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RESIDUAL SPECTRUM OF POWER BOUNDED OPERATORS

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Abstract. The residual spectrum of a power bounded operator lies in the open unit disk.

1. Introduction

A well-known open question on Hilbert-space operators asks whether a power bounded operator has the residual spectrum included in the unit circle. The purpose of this paper is to offer an answer to this so far open question. The answer follows from known results and is obtained by elementary arguments.

It has been known for a long time that the above question has an affirmative answer if power bounded is restricted to contractions. All proofs for the contraction case are elementary. We first give a new but still elementary proof for the contraction case, which shows that attempts to extend the contraction case towards the power bounded case might lead to a false start. The power bounded case requires a different (although still elementary and based on a well-known result) start.

A possible new start that leads to a proof for the power bounded case is the Ergodic Theorem for power bounded operators. After this, there are certainly a few different (but similar) paths to proceed. We have chosen what we consider to be an elementary path in the proof of Theorem 4.1. Applications are explored in Sections 5 and 6.

2. Notation

Throughout the paper $\mathcal{H}$ will stand for an infinite-dimensional, complex (not necessarily separable) Hilbert space. The inner product in $\mathcal{H}$ will be denoted by $\langle\cdot\,;\cdot\rangle$. By an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ stand for the C^*-algebra of all operators on $\mathcal{H}$. Both norms in $\mathcal{H}$ or in $\mathcal{B}[\mathcal{H}]$ will be denoted by the same symbol $\|\cdot\|$. An operator $T\in\mathcal{B}[\mathcal{H}]$ is an isometry if $\|Tx\|=\|x\|$ for every $x\in\mathcal{H}$, it is unitary if it is an invertible isometry, a contraction if $\|Tx\|\le\|x\|$ for every $x\in\mathcal{H}$ (i.e., $\|T\|\le 1$), and power bounded if

$$\sup_n \|T^n\| < \infty.$$

Equivalently (by the Banach–Steinhaus Theorem), if $\sup_n \|T^n x\| < \infty$ for every $x\in\mathcal{H}$ (otherwise it is called power unbounded). It is clear that every isometry is a contraction, and every contraction is power bounded. A completely nonunitary contraction is an operator on $\mathcal{H}$ for which its restriction to any reducing subspace is not unitary. For any operator $T\in\mathcal{B}[\mathcal{H}]$, let $\mathcal{N}(T)=T^{-1}(\{0\})$ denote its kernel and let I stand for the identity in $\mathcal{B}[\mathcal{H}]$. The set $\sigma_P(T)=\big\{\lambda\in\mathbb{C}\colon \mathcal{N}(\lambda I-T)\ne\{0\}\big\}$ is the point spectrum of T (i.e., the set of all eigenvalues of T). Let $T^*\in\mathcal{B}[\mathcal{H}]$ stand for the adjoint of $T\in\mathcal{B}[\mathcal{H}]$. The residual spectrum of T is the set [3, p.5]

$$\sigma_R(T)=\sigma_P(T^*)^*\backslash\sigma_P(T).$$

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Here $\Lambda^* = \{\overline{\lambda} \in \mathbb{C} : \lambda \in \Lambda\}$ is the set of all complex conjugates of points in a set $\Lambda \subseteq \mathbb{C}$. Let $\mathbb{D}$ be the open unit disk (centered at the origin of the complex plane), and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle, where $\mathbb{D}^- = \mathbb{D} \cup \mathbb{T}$ is the closed unit disk.

3. Preliminaries

Consider the above set-up. Take an arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$.

Proposition 3.1. *If T is a contraction, then $\sigma_R(T) \subseteq \mathbb{D}$.*

Proof. Take a contraction T acting on a Hilbert space. Consider the Nagy–Foiaş–Langer decomposition

$$T = C \oplus U$$

of T (see e.g., [3, Theorem 5.1]). Here C is a completely nonunitary contraction, U is unitary, and $\oplus$ stands for orthogonal direct sum. Since C is a completely nonunitary contraction,

$$\sigma_P(C) \cup \sigma_R(C) \subseteq \mathbb{D}$$

[3, Corollary 7.4 and Proposition 8.4]. Recall that the point spectrum of an orthogonal direct sum of operators is the union of the point spectra. Thus

$$\sigma_P(C \oplus U) = \sigma_P(C) \cup \sigma_P(U) \quad [\text{and} \quad \sigma_P((C \oplus U)^*) = \sigma_P(C^*) \cup \sigma_P(U^*)\,].$$

