}T^{k+j}x\,;x\rangle = \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^{k+j}x\|^2$$

for every $n \geq 1$ and every $j \geq 0$. Thus, since $\|Q_n{}^{\frac{1}{2}}T^j x\|^2 \to \|Q^{\frac{1}{2}}T^j x\|^2$ by (a), then

$$\tfrac{1}{n}\sum_{k=0}^{n-1}\|T^{k+j}x\|^2 \to \|Q^{\frac{3}{2}}x\|^2, \qquad \text{for every } x \in \mathcal{X} \text{ and every } j \geq 0.$$

(d) According to (c), $\|Q^{\frac{1}{2}}x\|^2 = \|Q^{\frac{1}{2}}T^k x\|^2 \leq \|Q\|\,\|T^k x\|^2$ for any $k \geq 1$ and $\|Q^{\frac{1}{2}}x\|^2 = \frac{1}{n}\sum_{k=0}^{n-1}\|Q^{\frac{1}{2}}x\|^2 \leq \|Q\|\,\frac{1}{n}\sum_{k=0}^{n-1}\|T^k x\|^2 \to \|Q\|\,\|Q^{\frac{1}{2}}x\|^2 \neq 0$ for every $x \in \mathcal{X}\backslash\mathcal{N}(Q)$. So

$$1 \leq \|Q\| \qquad \text{whenever} \qquad Q \neq O.$$

Since $Q = T^*Q\,T$, then $\|Q\| \leq \|Q\|\|T\|^2$. Hence $Q \neq O$ implies $1 \leq \|T\|$.

(e) $Q = O$ trivially implies $QT = TQ = O$, and $QT = O$ implies $Q = O$ by (c). If $TQ = O$, then $0 = \|Q_n{}^{\frac{1}{2}}TQx\| \to \|Q^{\frac{3}{2}}x\|$ for every $x \in \mathcal{X}$ by (c) again, and so $Q = O$.

(f) Since $O \leq Q$, then $QT = TQ$ if and only if $Q^{\frac{1}{2}}T = TQ^{\frac{1}{2}}$. If $Q^{\frac{1}{2}}T = TQ^{\frac{1}{2}}$, then according to (c) it follows that

$$\begin{aligned}\langle Qx\,;x\rangle = \|Q^{\frac{1}{2}}x\|^2 = \tfrac{1}{n}\sum_{k=1}^{n-1}\|Q^{\frac{1}{2}}x\|^2 &= \tfrac{1}{n}\sum_{k=1}^{n-1}\|Q^{\frac{1}{2}}T^k x\|^2 \\ &= \tfrac{1}{n}\sum_{k=1}^{n-1}\|T^kQ^{\frac{1}{2}}x\|^2 \to \|Qx\|^2 = \langle Q^2x\,;x\rangle\end{aligned}$$

for every $x \in \mathcal{X}$. So $Q = Q^2$. Conversely, $\langle QT^n x\,;T^nQx\rangle = \|Q^{\frac{1}{2}}T^n x\|^2 = \|Q^{\frac{1}{2}}x\|^2$ according to (c). Since $Q = Q^2$ if and only if $Q^{\frac{1}{2}} = Q$, then we get in this case

$$\begin{aligned}\|(QT^n - T^nQ)x\|^2 &= \|QT^n x\|^2 + \|T^nQx\|^2 - 2\|Q^{\frac{1}{2}}x\|^2 \\ &= \|T^nQx\|^2 - \|Qx\|^2 \leq (\sup_n\|T^n\|^2 - 1)\|Qx\|^2\end{aligned}$$

for all n and every $x \in \mathcal{X}$. In particular, for $n = 1$ we get for every $x \in \mathcal{X}$

$$\|(QT - TQ)x\|^2 \leq (\|T\|^2 - 1)\|Qx\|^2.$$

Thus, asymptotically (with the assumption $Q = Q^2$ still in force), we get by (c)

$$\lim_n \|(QT^n - T^nQ)x\|^2 = \lim_n \|T^nQ^{\frac{1}{2}}x\|^2 - \|Q^{\frac{1}{2}}x\|^2 = 0$$

for every $x \in \mathcal{X}$ (and so $\{QT^n - T^nQ\}$ is a bounded the sequence of operators disregarding whether T is power bounded or not). Again, as in the proof of Theorem 6.1, $O \leq Q = Q^2 \neq O$ implies Q is an nonzero orthogonal projection, and so $\|Q\| = 1$.

