Linear and Multilinear Algebra, **69** (2021) 515–525

Power bounded m-left invertible operators

B.P. Duggal and C.S. Kubrusly

05 April 2019

Abstract

A Hilbert space operator $S \in B(\mathcal{H})$ is left m-invertible by $T \in B(\mathcal{H})$ if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^j S^j = 0,$$

S is m-isometric if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} S^{*j} S^j = 0$$

and S is (m, C)-isometric for some conjugation C of $\mathcal{H}$ if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} S^{*j} C S^j C = 0.$$

If a power bounded operator S is left invertible by a power bounded operator T, then S (also, T^*) is similar to an isometry. Translated to m-isometric and (m, C)-isometric operators S this implies that S is 1-isometric, equivalently isometric, and (respectively) $(1, C)$-isometric.

1. Introduction

Given a complex infinite dimensional Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the algebra of bounded linear transformations, equivalently operators, on $\mathcal{H}$ into itself. Given operators $S, T \in B(\mathcal{H})$, let

$$P_m(S, T) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^j S^j.$$

We say that T is a left m-inverse of S (equivalently, S is left m-invertible by T) for some integer $m > 0$ if

$$P_m(S, T) = 0$$

[2, 11, 13]. Left m-invertible operators occur quite naturally, and the class of m-isometric operators, i.e., operators $S \in B(\mathcal{H})$ such that

$$P_m(S, S^*) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} S^{*j} S^j = 0,$$

AMS(MOS) subject classification (2010). Primary: Primary47A05, 47A55; Secondary 47A80, 47A10.
Keywords: Hilbert space, Power bounded operator, m-left invertible operator, m-isometric operator, (m, C)-isometric operator, similar to an isometry.

of Agler and Stankus [1] is an important widely studied example of operators left m-invertible by their adjoint. A generalization of m-isometric operators, which has been considered in the recent past [7], is that of (m, C)-isometric operators. Here an operator $S \in B(\mathcal{H})$ is (m, C)-isometric for some conjugation C of $\mathcal{H}$ if

$$P_m(CSC, S^*) = \sum_{j=0}^{m} (-1)^{m-j} \begin{pmatrix} m \\ j \end{pmatrix} S^{*j} C S^j C = 0.$$

(Recall that a conjugation C of $\mathcal{H}$ is an antilinear operator such that $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$.)

An operator $S \in B(\mathcal{H})$ is power bounded if there exists a scalar $M > 0$ such that

$$\sup_{n \in \mathbb{N}} ||S^n|| < M.$$

It is immediate from the definition that if $S \in B(\mathcal{H})$ is power bounded, then the spectral radius

$$r(S) = \lim_{n \to \infty} ||S^n||^{\frac{1}{n}} = 1$$

and the spectrum $\sigma(S)$ of S satisfies $\sigma(S) \subseteq \mathbb{D}$ $(= \{\lambda \in \mathbb{C} : |\lambda| \le 1\})$. Recall from [12, Theorem 2.4] that an m-isometric operator is power bounded if and only if it isometric.

Given a positive operator $A \in B(\mathcal{H})$, $A \ge 0$, let $||.||_A$ denote the semi-norm

$$||x||_A^2 = \langle x, x \rangle_A = \langle Ax, x \rangle, \quad x \in \mathcal{H}.$$

(Then $||.||_A$ is a norm on $\mathcal{H}$ if and only if A is injective.) An operator $A \in B(\mathcal{H})$ is said to be A-isometric if $S^*AS = A$ [5] and S is an (A, m)-isometry [12] if

$$P_m(A; S, S^*) = \sum_{j=0}^{m} (-1)^{m-j} \begin{pmatrix} m \\ j \end{pmatrix} S^{*j} A S^j = 0.$$

This paper considers left m-invertible operators such that both the operator $S \in B(\mathcal{H})$ and its left m-inverse $T \in B(\mathcal{H})$ are power bounded. It is proved that there exist positive invertible operators P_i, $P_i > 0$ $(i = 1, 2)$, such that $S = P_1 V_1 P_1^{-1}$ and $T^* = P_2 V_2 P_2^{-1}$ for some isometries V_i, $i = 1, 2$. Translated to m-isometric and (m, C)-isometric S this means that: *a power bounded m-isometric operator is isometric and a power bounded (m, C)-isometric operator is $(1, C)$-isometric.*