Since U is normal, $\sigma_R(U) = \sigma_R(U^*) = \varnothing$, and so

$$\sigma_P(U^*)^* = \sigma_P(U).$$

Therefore,

$$\sigma_R(T) = [\sigma_P(C^*)^* \cup \sigma_P(U)]\backslash[\sigma_P(C) \cup \sigma_P(U)] \subseteq \sigma_R(C) \subseteq \mathbb{D}. \qquad \square$$

For another proof, still more elementary than the above one see, for instance, [3, Proposition 8.5]. Similarity to a contraction implies power boundedness and similarity preserves the spectrum and its parts. In particular, if $\widetilde{T}$ is similar to T, then $\sigma_R(\widetilde{T}) = \sigma_R(T)$. Therefore Proposition 3.1 says

$$T \text{ is similar to a contraction} \quad \Longrightarrow \quad \sigma_R(T) \subseteq \mathbb{D}.$$

Thus it has been asked in [3, p.114] whether Proposition 3.1 (or its equivalent form in the above displayed implication) can be extended to power bounded operators: *does power boundedness imply inclusion of the residual spectrum in the open unit disk*? In other words, if $\sup_n \|T^n\| < \infty$, is it true that $\sigma_R(T) \subseteq \mathbb{D}$? The purpose of the note is to answer this question.

4. Answer

Consider the above set-up. Take an arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$.

Theorem 4.1. *If T is power bounded, then $\sigma_R(T) \subseteq \mathbb{D}$.*

Proof. Consider the Cesàro means associated with the operator $T \in \mathcal{B}[\mathcal{H}]$,

$$C_n = \tfrac{1}{n} \sum_{k=0}^{n-1} T^k \in \mathcal{B}[\mathcal{H}].$$

Recall: *If T is power bounded, then the sequence of Cesàro means $\{C_n\}$ converges strongly.* [In other words, every power bounded operator is (strongly) ergodic].

This is the well-known Mean Ergodic Theorem for power bounded operators (which holds on reflexive Banach spaces; see e.g., [1, Corollary VIII.5.4]). The $\mathcal{B}[\mathcal{H}]$-valued sequence $\{C_n\}$ converges strongly, which means the $\mathcal{H}$-valued sequence $\{C_n x\}$ converges in $\mathcal{H}$ for every $x \in \mathcal{H}$. The Banach–Steinhaus Theorem ensures the existence of an operator $E \in \mathcal{B}[\mathcal{H}]$ for which $C_n x \to Ex$ for every $x \in \mathcal{H}$. Notation:

$$C_n \xrightarrow{\;s\;} E.$$

Since each C_n is a polynomial in T, C_n and T commute, and so E and T commute. Since $C_n T = C_n + \frac{1}{n}T^n - \frac{1}{n}I$ and T is power bounded, $ET = E$. Therefore,

$$ET = TE = E.$$

If T is power bounded, then $r(T) \leq 1$ where $r(T)$ stands for the spectral radius of T (see e.g., [3, p.10]), which implies $\sigma_R(T) \subseteq \mathbb{D}^-$. Suppose

$$\sigma_R(T) \cap \mathbb{T} \neq \varnothing.$$

Take any $\lambda \in \sigma_R(T) \cap \mathbb{T}$. Thus $|\lambda| = 1$,

$$T^* x_0 = \overline{\lambda} x_0 \;\text{ for some nonzero vector } x_0 \in \mathcal{H} \qquad (\text{i.e., } \lambda \in \sigma_P(T^*)^*),$$

$$Tx \neq \lambda x \;\text{ for every nonzero vector } x \in \mathcal{H} \qquad (\text{i.e., } \lambda \notin \sigma_P(T)).$$