(g) The inequality is readily verified and the limit comes from (c): for any $x \in \mathcal{X}$,

$$\begin{aligned}\|(I - Q)Q_n{}^{\frac{1}{2}}x\|^2 &= \|Q_n{}^{\frac{1}{2}}x\|^2 + \|QQ_n{}^{\frac{1}{2}}x\|^2 - 2\|Q^{\frac{1}{2}}Q_n{}^{\frac{1}{2}}x\|^2 \\ &\leq \|Q_n{}^{\frac{1}{2}}x\|^2 + \|Q\|^2\|Q_n{}^{\frac{1}{2}}x\|^2 - 2\|Q\|\,\|Q_n{}^{\frac{1}{2}}x\|^2 \\ &= (\|Q\| - 1)^2\|Q_n{}^{\frac{1}{2}}x\|^2 \to (\|Q\| - 1)^2\|Q^{\frac{1}{2}}x\|^2,\end{aligned}$$

and since $I - Q = (I + Q^{\frac{1}{2}})(I - Q^{\frac{1}{2}})$ and $I + Q^{\frac{1}{2}}$ is invertible, then we get the second inequality form the above one.

(h) As we saw above, $\|QQ_n{}^{\frac{1}{2}}x\|^2 = \|(I - Q)Q_n{}^{\frac{1}{2}}x\|^2 + 2\|Q^{\frac{1}{2}}Q_n{}^{\frac{1}{2}}x\|^2 - \|Q_n{}^{\frac{1}{2}}x\|^2$, but $2\|Q^{\frac{1}{2}}Q_n{}^{\frac{1}{2}}x\|^2 - \|Q_n{}^{\frac{1}{2}}x\|^2 = \|(2Q - I)^{\frac{1}{2}}Q_n{}^{\frac{1}{2}}x\|^2$, for every $x \in \mathcal{X}$. So we get (h).

(i) As in the proof of Proposition 7.1, if $\|T^nx\| \to 0$, then $\frac{1}{n}\sum_{k=0}^{n-1}\|T^kx\| \to 0$, which means $\|Q^{\frac{1}{2}}x\| = 0$ by (c) or, equivalently, $x \in \mathcal{N}(Q^{\frac{1}{2}})$ (i.e., $x \in \mathcal{N}(Q)$). The converse requires power boundedness. If $\frac{1}{n}\sum_{k=0}^{n-1}\|T^kx\| \to 0$ (i.e., if $x \in \mathcal{N}(Q)$), then

$$0 \leq \liminf_n \|T^nx\| \leq \lim_n \inf_j \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^{k+j}x\| \leq \lim_n \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^kx\| = 0$$

(as we saw in Section 6). But if T power bounded, then $\liminf_n \|T^nx\| = 0$ implies $\lim_n \|T^nx\| = 0$ (as we saw in the proof of Theorem 6.1(i)).

(j) Suppose $\sup_n \|T^n\| \leq \beta$. Again, as in the proof of Proposition 7.1,

$$\|T^nx\| \to \beta\|x\| \implies \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^kx\|^2 \to \beta^2\|x\|.$$

According to (c),

$$\lim_n \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^kx\|^2 = \beta^2\|x\| \iff \|Q^{\frac{1}{2}}x\|^2 = \beta^2\|x\|^2 \iff \langle (Q - \beta^2 I)x\,;x\rangle,$$

and according to (b),

$$\langle (Q - \beta^2 I)x\,;x\rangle \iff \|(Q - \beta^2 I)^{\frac{1}{2}}x\| = 0 \iff x \in \mathcal{N}(Q - \beta^2 I)$$

since $\mathcal{N}(Q - \beta^2 I)^{\frac{1}{2}} = \mathcal{N}(Q - \beta^2 I)$. Conversely,

$$\lim_n \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^kx\|^2 = \beta^2\|x\| \implies \limsup_n \|T^nx\| = \beta\|x\|.$$