2. Results

Given an operator $A \in B(\mathcal{H})$, we write $A - \lambda$ for $A - \lambda I$, $\lambda \in \mathbb{C}$. A has SVEP, *the single-valued extension property*, at $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathbb{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to \mathcal{H}$ satisfying $(A - \lambda) f(\lambda) = 0$ is the function $f \equiv 0$ [3, 16]. Every operator A has SVEP at points in its resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and on the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. *We say that A has SVEP on a set Ξ if it has SVEP at every $\lambda \in \Xi$.* The *ascent of A*, $asc(A)$, is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$: If no such integer exists, then $asc(A) = \infty$. It is well known that $asc(A) < \infty$ implies A has SVEP at 0 [3, 16].

For $A, B \in B(\mathcal{H})$, let $\triangle_{A,B} \in B(B(\mathcal{H}))$ denote the elementary operator $\triangle_{A,B}(X) = AXB - X = (L_A R_B - I)(X)$, where L_A and $R_B \in B(B(\mathcal{H}))$ are the operators $L_A(X) = AX$ and $R_B(X) = XB$ (of left multiplication by A and, respectively, right multiplication by B). It is known, see for example [18], that if A, B are normal operators, then ($\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$, consequently) $\mathrm{asc}(\triangle_{A,B}) \leq 1$.

$A \in B(\mathcal{H})$ is a $C_{0.}$, respectively $C_{1.}$, operator if

$$\lim_{n\to\infty} ||A^n x|| = 0 \quad \text{for all} \quad x \in \mathcal{H},$$

$$\text{respectively,} \quad \inf_{n\in\mathbb{N}} ||A^n x|| > 0 \quad \text{for all} \quad 0 \neq x \in \mathcal{H};$$

$A \in C_{.0}$ (resp., $A \in C_{.1}$) if $A^* \in C_{0.}$ (resp., $A^* \in C_{1.}$) and $A \in C_{\alpha\beta}$ if $A \in C_{\alpha.} \cap C_{.\beta}$ ($\alpha, \beta = 0, 1$). It is well known [14] that every power bounded operator $A \in B(\mathcal{H})$ has an upper triangular matrix representation

$$A = \begin{pmatrix} A_1 & A_0 \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

for some decomposition $\mathcal{H} - \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$ such that $A_1 \in C_{0.}$ and $A_2 \in C_{1.}$. Recall that every isometry $V \in B(\mathcal{H})$ has a direct sum decomposition

$$V = V_c \oplus V_u \in B(\mathcal{H}_c \oplus \mathcal{H}_u), \quad V_c \in C_{10} \quad \text{and} \quad V_u \in C_{11}$$

into its completely non-unitary (i.e., unilateral shift) and unitary parts.

Let $\delta_{A,B} \in B(B(\mathcal{H}))$ denote the generalized derivation $\delta_{A,B}(X) = AX - XB$. Recall from [10] that $A \in B(\mathcal{H})$ satisfies (the Putnam-Fuglede) property PF($\triangle$) (resp., PF(δ)), $A \in$ PF($\triangle$) (resp., $A \in$ PF(δ)), if (either A is trivially unitary, or) for every isometry $V \in B(\mathcal{H})$ for which $\triangle_{A,V^*}(X) = 0$ (resp. $\delta_{A^*,V}(X) = 0$) has a non-trivial solution $X \in B(\mathcal{H})$, X is also a solution of $\triangle_{A^*,V}(X) = 0$ (resp., $\delta_{A^*,V}(X) = 0$). The following theorem is in [10, Corollary 2.5] (see also [17]). Let $d_{A,B}$ denote either of $\triangle_{A,B}$ and $\delta_{A,B}$, and, correspondingly, let PF(d) denote either of PF($\triangle$) and PF(δ). (Recall from [10, Theorem 2.1] that $A \in$ PF($\triangle$) if and only if $A \in$ PF(δ).)