Set

$$T^{\#} = \tfrac{1}{\overline{\lambda}} T^* \;\in\; \mathcal{B}[\mathcal{H}],$$

which is power bounded since T is and $|\lambda| = 1$. So there is an $E^{\#} \in \mathcal{B}[\mathcal{H}]$ such that

$$C_n^{\#} = \tfrac{1}{n}\sum_{k=0}^{n-1} T^{\#k} \xrightarrow{\;s\;} E^{\#}, \qquad E^{\#}T^{\#} = T^{\#}E^{\#} = E^{\#} \quad \text{and} \quad E^{\#}x_0 = C_n^{\#}x_0 = x_0.$$

(Indeed, since $T^{*k}x_0 = \overline{\lambda}^k x_0$ we get $C_n^{\#}x_0 = \frac{1}{n}\sum_{k=0}^{n-1} T^{\#k}x_0 = x_0$.) Moreover, set

$$y_0 = E^{\#*}x_0 \;\in\; \mathcal{H},$$

which is nonzero since x_0 is: $0 \neq \|x_0\| = \langle x_0\,;x_0\rangle = \langle E^{\#}x_0\,;x_0\rangle = \langle x_0\,;y_0\rangle$. Hence

$$\langle Ty_0\,;x\rangle = \langle TE^{\#*}x_0\,;x\rangle = \langle x_0\,;E^{\#}T^*x\rangle = \langle x_0\,;\overline{\lambda}E^{\#}T^{\#}x\rangle = \langle x_0\,;\overline{\lambda}E^{\#}x\rangle = \langle \lambda y_0\,;x\rangle$$

for every $x \in \mathcal{H}$, and so

$$Ty_0 = \lambda y_0.$$

Since $y_0 \neq 0$ this contradicts the fact that $Tx \neq \lambda x$ for every $0 \neq x \in \mathcal{H}$. Then the assumption $\sigma_R(T) \cap \mathbb{T} \neq \varnothing$ fails, and consequently $\sigma_R(T) \subseteq \mathbb{D}$. □

5. Remarks

Let $\sigma(T)$ be the spectrum of an operator T in $\mathcal{B}[\mathcal{H}]$ and consider its continuous spectrum $\sigma_C(T) = \sigma(T)\backslash[\sigma_P(T) \cup \sigma_R(T)]$ so that $\{\sigma_P(T), \sigma_R(T), \sigma_C(T)\}$ is a classical partition of $\sigma(T)$.

Recall: (i) similarity preserves the spectrum and its parts (i.e., if $W \in \mathcal{B}[\mathcal{H}]$ is invertible with an inverse W^{-1} in $\mathcal{B}[\mathcal{H}]$, then the parts of the spectrum of WTW^{-1} coincide with the respective parts of the spectrum of T), and (ii) similar to a power bounded is again power bounded. Hence Theorem 4.1 says

(a) $\qquad T$ is similar to a power bounded $\quad\Longrightarrow\quad \sigma_R(T) \subseteq \mathbb{D}$.

Thus everything that has been said above (and below) about power bounded operators applies "ipsi literis" to operators similar to a power bounded operator.

Theorem 4.1 also characterizes the peripheral spectrum (i.e., $\sigma(T)\cap\mathbb{T}$) of a power bounded operator:

(b) $\qquad T$ is power bounded $\quad\Longrightarrow\quad \sigma(T)\cap\mathbb{T}\subseteq\sigma_P(T)\cup\sigma_C(T)$.

However, the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$ of a power bounded operator is not necessarily included in $\sigma_P(T)\cup\sigma_C(T)$. For instance take a unilateral weighted shift $T=\text{shift}\{w_k\}$ in $\mathcal{B}[\ell^2_+]$ with weighting sequence $\{w_k\}$ such that $w_k=\frac{1}{k}$ for each positive integer k. This is a quasinilpotent compact contraction (thus power bounded) with $\|T\|=1$ for which

$$\partial\sigma(T)=\sigma(T)=\sigma_R(T)=\{0\}.$$

On the other hand Theorem 4.1 leads to a conjugate symmetry between the intersection of the unit circle with the point spectra of T and T^* (so that eigenvalues in the unit circle of a power bounded operator are normal eigenvalues), namely,

(c) $\qquad T$ is power bounded $\quad\Longrightarrow\quad \sigma_P(T^*)^*\cap\mathbb{T}=\sigma_P(T)\cap\mathbb{T}$.