Indeed, since $\sup_n \|T^n\| \leq \beta$, then as we saw in Section 6

$$\beta^2\|x\|^2 = \lim_n \tfrac{1}{n}\sum_{k=0}^{n-1}\|T^kx\|^2 \leq \limsup_n \|T^nx\|^2 \leq \sup_n \|T^nx\|^2 \leq \beta^2\|x\|^2.$$

Thus, as in the proof of Theorem 6.1(j), for $x \neq 0$ and since $\beta > 0$,

$$\limsup_n \|T^nx\| = \beta\|x\| \implies \beta \geq 1 \text{ and } \|x\| \leq \liminf_n \|T^nx\|.$$

If $\lim_n \|T^nx\| = \beta\|x\|$ for every $x \in \mathcal{X}$, then $\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\|T^kx\|^2 = \beta^2\|x\|$ for every $x \in \mathcal{X}$ as we saw above, meaning $Q = \beta^2 I$, which implies $T^*T = I$ by (c), and so

$Q_n = I = Q$. Conversely, if $Q = I$, then $\|T^n x\| = \|x\|$ for every $x \in \mathcal{X}$ by (c) again (i.e., T is an isometry), and so $\lim_n \|T^n x\| = \beta\|x\|$ for every $x \in \mathcal{X}$ with $\beta = 1$.

Suppose T is power bounded. If T is of class $C_{1\cdot}$ (i.e.,if $\|T^n x\| \not\to 0$ if $x \neq 0$), then as we saw in the proof of Theorem 6.1(i) $0 < \liminf_n \|T^n x\|$ for $x \neq 0$. The converse is trivial. Since $\liminf_n \|T^n x\|^2 \le \lim_n \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x\|^2$ (as we saw in Section 6) and $\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x\|^2 = \|Q_n x\|^2 \to \|Qx\|^2 = \|x\|_Q^2$ by (c), then $0 < \liminf_n \|T^n x\|$ implies $0 < \|x\|_Q$ for $x \neq 0$, which means $Q > O$. Thus if T is power bounded, then

$$\langle\,\cdot\,;\cdot\,\rangle_Q \text{ is an inner product } \iff T \text{ is of class } C_{1\cdot} \iff Q \text{ is positive.}$$

(k) This follows as in the proof of Theorem 6.1(k) with A_φ replaced by Q.

Moreover, the equivalences among the assertions (1) to (4), depend on the new inner product $\langle\cdot\,;\cdot\rangle_Q$ generated by the positive Q, and so they follow by Propositions 4.1 and 4.2 by using the same argument of Theorem 6.1, with $\varphi(\{\|T^n x\|^2\}) = \|x\|_\varphi^2$ replaced by $\lim_n \|Q_n{}^{\frac{1}{2}} x\|^2 = \lim_n \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x\|^2 \to \|Qx\|^2 = \|x\|_Q^2$. $\square$

Remark 7.1. Even a power unbounded operator may have a Cesàro asymptotic limit (see, e.g., [11, Example 3]), while there is no φ-asymptotic limit for power unbounded operators. From now on suppose T is power bounded.

(a) Thus for every Banach limit φ there exists a φ-asymptotic limit A_φ for T. Even in this case of a power bounded operator, the Cesàro asymptotic limit Q may not exist (even in the weak sense; see [11, Example 2]).

(b) Moreover, even when Q exists it may not coincide with A_φ. Indeed, it was exhibited in [11, Example 1] a power bounded unilateral weighted shift T such that $\|T^n\| = \beta = \sqrt{2}$ for all n and $\|T^n e_1\|^2$ is either β^2 or 1 depending on n, with Cesàro asymptotic limit $Q = I$, which does not coincide with an arbitrary φ-asymptotic limit A_φ (i.e., $Q \neq A_\varphi$ for a specific Banach limit φ — actually, there exist Banach limits φ for which $\|A_\varphi{}^{\frac{1}{2}} e_1\|^2$ lies anywhere in the interval $[1, \beta^2]$).