Theorem 2.1 *A power bounded operator satisfies property* PF(d) *if and only if it is the direct sum of a unitary with a* $C_{.0}$ *operator.*

The PF(d) property implies range-kernel orthogonality (i.e., if $d^{-1}_{A,V^*}(0) \subseteq d^{-1}_{A^*,V}(0)$, then $d^{-1}_{A,V^*}(0) \perp d_{A,V^*}(B(\mathcal{H}))$ in the sense of G. Birkhoff [6]), hence d_{A,V^*} has finite ascent [9, Proposition 2.6].

Theorem 2.2 *If* $d^{-1}_{A,V^*}(0) \subseteq d^{-1}_{A^*,V}(0)$, *then* $\mathrm{asc}(d_{A,V^*}) \leq 1$.

The following result from [8] will be used in some of our argument below.

Theorem 2.3 *If* $A, B \in B(\mathcal{H})$, *then the following statements are pairwise equivalent.*

(i) ran(A) $\subseteq$ ran(B).

(ii) *There is a* $\mu \geq 0$ *such that* $AA^* \leq \mu^2 BB^*$.

(iii) *There is an operator* $C \in B(\mathcal{H})$ *such that* $A = BC$.

Furthermore, if these conditions hold, then the operator C *may be chosen so that*
(a) $||C||^2 = \inf\{\lambda : AA^* \leq \lambda BB^*\}$; (b) $\ker(A) = \ker(C)$; (c) $\mathrm{ran}(C) \subseteq \ker(B)^\perp$.

We note for future reference that $P_m(S,T)=0$ implies $P_m(S^n,T^n)=0$, i.e., *$S\in B(\mathcal{H})$ left m-invertible by $T\in B(\mathcal{H})$ implies S^n left m-invertible by T^n*, for all $n\in\mathtt{N}$ [11].

Our main result considers left m-invertible operators S such that both S and its left m-inverse T are power bounded to prove that such operators are A isometric for some $A>0$.

Theorem 2.4 *If $S\in B(\mathcal{H})$ is left m-invertible by a power bounded operator $T\in B(\mathcal{H})$, then the following statements are mutually equivalent.*

(i) *S is power bounded.*

(ii) *There exists a positive invertible operator $P\in B(\mathcal{H})$ and an isometry $V\in B(\mathcal{H})$ such that $S=PVP^{-1}$.*

(iii) *There exists a positive invertible operator $A\in B(\mathcal{H})$ such that $T=A^{-1}S^*A$ is a power bounded left m-inverse of S.*

Furthermore, if either of the statements (i)*,* (ii) *and* (iii) *above holds, and S^* has SVEP at* 0 (*or, S has a dense range*)*, then S and T are* (*respectively*) *similar to some unitaries U_1 and U_2 such that $U_1=PU_2P^{-1}$ for some invertible operator P.*

Proof. (i) $\Longrightarrow$ (ii). Let $P_m(S,T)=0$. The hypothesis S and T are power bounded implies the existence of a scalar $M_1>0$ such that

$$\sup_{n\in\mathtt{N}}\{||S^n||,||T^n||\}\le M_1.$$

The left m-invertibility of S by T implies the left invertibility of S^n by T^n for all $n\in\mathtt{N}$, i.e.,

$$P_m(S,T)=0\implies P_m(S^n,T^n)=0,\ n\in\mathtt{N}$$

[11]. Define Z_n by

$$Z_n=(-1)^{m+1}\sum_{j=0}^{m}(-1)^{m-j}\begin{pmatrix} m\\ j\end{pmatrix}T^{nj}S^{n(j-1)}.$$

Then

$$P_m(S^n,T^n)=0\iff Z_nS^n=I$$

for all $n\in\mathtt{N}$ and this, since

$$||Z_n||\le\left\{1+\begin{pmatrix} m\\ 1\end{pmatrix}+\cdots+\begin{pmatrix} m\\ m-1\end{pmatrix}\right\}M_1^2\le 2^mM_1^2=M$$

for some scalar $M>0$, implies

$$||x||=||Z_nS^nx||\le M||S^nx||$$

for all $x\in\mathcal{H}$. Already

$$||S^nx||\le||S^n||||x||\le M_1||x||;$$

hence

$$\frac{1}{M}||x||\le||S^nx||\le M_1||x||$$

for all $x\in\mathcal{H}$ and integers $n\ge 1$. Thus, see for example [15], S is similar to an isometry V_1. Let

