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Operators satisfying a similarity condition

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Abstract

Given Hilbert space operators $A, S \in B(\mathcal{H})$ such that $0 \notin \overline{W(S)}$ (= the closure of the numerical range of S), the similarities $ASA^* = S$ for invertible A and $AS = SA^*$ have been considered by a number of authors over past few decades. A classical result of C.R. De Prima (resp., I.H. Sheth) says that if A and A^{-1} are normaloid or convexoid (resp., A is hyponormal), then $ASA^* = S$ implies A is unitary (resp., $AS = SA^*$ implies A is self-adjoint). This paper uses (Putnam-Fuglede theorem type) commutativity results to obtain generalizations of existing results on similarities of the above type. Amongst other results, it is proved that if $ASA^* = S$ with A invertible and $0 \notin \overline{W(S)}$, then: (i) A normaloid implies either A is unitary or $\sigma_p(A) = \emptyset$; (ii) operators A satisfying the positivity condition $|A^2|^2 - 2|A|^2 + I \geq 0$ are unitary. If the operator A in $ASA^* = S$ (resp., $AS = SA^*$) is w-hyponormal or class $\mathcal{A}(1,1)$ with $A^{-1}(0) \subseteq A^{*-1}(0)$, then a sufficient condition for A to be unitary (resp., A to be self-adjoint) is that $0 \notin W(X)$; furthermore, one may drop the hypothesis $A^{-1}(0) \subseteq A^{*-1}(0)$ in the case in which $0 \notin \overline{W(X)}$.

1. Introduction

Given a complex infinite dimensional Hilbert space $\mathcal{H}$ with inner product $(.,.)$, let $B(\mathcal{H})$ denote the algebra of bounded linear transformations, equivalently operators, on $\mathcal{H}$ into itself. For $A, B \in B(\mathcal{H})$, let $\delta_{A,B}$ and $\triangle_{A,B} \in B(B(\mathcal{H}))$ denote, respectively, the generalized derivation $\delta_{A,B}(X) = AX - XB$ and the elementary operator $\triangle_{A,B}(X) = AXB - X$. Let $\overline{W(A)}$ denote the closure of the numerical range

$$W(A) = \{\lambda \in \mathbb{C} : \lambda = (Ax, x), x \in \mathcal{H}, ||x|| = 1\}$$

of A. Given $S \in B(\mathcal{H})$ such that $0 \notin \overline{W(S)}$, the problem of determining the properties of an operator $A \in B(\mathcal{H})$ such that $\delta_{A,A^*}(S) = 0$, or $\triangle_{A,A^*}(S) = 0$, has been considered over the past decades by a number of authors. Although, with A and S as above, $\delta_{A,A^*}(S) = 0$ implies A is similar to A^* (indeed, with similarity implemented by a positive operator) and $\triangle_{A,A^*}(S) = 0$ implies A^* is similar to an isometry, the equation $\delta_{A,A^*}(S) = 0$ does not in general imply A is self-adjoint (even, normal: consider for example a non-normal operator A such that AP is self-adjoint for some $P > 0$) and the equation $\triangle_{A,A^*}(S) = 0$ does not in general imply A is unitary (even, invertible: consider for example $P = I$ the identity operator and $A^* = V$, the forward unilateral shift). Additional hypotheses on A are required.