(c) As we have seen in Theorems 6.1(d) and 7.1(d), if the asymptotic limits are not null, then for every Banach limit φ we get $1 \le \|A_\varphi\|$, $1 \le \|Q\|$, and $1 \le \|T\|$. These norms, however, are not related. For instance, if $T = \text{shift}\{\beta, 1, 1, 1, \dots\}$ is the unilateral weighted shift with $\beta > 1$ as in the proofs of Theorems 6.1(b) and 7.1(b), then $\|A_\varphi\| = \|Q\| = \beta^2$ and $\|T\| = \beta$. On the other hand, if $T = \left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right) \oplus I$, then $T^n = O \oplus I$ for all $n \ge 2$ and $A_\varphi = Q = O \oplus I$ for all Banach limits φ and so $\|T\| = \beta$ with $\|A_\varphi\| = \|Q\| = 1$. Actually, as we saw in item (b) above, it was exhibited in [11, Example 1] a power bounded unilateral weighted shift T such that there is a maximum Banach limit φ_+ for which $\|A_{\varphi_+}\| \ge 2$, while $\|T\| = \sqrt{2}$ and $\|Q\| = 1$.

(d) The inclusions in Theorems 6.1(j) and 7.1(j) may also be proper (e.g., for the unilateral weighted shift T form [11, Example 1], as in item (b) above, $\beta^2 = 2$ and $Q = I$ so that $\mathcal{N}(\beta^2 I - Q) = \{0\}$ while $\|T^n e_1\|$ oscillates between 1 and β).

For a power bounded operator on a finite-dimensional space, the Cesàro asymptotic limit Q exists and coincides with the φ-asymptotic limit A_φ for every Banach limit φ [11, Theorem 2.1]. The next theorem gives a condition for $Q = A_\varphi$ on an infinite-dimensional space. As we saw in the proof of Theorem 7.1(c), if the sequence $\{Q_n\}$ of Cesàro means converges weakly to, say Q, then the sequence $\{T^{*j} Q_n T^j\}$

of Cesàro means converges weakly (again to Q) for every positive integer j. If such weak convergence holds uniformly in j, then $Q = A_\varphi$ for all Banach limits φ.

Theorem 7.2. *If T is a Hilbert-space power bounded operator for which the sequence $\{T^{*j}Q_nT^j\}$ of Cesàro means converges weakly and uniformly in j,*

$$T^{*j}Q_nT^j = \tfrac{1}{n}\sum\nolimits_{k=0}^{n-1} T^{*k+j}T^{k+j} \xrightarrow{\;w\;} Q \qquad \textit{uniformly in } j,$$

then the Cesàro asymptotic limit coincides with the φ-asymptotic limit,

$$Q = A_\varphi,$$

for all Banach limits $\varphi\colon \ell^\infty_+ \to \mathbb{C}$.

Proof. By Theorem 7.1(c), $Q_n \xrightarrow{\;w\;} Q$ if and only if $T^{*j}Q_nT^j \xrightarrow{\;w\;} Q$ which means

$$\tfrac{1}{n}\sum\nolimits_{k=0}^{n-1} \|T^{k+j}x\|^2 \to \|Q^{\frac{1}{2}}x\|^2 \qquad \text{for every } x \in \mathcal{X} \text{ and every } j \ge 0,$$

as $Q \ge O$. If the weak convergence of $\{T^{*j}Q_nT^j\}$ holds uniformly in j, then so does the above convergence. But the real-valued sequence $\big\{\frac{1}{n}\sum_{k=0}^{n-1}\|T^{k+j}x\|^2\big\}$ of Cesàro means converges uniformly in j if and only if all Banach limits $\varphi \in \mathcal{X}^*$ coincide at the sequence $\{\|T^nx\|^2\}$ and are equal to $\|Q^{\frac{1}{2}}x\|^2$ [29, Theorem 1] (also [34]); that is,

$$\varphi(\{\|T^nx\|^2\}) = \|Q^{\frac{1}{2}}x\|^2$$

for all Banach limits $\varphi\colon \ell^\infty_+ \to \mathbb{C}$. In particular, this holds for the arbitrary Banach limit φ of Theorem 6.1 since T is power bounded. For that Banach limit we got