$$S=EV_1E^{-1}\iff V_1=E^{-1}SE$$

for some (invertible operator $E\in B(\mathcal{H})$ and) isometry V_1. Then

$$S^*|E^{-1}|^2S=|E^{-1}|^2 \iff S^*P^2S=P^2, \quad P=|E|^{-1},$$

implies (by Theorem 2.3) the existence of an isometry $V\in B(\mathcal{H})$ such that

$$S^*P=PV^* \iff S=P^{-1}VP.$$

(ii) $\Longrightarrow$ (iii). If $S=P^{-1}VP$, P and V as above, then $S^{*n}P^2S^n=P^2$ for all $n\in\mathtt{N}$. Hence

$$\sum_{j=0}^{m}(-1)^{m-j}\left(\begin{array}{c}m\\ j\end{array}\right)(P^{-2}S^*P^2)^jS^j=\sum_{j=0}^{m}(-1)^{m-j}\left(\begin{array}{c}m\\ j\end{array}\right)I=0,$$

i.e., $P^{-2}S^*P^2=T$ is a power bounded left m-inverse of S.

(iii) $\Longrightarrow$ (i). If $T=A^{-1}S^*A$ is a power bounded left m-inverse of S, the

$$||S^n||=||S^{*n}||=||AT^nA^{-1}||\leq||A||\,||A^{-1}||\,||T^n||$$

implies S is power bonded.

Assume next that (i) is satisfied, and hence that $S=P_1U_1P_1^{-1}$ for some isometry U_1 and $P_1>0$. If S^* has SVEP (or, S has a dense range), then the left invertibility of S implies S is invertible [3], and this in turn implies that the isometry U_1 is indeed a unitary. Since

$$P_m(S,T)=0 \iff P_m(T^*,S^*)=P_m(S,T)^*=0,$$

and the operators T^* and S^* are power bounded, the equivalence (i) $\iff$ (iii) implies the existence of a positive invertible operator P_2 and an isometry U_2^* such that $T^*=P_2^{-1}U_2^*P_2$. Evidently, T^* is (m-left invertible, hence) left invertible. We prove that T^*, hence U_2, is invertible. Clearly,

$$\begin{aligned}P_m(T^*,S^*)\;=\;0 &\iff \sum_{j=0}^{m}(-1)^{m-j}\left(\begin{array}{c}m\\ j\end{array}\right)P_1^{-1}U_1^{*j}P_1T^{*j}\;=0\\ &\iff \sum_{j=0}^{m}(-1)^{m-j}\left(\begin{array}{c}m\\ j\end{array}\right)U_1^{*j}P_1T^{*j}\;=\;0\\ &\iff \sum_{j=0}^{m}(-1)^{m-j}\left(\begin{array}{c}m\\ j\end{array}\right)T^jP_1U_1^j\;=\;0\end{aligned}$$

The operator T^* being power bounded has an upper triangular matrix representation

$$T^*=\left(\begin{array}{cc}T_1^* & T_0^*\\ 0 & T_2^*\end{array}\right)\in B(\mathcal{H}_1\oplus\mathcal{H}_2),$$

where $T_1^*\in C_{0.}$ and $T_2^*\in C_{1.}$ [14]. Clearly, $U_2^*=U_{21}^*\oplus U_{22}^*\in B(\mathcal{H}_c\oplus\mathcal{H}_u)$ for some decomposition $\mathcal{H}=\mathcal{H}_c\oplus\mathcal{H}_u$ of $\mathcal{H}$ such that $U_{21}^*=U_2^*|_{\mathcal{H}_c}$ is the backward unilateral shift and $U_{22}^*=U_2^*|_{\mathcal{H}_u}$ is unitary; let $Q=P_2\in B(\mathcal{H}_c\oplus\mathcal{H}_u,\mathcal{H}_1\oplus\mathcal{H}_2)$ have the matrix representation $Q=\left(\begin{array}{cc}Q_{11} & Q_{12}\\ Q_{12}^* & Q_{22}\end{array}\right)$. Then since $T^*=P_2^{-1}U_2^*P_2$.