Indeed, set $A=\sigma_P(T)$ and $B=\sigma_P(T^*)$ so that $\sigma_R(T)=B^*\backslash A$ and $\sigma_R(T^*)=A^*\backslash B$. Suppose T is power bounded. Thus T^* is again power bounded. By Theorem 4.1, $(B^*\backslash A)\cap\mathbb{T}=(A^*\backslash B)\cap\mathbb{T}=\varnothing$. Equivalently, $(B^*\backslash A)\cap\mathbb{T}=\varnothing$ and $(A^*\backslash B)^*\cap\mathbb{T}=\varnothing$, (i.e., $(A\backslash B^*)\cap\mathbb{T}=\varnothing$). Therefore $(B^*\cap\mathbb{T})\backslash(A\cap\mathbb{T})=\varnothing$ and $(A\cap\mathbb{T})\backslash(B^*\cap\mathbb{T})=\varnothing$, which means $B^*\cap\mathbb{T}=A\cap\mathbb{T}$.

6. Applications

Take an arbitrary $T\in\mathcal{B}[\mathcal{H}]$. Let $\mathcal{R}(\lambda I-T)=(\lambda I-T)(\mathcal{H})$ denote the range of $\lambda I-T$ for an arbitrary $\lambda\in\mathbb{C}$ and consider the set

$$\mathcal{M}(T)=\big\{u\in\mathcal{H}\colon \sup_n\big\|\textstyle\sum_{k=0}^n T^k u\big\|<\infty\big\},$$

which are linear manifolds of $\mathcal{H}$ invariant under T. Take $y\in\mathcal{R}(I-T)$ arbitrary so that $y=Tx-x$ for some $x\in\mathcal{H}$. Then for each nonnegative integer n

$$\textstyle\sum_{k=0}^n T^k y=T^{n+1}x-x. \qquad (*)$$

Proposition 6.1. *$\mathcal{R}(I-T)\subseteq\mathcal{M}(T)$ if and only if T is power bounded.*

Proof. Take any $x\in\mathcal{H}$ and set $y=Tx-x$ so that $y\in\mathcal{R}(I-T)$. If $\mathcal{R}(I-T)\subseteq\mathcal{M}(T)$, then $\sup_n\|T^{n+1}x-x\|<\infty$ according to $(*)$, and hence $\sup_n\|T^n x\|<\infty$. Since this holds for every $x\in\mathcal{H}$, T is power bounded by the Banach–Steinhaus Theorem. Conversely, if T is power bounded, then $\mathcal{R}(I-T)\subseteq\mathcal{M}(T)$ by $(*)$. □

Corollary 6.1. *If $\mathcal{R}(I-T)\subseteq\mathcal{M}(T)$, then $\sigma_R(T)\subseteq\mathbb{D}$.*

Proof. This follows by Proposition 6.1 and Theorem 4.1. □

By Proposition 6.1 T is power unbounded if and only if $\mathcal{R}(I-T)\not\subseteq\mathcal{M}(T)$.

> *In particular, if a power unbounded operator is such that* $\sup_n\|T^n x\|=\infty$ *for every nonzero $x\in\mathcal{H}$, then $\mathcal{M}(T)\cap\mathcal{R}(I-T)=\{0\}$.*

Indeed, if $\mathcal{M}(T)\cap\mathcal{R}(I-T)\neq\{0\}$ and $0\neq y\in\mathcal{M}(T)\cap\mathcal{R}(I-T)$, then by $(*)$ there exists $0\neq x\in\mathcal{H}$ such that $\|T^{n+1}x-x\|=\|\sum_{k=0}^n T^k y\|$ for each nonnegative integer n, and hence $\sup_n\|T^n x\|<\infty$ because $y\in\mathcal{M}(T)$.

More particularly, there are power unbounded operators with $\sup_n \|T^n x\| = \infty$ *for every nonzero* $x \in \mathcal{H}$ *for which* $\{0\} = \mathcal{M}(T) \subset \mathcal{R}(I-T) = \mathcal{H}$.

(Set $T = 2I$.) On the other hand,

there exist power bounded operators for which $\mathcal{R}(I-T) = \mathcal{M}(T)$.

(Trivial examples: $T = O, I, \frac{1}{2}I$ — less trivial: $T = (I \oplus \frac{1}{2}I), (O \oplus \frac{1}{2}I)$.)