$$\varphi(\{\|T^nx\|^2) = \|A_\varphi{}^{\frac{1}{2}}x\|^2$$

where $A_\varphi \ge O$ is the φ-asymptotic limit of T (associated with φ). Thus $\|Q^{\frac{1}{2}}x\|^2 = \|A_\varphi{}^{\frac{1}{2}}x\|^2$ or, equivalently, $\langle (Q - A_\varphi)x, ; x\rangle = 0$, for every $x \in \mathcal{X}$. This means

$$Q = A_\varphi$$

(either because the Hilbert space is complex or because $Q - A_\varphi$ is self-adjoint). □

Remark 7.2. (a) A class of operators that satisfies the assumption of Theorem 7.2 is the class of quasinormal operators. A Hilbert-space operator T is quasinormal if it commutes with T^*T. If $T \in \mathcal{B}[\mathcal{X}]$ is quasinormal on a Hilbert space $\mathcal{X}$, then by two trivial inductions we get $T^*T\,T^k = T^kT^*T$ for every $k \ge 1$, and consequently $T^{*k}T^k = (T^*T)^k$ for every $k \ge 1$. This in fact is equivalent to quasinormality — see, e.g., [14, Proposition 13] and [13, Theorem 3.6]. Therefore

> *there is an operator S for which $T^{*k}T^k = S^k$ for every $k \ge 1$ if and only if T is quasinormal, and such an operator is unique and given by $S = |T|^2 = T^*T$.*

In this case, $T^{*k+j}T^{k+j} = (T^*T)^{k+j} = S^{k+j}$ for every nonnegative integers j, k. If T is power bounded, then so is S, and the Mean Ergodic Theorem for power bounded operators (which holds in reflexive Banach spaces — see, e.g., [7, Corollary VIII.5.4]) ensures strong convergence for the sequence of Cesàro means $\big\{\frac{1}{n}\sum_{k=0}^{n-1} S^k\big\}$ whose strong limit Q lies in $\mathcal{B}[\mathcal{X}]$ by the Banach–Steinhaus Theorem. Thus

$$Q_n = \tfrac{1}{n}\sum\nolimits_{k=0}^{n-1} T^{*k}T^k = \tfrac{1}{n}\sum\nolimits_{k=0}^{n-1} S^k \xrightarrow{\;s\;} Q,$$

where $Q \ge O$ is the Cesàro asymptotic limit of T (cf. proof Theorem 7.1). Hence

$$\tfrac{1}{n}\sum\nolimits_{k=0}^{n-1} S^{k+j} = \tfrac{1}{n}\sum\nolimits_{k=0}^{n-1} T^{*k+j}T^{k+j} = T^{*j}Q_nT^j \xrightarrow{\;s\;} T^{*j}QT^j = Q = S^jQ = QS^j$$

for every j according to Theorem 7.1 (as strong convergence implies weak convergence). Take an arbitrary $x \in \mathcal{X}$. By the above strong convergence

$$\begin{aligned} \sup_j \Big\| \Big(\tfrac{1}{n} \sum_{k=0}^{n-1} T^{*k+j} T^{k+j} - Q \Big) x \Big\| &= \sup_j \Big\| S^j \Big(\tfrac{1}{n} \sum_{k=0}^{n-1} S^k - Q \Big) x \Big\| \\ &\le \sup_j \|S^j\| \, \Big\| \Big(\tfrac{1}{n} \sum_{k=0}^{n-1} S^k - Q \Big) x \Big\| \to 0. \end{aligned}$$

Then $\frac{1}{n}\sum_{k=0}^{n-1} T^{*k+j} T^{k+j} \xrightarrow{\;s\;} Q$ uniformly in j. Hence (again, since strong convergence implies weak convergence) $\frac{1}{n}\sum_{k=0}^{n-1} T^{*k+j} T^{k+j} \xrightarrow{\;w\;} Q$ uniformly in j. (Indeed, $\sup_j |\langle (\frac{1}{n}\sum_{k=0}^{n-1} T^{*k+j} T^{k+j} - Q) x\,; x\rangle| \le \sup_j \|(\frac{1}{n}\sum_{k=0}^{n-1} T^{*k+j} T^{k+j} - Q) x\| \, \|x\|$.) So

$$\tfrac{1}{n} \sum_{k=0}^{n-1} \|T^{k+j} x\|^2 \to \|Q^{\frac{1}{2}} x\|^2 \quad \text{uniformly in } j.$$

Thus, according to Theorem 7.2, $Q = A_\varphi$ for all Banach limits φ, where A_φ is the φ-asymptotic limit for the power bounded operator T as in Theorem 6.1.