$$Q_{12}^*T_1^*=U_{22}^*Q_{12}^*, \quad Q_{11}T_0^*+Q_{12}T_2^*=U_{21}^*Q_{12}, \quad Q_{11}T_1^*=U_{21}^*Q_{11}.$$

Since $Q_{12}^*T_1^*=U_{22}^*Q_{12}^*$ implies $Q_{12}^*T_1^{*n}=U_{22}^{*n}Q_{12}^*$ for all $n\in\mathtt{N}$, since U_{22}^* is unitary and $T_1^*\in C_{0.}$,

$$\begin{aligned}||Q_{12}^*x|| &= \lim_{n\longrightarrow\infty}||U_{22}^{*n}Q_{12}^*x||\\ &= \lim_{n\to\infty}||Q_{12}^*T_1^{*n}x||\leq||Q_{12}^*||\lim_{n\to\infty}||T_1^{*n}x||=0\end{aligned}$$

for all $x \in \mathcal{H}_c$. Thus $Q_{12} = 0$, and then Q_{11}, Q_{22} are invertible positive operators and $Q_{11}T_0^* + Q_{12}T_2^* = U_{21}^*Q_{12}$ implies $T_0^* = 0$. Considering now the equation $Q_{11}T_1^* = U_{21}^*Q_{11}$, we have

$$||T_1^n x|| = ||Q_{11}U_{21}^n Q_{11}^{-1}x|| \leq ||Q_{11}||\, ||U_{12}^n Q_{11}^{-1}x||$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}_c$. Since $U_{21} \in C_{0\cdot}$, we conclude that $T_1 \in C_{0\cdot} \cap C_{\cdot 0} = C_{00}$. Hence T is a power bounded operator which is the direct sum of a C_{00} operator with $T_2 = Q_{22}^{-1}U_{22}Q_{22}$ (where U_{22} is unitary and Q_{22} is positive invertible).

Define the operator $A \in B(\mathcal{H}_c \oplus \mathcal{H}_u)$, $X \in B(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_c \oplus \mathcal{H}_u)$ and $E \in B(\mathcal{H}, \mathcal{H}_c \oplus \mathcal{H}_u)$ by

$$A = T_1 \oplus U_{22}, \quad X = I \oplus Q_{22} \quad \text{and} \quad E = XP_1.$$

Then

$$\begin{aligned} & \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^j P_1 U_1^j = 0 \\ \iff & \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} X^{-1}A^j X P_1 U_1^j = 0 \\ \iff & \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} A^j E U_1^j = 0 \\ \iff & (L_A R_{U_1} - 1)^m(E) = \triangle_{A,U_1}^m(E) = 0. \end{aligned}$$

Since the operator $A = XTX^{-1}$ is a power bounded operator which is the direct sum of a unitary with a $C_{\cdot 0}$ operator and the operator U_1 is unitary, it follows from an application of Theorems 2.1 and 2.2 that

$$\triangle_{A,U_1}(E) = 0.$$

Equivalently,

$$AEU_1 - E = 0 \iff TP_1U_1 - P_1 = 0 \iff T = P_1U_1^*P_1^{-1}.$$

This implies T is invertible, hence (U_2 is unitary and)

$$P_1^{-1}U_1P_1 = P_2U_2P_2^{-1} \iff U_1 = P_1P_2U_2P_2^{-1}P_1^{-1}.$$

Now define (the invertible operator) P by $P = P_2P_1^{-1}$ to complete the proof. □

It is immediate from the theorem that if a power bounded operator $S_1 \in B(\mathcal{H})$ is left m-invertible by a power bounded operator $S_2^* \in B(\mathcal{H})$, then there exist invertible operators $A_i > 0$ in $B(\mathcal{H})$ such that S_i is an (A_i, m)-isometry; $i = 1, 2$.