The noninclusion $\mathcal{M}(T) \not\subseteq \mathcal{R}(I-T)$ can be characterized as follows.

Proposition 6.2. $\mathcal{M}(T) \not\subseteq \mathcal{R}(I-T)$ *if and only if there exists a nonzero operator* $L \in \mathcal{B}[\mathcal{H}]$ *for which* $\mathcal{R}(L) \cap \mathcal{R}(I-T) = \{0\}$ *and* $\mathcal{R}(L) \subseteq \mathcal{M}(T)$.

Proof. If $T \in \mathcal{B}[\mathcal{H}]$ is such that $\mathcal{M}(T) \not\subseteq \mathcal{R}(I-T)$, then take an arbitrary nonzero $y \in \mathcal{M}(T)\backslash\mathcal{R}(I-T)$, and consider the orthonormal projection $E \in \mathcal{B}[\mathcal{H}]$ for which $\mathcal{R}(E) = \mathrm{span}\{y\}$ so that $\mathcal{R}(E) \cap \mathcal{R}(I-T) = \{0\}$ and $\mathcal{R}(E) \subseteq \mathcal{M}(T)$ (since $\mathcal{R}(E)$, $\mathcal{R}(I-T)$, and $\mathcal{M}(T)$ are linear manifolds of $\mathcal{H}$). Conversely, if L and T in $\mathcal{B}[\mathcal{H}]$ are such that $\{0\} \neq \mathcal{R}(L) \subseteq \mathcal{M}(T) \subseteq \mathcal{R}(I-T)$, then $\mathcal{R}(L) \cap \mathcal{R}(I-T) \neq \{0\}$. □

And the inclusion $\mathcal{R}(L) \subseteq \mathcal{M}(T)$ is characterized as follows.

Proposition 6.3. $\mathcal{R}(L) \subseteq \mathcal{M}(T)$ *if and only if* $\sup_n \big\| \sum_{k=0}^n T^k L \big\| < \infty$.

Proof. Take arbitrary operators L and T in $\mathcal{B}[\mathcal{H}]$. By the Banach–Steinhaus Theorem, $\sup_n \big\| \sum_{k=0}^n T^k L \big\| < \infty$ if and only if $\sup_n \big\| \sum_{k=0}^n T^k L x \big\| < \infty$ for every $x \in \mathcal{H}$. Equivalently, $\sup_n \big\| \sum_{k=0}^n T^k y \big\| < \infty$ for every $y \in \mathcal{R}(L)$, which means $\mathcal{R}(L) \subseteq \mathcal{M}(T)$. □

As it is well known, the spectral radius $r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$ of an operator T on a Banach space is less than 1 if and only if the operator T is uniformly stable (notation: $T^n \xrightarrow{u} O$), which completely characterizes uniform stability; that is,

$$\|T^n\| \to 0 \quad \text{if and only if} \quad r(T) < 1.$$

Clearly, uniform stability implies power boundedness, which implies $r(T) \leq 1$. The above equivalence has been extended in [4] where it was given a complete characterization (in terms of an ergodic condition and the peripheral spectrum) for uniform convergence to zero of $\{T^n L\}$ for operators L in the commutant of a power bounded operator T. We state the result from [4] below. Take any operator $T \in \mathcal{B}[\mathcal{H}]$ and let $\{T\}'$ denote its commutant (the subalgebra of $\mathcal{B}[\mathcal{H}]$ consisting of all operators $L \in \mathcal{B}[\mathcal{H}]$ that commute with T).

Proposition 6.4. [4] *If* $T \in \mathcal{B}[\mathcal{H}]$ *is a power bounded operator and* L *lies in* $\{T\}'$, *then* $\|T^n L\| \to 0$ *if and only if* $\frac{1}{n+1}\big\| \sum_{k=0}^n \big(\frac{T}{\lambda}\big)^k L \big\| \to 0$ *for every* $\lambda \in \sigma(T) \cap \mathbb{T}$.