(b) A normed-space operator T is normaloid if $\|T^n\| = \|T\|^n$ for every integer $n \ge 0$. By the Gelfand–Beurling formula, on a complex Banach-space a normaloid is an operator T for which spectral radius coincides with norm: $r(T) = \|T\|$. Since power boundedness implies $r(T) \le 1$, then it follows at once that

> *a power bounded operator is normaloid if and only if it is a normaloid contraction.*

(In fact, *if a normaloid operator is similar to a power bounded operator, then it is a contraction* [26, Proposition 1].) Quasinormal is a class of operators including the normal operators and the isometries, and it is included in the class of subnormal operators, which is included in the class of hyponormal operators, which in turn is included in the class of paranormal operators, which are all normaloid. So all these Hilbert-space normaloid operators, when power bounded, are contractions and so they naturally fit to Proposition 7.1 (and consequently they trivially fit to Theorem 7.2 — see Remark 6.2).

Corollary 7.1. *Let T be a Hilbert-space power bounded operator. If the sequence $\{Q_n\}$ of Cesàro means converges uniformly,*

$$Q_n = \tfrac{1}{n} \sum_{k=0}^{n-1} T^{*k} T^k \xrightarrow{\;u\;} Q,$$

then the Cesàro asymptotic limit coincides with the φ-asymptotic limit,

$$Q = A_\varphi,$$

for all Banach limits $\varphi\colon \ell^\infty_+ \to \mathbb{C}$.

Proof. Consider the setup of Theorem 7.1. Recall that $Q = T^{*j} Q T^j$ for every $j \ge 1$. If $Q_n = \frac{1}{n}\sum_{k=0}^{n-1} T^{*k} T^k \xrightarrow{\;u\;} Q$, then

$$\sup_j \Big\| \tfrac{1}{n} \sum_{k=0}^{n-1} T^{*k+j} T^{k+j} - Q \Big\| \le \sup_j \|T^j\|^2 \Big\| \tfrac{1}{n} \sum_{k=0}^{n-1} T^{*k} T^k - Q \Big\| \to 0.$$

Thus $\frac{1}{n}\sum_{k=0}^{n-1} T^{*k+j} T^{k+j} \xrightarrow{\;u\;} Q$ (and so $\frac{1}{n}\sum_{k=0}^{n-1} T^{*k+j} T^{k+j} \xrightarrow{\;w\;} Q$) uniformly in j. If T is power bounded, then $Q = A_\varphi$ for all φ by Theorem 7.2. □

For instance, let T is a uniformly stable noncontraction (i.e., $r(T) < 1 < \|T\|$) acting on any Hilbert space $\mathcal{X}$. Then $T^n \xrightarrow{\;u\;} O$ (so that T is power bounded) or, equivalently, $T^{*n}T^n \xrightarrow{u} O$, and so $Q_n = \frac{1}{n}\sum_{k=0}^{n-1}T^{*k}T^k \xrightarrow{u} Q = O = A_\varphi$ for all Banach limits $\varphi\colon \ell_+^\infty \to \mathbb{C}$ (in accordance with Corollary 7.1).

Remark 7.3. If T is a power bounded operator on a finite-dimensional space, then $Q = A_\varphi$ for all Banach limits $\varphi\colon \ell_+^\infty \to \mathbb{C}$. Indeed, for power bounded operators on a finite-dimensional space (where weak, strong, and uniform convergences coincide), the Cesàro asymptotic limit exists [11, Theorem 2.1]. Thus $Q = A_\varphi$ for all Banach limits φ by Corollary 7.1.

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