Recall that an operator $A \in B(\mathcal{H})$ is said to be *supercyclic* if there exists a vector $x \in \mathcal{H}$ such that the projective orbit of x under A,

$$\mathcal{O}_A(\text{span}\{x\}) = \{\alpha A^n x \in \mathcal{H} \colon \alpha \in \mathbb{C}, \ n \geq 0\},$$

is dense in $\mathcal{H}$. Similarities preserve supercyclicity, and power bounded operators of class $C_{1\cdot}$ can not be supercyclic [4, Theorem 2.1]. Since isometries are $C_{1\cdot}$ operators:

Corollary 2.5 *If a power bounded operator $S \in B(\mathcal{H})$ is left m-invertible by a power bounded operator $T \in B(\mathcal{H})$, then neither of S and T is supercyclic.*

Theorem 2.4 is a generalization of the result: m-isometric power bounded operators are isometric [12]. That Theorem 2.4 does indeed imply this result is the content of the following proposition. (We remark here that the argument proving the proposition below differs radically from the argument used in [12].)

Proposition 2.6 *Power bounded m-isometric operators $S \in B(\mathcal{H})$ are isometric.*

Proof. If $S \in m$-isometric is power bounded, then (as seen above) $S = P^{-1}VP$ for some isometry V and $P > 0$. We prove that $[P, S] = PS - SP = 0$ (and this would then imply that $S = V$). Evidently, $S = P^{-1}VP$ implies $S^*P = PV^*$. Decompose V into its completely non-unitary (i.e., unilateral shift) and unitary parts by

$$V = V_c \oplus V_u \in B(\mathcal{H}_c \oplus \mathcal{H}_u).$$

The operator S being power bounded has an upper triangular matrix representation

$$S = \begin{pmatrix} S_1 & S_0 \\ 0 & S_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

where $S_1 \in C_{0.}$ and $S_2 \in C_{1.}$ [14]. Let $P \in B(\mathcal{H}_c \oplus \mathcal{H}_u, \mathcal{H}_1 \oplus \mathcal{H}_2)$ have the representation

$$P = \begin{pmatrix} P_1 & P_3 \\ P_3^* & P_2 \end{pmatrix}.$$

Then $S^*P = PV^*$ implies

$$S_1^*P_3 = P_3V_u^* \iff V_uP_3^* = P_3^*S_1 \implies V_u^nP_3^* = P_3^*S_1^n$$

for all $n \in \mathtt{N}$. Hence

$$||P_3^*x|| = ||V_u^nP_3^*x|| = ||P_3^*S_1^nx|| \leq ||P_3^*||\,||S_1^nx|| \to 0 \text{ as } n \to \infty$$

for all $x \in \mathcal{H}_1$. Thus $P_3 = 0$ and $P_1, P_2 > 0$. Since $S^*P = PV^*$ now implies $S_0^*P_1 = 0$, we must have $S_0 = 0$. Hence

$$S = S_1 \oplus S_2, \quad S_2 = P_2^{-1}V_uP_2 \quad \text{and} \quad S_1 \in C_{0.}.$$

Evidently, $S \in m$-isometric implies

$$(L_{S_2^*}R_{S_2} - I)^m(I) = 0 \iff (L_{V_u^*}R_{V_u} - I)^m(P_2^{-2}) = 0$$

Applying Theorems 2.1 and 2.2 it follows that

$$\begin{aligned} (L_{S_2^*}R_{S_2} - I)^m(I) = 0 &\iff (L_{V_u^*}R_{V_u} - I)^m(P_2^{-2}) = 0 \\ &\iff (L_{V_u^*}R_{V_u} - I)(P_2^{-2}) = 0 \quad (\text{i.e., } asc(L_{V_u^*}R_{V_u} - I) \leq 1) \\ &\iff [V_u, P_2^{-2}] = 0 \iff [V_u, P_2^{-1}] = 0 \\ &\implies S_2 = V_u. \end{aligned}$$

Conclusion: S^* is the direct sum of a $C_{.0}$ and a unitary operator. Applying Theorem 2.1 to $S^*P = PV^*$, V isometric, it follows that $SP = PV$. Hence

$$PS^*P = P^2V^* \iff [V^*, P^2] = 0 \iff [V, P] = 0 \iff [S, P] = 0,$$

and the proof is complete. □

The following corollary is immediate from Proposition 2.6, since either of the hypotheses S has a dense range and S^* has SVEP at 0 for an m-isometric operator S implies the invertibility of S.