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A well known result of I.H. Sheth [22], see also [28], says that if A is a hyponormal Hilbert space operator such that $\delta_{A,A^*}(S) = 0$, $0 \notin \overline{W(S)}$, then $A = A^*$. Hyponormal operators are normaloid (also, convexoid), with the property that the inverse operator (whenever it exists) is again hyponormal. Hyponormal operators have SVEP (*the single-valued extension property*) and satisfy the property that if $\triangle_{A,A^*}(S) = 0$, $0 \notin \overline{W(S)}$, then A is unitary: This follows from a corrected version of a result of Singh and Mangla [23] by DePrima [5, Theorem 1], which states that if $\triangle_{A,A^*}(S) = 0$, $0 \notin \overline{W(S)}$, for an invertible operator A such that A, A^{-1} are normaloid or convexoid (all combinations allowed), then A is unitary. Sheth's result has recently been extended to classes of Hilbert space operators more general than the class of hyponormal operators. Thus Jeon *et al* [19, Theorem 6] prove that if $A \in B(\mathcal{H})$ is a quasi class A operator (i.e., if $A^*(|A^2| - |A|^2)A \geq 0$), $\delta_{A,A^*}(S) = 0$, where $S \in B(\mathcal{H})$ and $0 \notin \overline{W(S)}$, then A is self-adjoint. Dehimi and Mortad [4, Theorem 8] extend Sheth's result to unbounded hyponormal operators to prove that if $S \in B(\mathcal{H})$ satisfies $0 \notin \overline{W(S)}$ and $SA^* \subset AS$ for an unbounded closed densely defined (on $\mathcal{H}$) hyponormal operator A, then A is self-adjoint.

If $A, B \in B(\mathcal{H})$ are hyponormal operators, and $\delta_{A,B^*}(X) = 0$ (resp., $\triangle_{A,B^*}(X) = 0$) for a quasi-affinity $X \in B(\mathcal{H})$, then the (Putnam-Fuglede) commutativity theorem for hyponormal operators says that $\delta_{A^*,B}(X) = 0$ (resp., $\triangle_{A^*,B}(X) = 0$), see [25], [10], [13], and hence that A, B^* (resp., A, B^{*-1}) are unitarily equivalent normal operators. If it so happens that X can be chosen to be an injective positive operator, then $A = B^*$ (resp., $A = B^{*-1}$). The particular choice of $B = A$ and X such that $0 \notin \overline{W(X)}$ guarantees the existence of a positive invertible operator P such that $\delta_{A,A^*}(P) = 0$ (resp., $\triangle_{A,A^*}(P) = 0$). Thus the *(loc.cit.)* result of Sheth [22] is a straightforward consequence of the commutativity theorem for hyponormal operators; that a similar application of the commutativity theorem results in the unitarily of A in the case in which $\triangle_{A,A^*}(X) = 0$ and both A and A^{-1} are normaloid (or, convexoid) is consequent from a result of Stampfli [24] (which, as we shall see, implies that such an operator A is normal). Our aim in this paper is to use *commutativity theorems* to obtain generalizations of extant results on the similarities $\delta_{A,A^*}(X) = 0$ and $\triangle_{A,A^*}(X) = 0$, where $0 \notin \overline{W(X)}$. We prove, amongst other results, that if $\triangle_{A,A^*}(X) = 0$, $0 \notin \overline{W(X)}$, then: (a) invertible operators A such that $|A^2|^2 - 2|A|^2 + I \geq 0$ are unitary; (b) A invertible and normaloid implies either $\sigma_p(A) = \emptyset$, or, A is unitary. (Most of our notation is standard; however, all our currently undefined notation is defined in the following section.) If the operator A in $\triangle_{A,A^*}(X) = 0$ (resp., $\delta_{A,A^*}(X) = 0$) is w-hyponormal or class $\mathcal{A}(1,1)$ with $A^{-1}(0) \subseteq A^{*-1}(0)$, then a sufficient condition for A to be unitary (resp., A to be self-adjoint) is that $0 \notin W(X)$; furthermore, one may drop the hypothesis $A^{-1}(0) \subseteq A^{*-1}(0)$ in the case in which $0 \notin \overline{W(X)}$. For densely defined closed operators A we prove that if A has SVEP, $\delta^{-1}_{(A-\lambda)^{-1},(A^*-\lambda)^{-1}}(0) \subseteq \delta^{-1}_{(A-\lambda)^{*-1},(A^*-\lambda)^{*-1}}(0)$ for a λ in the resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and $SA^* \subset AS$ for an operator $S \in B(\mathcal{H})$ with $0 \notin \overline{W(S)}$, then A is self-adjoint.