The above statement is vacuous for the particular case of L being an isometry in $\{T\}'$ (e.g., for $L = I$). Indeed, if L is an isometry then $\|T^n L\| = \|T^n\|$, and $r(T) < 1$ makes the peripheral spectrum $\sigma(T) \cap \mathbb{T}$ empty. In fact, $r(T) < 1$ if and only if $\|T^n\| \to 0$, which implies $\|T^n L\| \to 0$ for all $L \in \mathcal{B}[\mathcal{H}]$. Thus we assume the power bounded operator T is not uniformly stable; equivalently, the power bounded operator T is such that $r(T) = 1$. Also, if T is an isometry and $\|T^n L\| \to 0$, then $L = O$. Actually, if L is not injective (i.e., if $\mathcal{N}(L) \neq \{0\}$), then there exists $0 \neq x_0$ in $\mathcal{H}$ for which $\|T^n L x_0\| = \big\| \sum_{k=0}^n (\frac{T}{\lambda})^k L x_0 \big\| = 0$ for every $n \geq 0$ and every $\lambda \in \mathbb{C}$.

Corollary 6.2. *Let $T \in \mathcal{B}[\mathcal{H}]$ be a power bounded operator with $r(T) = 1$ and let L be an injective operator in $\{T\}'$. Then*

$$\|T^n L\| \to 0 \text{ if and only if } \tfrac{1}{n+1}\Big\|\sum_{k=0}^{n} \big(\tfrac{T}{\lambda}\big)^k L\Big\| \to 0 \text{ for every } \lambda \in \sigma_C(T) \cap \mathbb{T},$$

and $\sigma(T) \cap \mathbb{T} = \sigma_C(T) \cap \mathbb{T} \neq \varnothing$ whenever any of the above limits hold.

Proof. If $r(T) = 1$, then $\sigma(T) \cap \mathbb{T} \neq \varnothing$. If T is a power bounded, then $\sigma_R(T) \cap \mathbb{T} = \varnothing$ according to Theorem 4.1. If $L \in \{T\}'$ is injective and the claimed equivalence holds, then $\sigma_P(T) \cap \mathbb{T} = \varnothing$. Indeed, if $L \in \{T\}'$ and $\sigma_P(T) \cap \mathbb{T} \neq \varnothing$, then there is an $\lambda \in \mathbb{T}$ such that $Tx_0 = \lambda x_0$ for some $0 \neq x_0 \in \mathcal{H}$ (thus $x_0 = T^k x_0/\lambda^k$). Hence $\|Lx_0\| = \|T^n L x_0\| \to 0$ (also $\|Lx_0\| = \frac{1}{n+1}\big\|\sum_{k=0}^{n} Lx_0\big\| = \frac{1}{n+1}\big\|\sum_{k=0}^{n}\big(\frac{T}{\lambda}\big)^k Lx_0\big\| \to 0$) and so $0 \neq x_0 \in \mathcal{N}(L)$, which implies L is not injective. Thus

$$\sigma(T) \cap \mathbb{T} = \sigma_C(T) \cap \mathbb{T} \neq \varnothing,$$

and the claimed result follows from Proposition 6.4. □

An important particular case of Proposition 6.4 is that of $L = I - T$ so that $T^n L = T^n - T^{n+1}$ for every integer $n \geq 0$. This leads to a classical result from [2] (which holds for contractions acting on Banach spaces).

Proposition 6.5. [2] *Let $T \in \mathcal{B}[\mathcal{H}]$ be a contraction. Then $\|T^n - T^{n+1}\| \to 0$ if and only if $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$.*

If the contraction T is uniformly stable (i.e., if $r(T) < 1$), then $\sigma(T) \cap \mathbb{T} = \varnothing$. Otherwise it is a normaloid contraction of unit norm, which means $r(T) = \|T\| = 1$.

Corollary 6.3. *If $r(T) = 1$, then the above unique point $\lambda = 1$ in the peripheral spectrum $\sigma(T) \cap \mathbb{T}$ is either an eigenvalue* (i.e., *lies in $\sigma_P(T)$*), *or lies in $\sigma_C(T)$.*

Proof. If $\mathcal{N}(I - T) \neq \{0\}$ (i.e., if $I - T$ is not injective), then $1 \in \sigma_P(T)$. Otherwise $1 \in \sigma_C(T)$ because $\sigma_R(T) \cap \mathbb{T} = \varnothing$ by Proposition 3.1 since T is a contraction (or by Theorem 4.1 since contractions are power bounded). □

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