Corollary 2.7 *If a power bounded m-isometric operator is such that either S^* has SVEP at 0 or S has a dense range, then S is unitary.*

M. Chō et al [7, Theorem 3.15] prove that "if $S \in B(\mathcal{H})$ is a power bounded (m, C)-isometric operator such that $P_1(CSC, S^*)$ is normaloid (i.e., its norm equals its spectral radius), then S is $(1, C)$-isometric. The following proposition shows that the hypothesis $P_1(CSC, S^*)$ is normaloid is redundant, and that the power boundedness of S is sufficient to guarantee that S is $(1, C)$-isometric.

Proposition 2.8 *Power bounded (m, C)-isometric operators are $(1, C)$-isometric.*

Proof. By definition,

$$\begin{aligned} S \text{ is } (m,C)\text{-isometric} &\iff P_m(CSC, S^*) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} S^{*j} C S^j C = 0 \\ &\iff P_m(S, CS^*C) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} C S^{*j} C S^j = 0. \end{aligned}$$

Arguing as in the proof of Proposition 2.6, it follows from the left m-invertibility and the power boundedness of S (consequently, also that of S^* and CS^*C) that there exists a decomposition

$$S = S_1 \oplus S_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad S_1 \in C_{0.} \quad \text{and} \quad S_2 \in C_{1.},$$

of S and a positive invertible operator $Q = Q_1 \oplus Q_2 \in B(\mathcal{H}_c \oplus \mathcal{H}_u, \mathcal{H}_1 \oplus \mathcal{H}_2)$ such that

$$S_1 = Q_1^{-1} V_c Q_1 \quad \text{and} \quad S_2 = Q_2^{-1} V_u Q_2$$

for some unilateral shift $V_c \in B(\mathcal{H}_c)$ and unitary $V_u \in B(\mathcal{H}_u)$. Set

$$Q_1 \oplus Q_2 = Q, \quad V_c \oplus V_u = V.$$

Evidently,

$$S \text{ is } (m,C)\text{-isometric} \iff \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} S^{*j} C S^j = (L_{S^*} R_S - I)^m (C) = 0.$$

We prove that $C \colon \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ has a decomposition $C = C_{11} \oplus C_{22}$. Let $C \colon \mathcal{H}_1 \oplus \mathcal{H}_2$ into itself have the matrix representation

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2).$$

Since

$$\begin{aligned} (L_{S^*} R_S - I)^m (C) = 0 &\iff Q\{(L_{V^*} R_V - I)^m (Q^{-1} C Q^{-1})\} Q = 0 \\ &\iff (L_{V^*} R_V - I)^m (Q^{-1} C Q^{-1}) = 0, \end{aligned}$$

$$\begin{aligned} &(L_{S^*} R_S - I)^m (C) = 0 \\ \iff &\begin{pmatrix} (L_{V_c^*} R_{V_c} - I)^m (Q_1^{-1} C_{11} Q_1^{-1}) & (L_{V_c^*} R_{V_u} - I)^m (Q_1^{-1} C_{12} Q_2^{-1}) \\ (L_{V_u^*} R_{V_c} - I)^m (Q_2^{-1} C_{21} Q_1^{-1}) & (L_{V_u^*} R_{V_u} - I)^m (Q_2^{-1} C_{22} Q_2^{-1}) \end{pmatrix} = 0. \end{aligned}$$

Set

$$(L_{V_c^*}R_{V_u}-I)^{m-1}(Q_1^{-1}C_{12}Q_2^{-1})=Z_{m-1}.$$

Then, V_u (V_c^*) being unitary (resp. $C_{0.}$),

$$\begin{aligned}&(L_{V_c^*}R_{V_u}-I)^m(Q_1^{-1}C_{12}Q_2^{-1})=0\\ \iff\ &(L_{V_c^*}R_{V_u})(Z_{m-1})=V_c^*Z_{m-1}V_u-Z_{m-1}=0\\ \implies\ &Z_{m-1}=V_c^{*n}Z_{m-1}V_u^n \implies ||Z_{m-1}x||=||V_c^{*n}Z_{m-1}V_u^nx||\to 0 \text{ as } n\to\infty\end{aligned}$$