2. Some notation and terminology

We start by introducing further notation and explaining some of our terminology. An operator $A \in B(\mathcal{H})$ is normaloid if $||A||$ equals $r(A)$ (= the spectral radius of A), A is convexoid if $\overline{W(A)} = \text{con}\sigma(\text{A})$ (= the convex hull of the spectrum of A) and A is spectraloid if $r(A) = w(A)$ (= the numerical radius of A). It is well known that the classes consisting of normaloid and convexoid operators are independent of each other, but that both these classes are contained in the class of spectraloid operators. We shall denote the point spectrum of A by $\sigma_p(A)$, the peripheral spectrum $\{\lambda \in \sigma(A) : |\lambda| = r(A)\}$ by $\sigma_\pi(A)$, the boundary of the unit disc in the complex plane $\mathbb{C}$ by ∂D and the commutator $AB - AB$ of $A, B \in B(\mathcal{H})$ by $[A, B]$. Recall that every contraction $A \in B(\mathcal{H})$, i.e., operators $A \in B(\mathcal{H})$ such that $||A|| \leq 1$, has a direct sum decomposition $A = A_u \oplus A_c$, where A_u is unitary and A_c is completely non-unitary (henceforth abbreviated to cnut – thus A_c is the cnut-part of A). An operator $A \in B(\mathcal{H})$ is of the class $C_{0.}$ (resp., $C_{1.}$) if the sequence $\{||A^n ix||\}$ converges to 0 for all $x \in \mathcal{H}$ (resp., the sequence $\{||A^n ix||\}$ does not converges to 0 for all non-zero $x \in \mathcal{H}$); A is of class $C_{.0}$ (resp., $C_{.1}$) if $A^* \in C_{0.}$ (resp., $A^* \in C_{1.}$). All combinations are possible, leading to classes C_{11}, C_{00}, C_{10} and C_{01}.

An operator $A \in B(\mathcal{H})$ has SVEP, *the single-valued extension property*, at a point $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathbf{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathbf{D}_{\lambda_0} \longrightarrow \mathcal{H}$ satisfying $(A - \lambda)f(\lambda) = 0$ is the function $f \equiv 0$. (Here, and in the sequel, we have shortened $A - \lambda I$ to $A - \lambda$.) Evidently, every A has SVEP at points in the resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. *We say that A has SVEP on a set S if it has SVEP at every $\lambda \in S$.* The *ascent* of A at $\lambda \in \mathbb{C}$, $\mathrm{asc}(A - \lambda)$, is the least non-negative integer n such that $(A - \lambda)^{-n}(0) = (A - \lambda)^{-(n+1)}(0)$; if no such lambda exists, then $\mathrm{asc}(A - \lambda) = \infty$. It is well known [1] that finite ascent at a point implies SVEP at the point. The *deficiency indices* of $A \in B(\mathcal{H})$ at a point $\lambda \in \mathbb{C}$ are the non-negative integers $\alpha(A - \lambda) = \dim(A - \lambda)^{-1}(0)$ and $\beta(A - \lambda) = \dim(\mathcal{H} \setminus (A - \lambda)\mathcal{H})$.

Given (not necessarily bounded) linear transformations A and B of $\mathcal{H}$ into itself with domains dom(A) and dom(B), and an operator $X \in B(\mathcal{H})$, the relation $A \subset B$ means B is an extension of A (Thus: dom(A) $\subset$ dom(B) and $Bx = Ax$ for all $x \in$ dom(A)), and the relation $XA \subset BX$ means X maps dom(A) into dom(B) and $XAx = BXx$ for all $x \in$ dom(A). The linear transformation A is boundedly invertible if there is a bounded linear operator $A^{-1} \in B(\mathcal{H})$ such that $AA^{-1} = I$ and $A^{-1}A \subset I$. It is easily seen that if A, B are boundedly invertible linear transformations such that $XA \subset BX$ for an operator $X \in B(\mathcal{H})$, then $B^{-1}X = XA^{-1}$.

The following result from [8] is essential to some of our argument below.

Theorem 2.1. *If $A, B \in B(\mathcal{H})$, then the following statements are pairwise equivalent.*

(i) $\mathrm{ran}(A) \subseteq \mathrm{ran}(B)$.