for all $x\in\mathcal{H}_1$. Hence $Z_{m-1}=0$. Repeating this argument, considering next $(L_{V_c^*}R_{V_u}-I)$ $(Z_{m-2})=0$, a finite number of times it follows that

$$Q_1^{-1}C_{12}Q_2^{-1}=0 \iff C_{12}=0.$$

A similar argument applied to

$$(L_{V_u^*}R_{V_c}-I)^m(T)=0 \iff (L_{V_c^*}R_{V_u}-I)^m(T^*)=0, \quad T=Q_2^{-1}C_{21}Q_1^{-1},$$

implies that

$$T=Q_2^{-1}C_{21}Q_1^{-1}=0 \iff C_{21}=0.$$

Hence

$$C=C_{11}\oplus C_{22}.$$

Considering next the equality $(L_{V_u^*}R_{V_u}-I)^m(Q_2^{-1}C_{22}Q_2^{-1})=0$. Set $Q_2^{-1}C_{22}Q_2^{-1}=H$. Then $(L_{V_u^*}R_{V_u}-I)^m(HC_{22})=0$. Since $C_{22}V_uC_{22}$ and V_u^* are unitary and $HC_{22}\in B(\mathcal{H}_2)$, $(L_{V_u^*}R_{C_{22}V_uC_{22}}-I)^m(HC_{22})=0$ if and only if $(L_{V_u^*}R_{C_{22}V_uC_{22}}-I)(HC_{22})=0$. Hence

$$\begin{aligned}&(L_{V_u^*}R_{V_u}-I)^m(Q_2^{-1}C_{22}Q_2^{-1})=0\\ \iff\ &(L_{V_u^*}R_{V_u}-I)(Q_2^{-1}C_{22}Q_2^{-1})=0\\ \iff\ &V_u^*Q_2^{-1}C_{22}Q_2^{-1}V_u=Q_2^{-1}C_{22}Q_2^{-1}\\ \iff\ &(Q_2V_u^*Q_2^{-1})C_{22}(Q_2^{-1}V_uQ_2)C_{22}=I\\ \iff\ &S_2^*C_{22}S_2C_{22}-1=0.\end{aligned}$$

To complete the proof, we prove next that $S_1^*C_{11}S_1C_{11}-I=0$: This would then imply that

$$0=(S_1\oplus S_2)^*(C_{11}\oplus C_{22})(S_1\oplus S_2)(C_{11}\oplus C_{22})-I=S^*CSC-I.$$

Set

$$(L_{S_1^*}R_{S_1}-I)^{m-1}(C_{11})=X_{m-1}.$$

Then, since S_1^* is power bounded and $S_1\in C_{0.}$,

$$\begin{aligned}&(L_{S_1^*}R_{S_1}-I)(X_{m-1})=(L_{S_1^*}R_{S_1}-I)^m(C_{11})=0\\ \implies\ &||X_{m-1}x||=||S_1^{*n}X_{m-1}S_1^nx||\le||S_1^{*n}||\,||X_{m-1}||\,||S_1^nx|\to 0 \text{ as } n\to\infty\end{aligned}$$

for all $x\in\mathcal{H}_1$. Hence $X_{m-1}=0$. Repeating the argument, see above, it follows eventually that $X_1=S_1^*C_{11}S_1-C_{11}=0$. Hence $S_1^*C_{11}S_1C_{11}-I=0$. □

Remark 2.9 As an immediate consequence of Corollary 2.5, we remark that (just as for m-isometries) the similarity of power bounded (m,C)-isometric $B(\mathcal{H})$ operators implies that such operators can not be supercyclic.

The authors are grateful to a referee for his remarks on the original draft of the paper; in particular, they are thankful to the referee for saving them from committing a very silly mistake.

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B.P. Duggal, Faculty of Mathematics, Visgradska 33, 1800 Niš, Serbia.
e-mail: bpduggal@yahoo.co.uk

C.S. Kubrusly, Applied Mathematics Department, Federal University, Rio de Janeiro, Brazil.
e-mail: carloskubrusly@gmail.com