(ii) *There is a $\mu \geq 0$ such that $AA^* \leq \mu^2 BB^*$.*

(iii) *There is an operator $C \in B(\mathcal{H})$ such that $A = BC$.*

Furthermore, if these conditions hold, then the operator C may be chosen so that (a) $||C||^2 = \inf\{\lambda : AA^* \leq \lambda BB^*\}$*;* (b) $\ker(A) = \ker(C)$*;* (c) $\mathrm{ran}(C) \subseteq \ker(B)^\perp$.

We shall also require the following well known lemma (see, for example, [11]).

Lemma 2.2. *Let $A, B, X \in B(\mathcal{H})$. If:*

(i) $X \in \delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$*, then* $[A, |X^*|] = 0$*,* $\overline{\mathrm{ran}(X)}$ *reduces* A*,* $\ker(X)^\perp$ *reduces* B*, and* $A|_{\overline{\mathrm{ran}(X)}}$ *and* $B|_{\ker(X)^\perp}$ *are unitarily equivalent normal operators.*

(ii) $X \in \triangle_{A,B}^{-1}(0) \subseteq \triangle_{A^*,B^*}^{-1}(0)$*, then* $[A, |X^*|] = 0$*,* $\overline{\mathrm{ran}(X)}$ *reduces* A*,* $\ker(X)^\perp$ *reduces* B*, and* $A|_{\overline{\mathrm{ran}(X)}}$ *and* $(B|_{\ker(X)^\perp})^{-1}$ *are unitarily equivalent normal operators.*

3. Results

We start by considering the equation $\triangle_{A,A^*}(S) = 0$, where $0 \notin \overline{W(S)}$. Along with proving some well known results (*albeit* using a somewhat different argument), we prove that the only way an invertible normaloid A can fail to be unitary is if it has an empty point spectrum.

Let $A, S \in B(\mathcal{H})$ be such that $\triangle_{A,A^*}(S) = 0$, A has SVEP at 0 and $0 \notin \overline{W(S)}$. Then, the operator S being invertible, $\triangle_{A,A^*}(S) = 0$ implies the right invertibility of A, and this since A has SVEP at 0 implies that A is invertible [1, Corollary 2.24]. Since $\overline{W(S)}$ is a convex set, the hypothesis $0 \notin \overline{W(S)}$ implies, upon replacing S by $e^{i\theta}S$ if need be, that we may separate 0 from $\overline{W(S)}$ by a half plane $\mathtt{Re}z \geq \epsilon$ for some $\epsilon > 0$. Then $P = \frac{S+S^*}{2} > 0$, and

$$\begin{aligned}\triangle_{A,A^*}(S) = 0 &\iff ASA^* - S = 0 \Longrightarrow APA^* - P = 0\\ &\iff (AP^{1/2})(AP^{1/2})^* = (\alpha P^{1/2})(\alpha P^{1/2})^*\end{aligned}$$

for every scalar α such that $|\alpha| = 1$. This, see Theorem 2.1, implies the existence of an isometry V, indeed a unitary since A is invertible, such that

$$AP^{1/2} = P^{1/2}(\alpha V^*) \iff \delta_{A,\alpha V^*}(P^{1/2}) = 0$$

for every choice of α such that $|\alpha| = 1$. Consequently, the operator A satisfies

$$||A^n|| \leq ||P^{1/2}||||P^{-1/2}||$$

for all integers n, and hence is a power bounded C_{11} operator. Evidently, a necessary and sufficient condition for A to be unitary is that $[A, P^{1/2}] = 0$. Equivalently, a necessary and sufficient condition for A to be unitary is that $\delta_{A,\alpha V^*}(P^{1/2}) = 0 \Longrightarrow \delta_{A^*,\overline{\alpha}V}(P^{1/2}) = 0$. The following theorem gives some further necessary and sufficient conditions for A to be unitary. (We remark here that condition (i) appears in [5] and condition (ii) appears in [4].)

Theorem 3.1. *If A, S are operators in $B(\mathcal{H})$ such that A has SVEP at 0, $0 \notin \overline{W(S)}$ and $\triangle_{A,A^*}(S) = 0$, then either of the following conditions is both necessary and sufficient for A to be unitary.*

(i) *A and A^{-1} are normaloid or convexoid or spectraloid (all combinations allowed).*

(ii) $[A, |S|] = 0$.

(iii) *A satisfies the positivity condition* $0 \leq |A^2|^2 - 2|A|^2 + 1$. (*Operators A satisfying this positivity condition have been described in the literature,* [14]*, as class $\mathcal{Q}$ operators.*)

Proof. (i) As seen above, the hypotheses imply the existence of a positive invertible operator P and a unitary operator V such that $AP^{1/2} = P^{1/2}(\alpha V^*)$. Evidently, $\sigma(A)$ and $\sigma(A^{-1})$ lie in the unit circle $\partial\mathcal{D}$. Since either of the hypotheses A (resp., A^{-1}) is normaloid or convexoid or spectraloid implies $r(A) = w(A) = 1$ (resp., $r(A^{-1}) = w(A^{-1}) = 1$), it follows that if (any of the combinations of hypothesis) (i) holds then

$$W(A^{\pm 1}) \subseteq \mathrm{con}\sigma(A^{\pm 1}).$$

This, see [24], implies that A is normal, and hence it follows from an application of the Putnam-Fuglede commutativity theorem [16] for normal operators that

$$AP^{1/2} = P^{1/2}(\alpha V^*) \iff A^*P^{1/2} = P^{1/2}(\alpha V^*)^*.$$

(ii) Assume now that $[A, |S|] = 0$. Then $[A, S^*S] = 0 = [A^*, S^*S]$, and

$$ASA^* = S \iff A^*S^*ASA^* = A^*S^*S = S^*SA^* \iff A^*S^*A = S^* \iff A^*SA = S.$$

But then

$$ASA^* = S \text{ and } A^*SA = S \Longrightarrow APA^* = P = A^*PA,$$

and hence

$$A^*P^2 = A^*PAPA^* = P^2A^* \iff [A^*, P] = 0 \iff [A, P^{1/2}] = 0.$$

Once again, A is unitary.

(iii) Recall the (easily proved) fact that an operator $A \in B(\mathcal{H})$ is unitary if and only if A and A^{-1} are contractions: We prove in the following that if the operator A (of the statement of the theorem) is a class $\mathcal{Q}$ operator, then A and A^{-1} are contractions. Start by recalling from [14] that an operator $A \in \mathcal{Q}$ if and only if $A^{-1} \in \mathcal{Q}$ (whenever A^{-1} exists). It is clear from $AP^{1/2} = P^{1/2}V^*$ that $||A^n|| \le ||P^{1/2}||||P^{-1/2}||$ for all integers n, i.e., A is power bounded. Assume that there exists an $x \in \mathcal{H}$ such that $0 < a < ||Ax||^2 - ||x||^2$, i.e., A is not a contraction. Then, since $A \in \mathcal{Q}$,

$$0 < a < ||Ax||^2 - ||x||^2 \le ||A^2x||^2 - ||Ax||^2.$$

We use induction to prove the hypothesis A is not a contraction leads to a contradiction. Assume then that $0 < a < ||A^mx||^2 - ||A^{m-1}x||^2$ for some integer $m > 2$. Then

$$\begin{aligned} A \in \mathcal{Q} &\iff 0 \le |A^2|^2 - 2|A|^2 + 1 \Longrightarrow 0 \le |A^{m+1}|^2 - 2|A^m|^2 + |A^{m-1}|^2 \\ &\Longrightarrow 0 < a < ||A^mx||^2 - ||A^{m-1}x||^2 \le ||A^{m+1}x||^2 - ||A^mx||^2. \end{aligned}$$

Thus

$$0 < a < ||A^nix||^2 - ||A^{n-1}x||^2$$

for all integers $n \ge 1$. But then

$$0 < na < ||A^nix||^2 - ||x||^2,$$

which implies that A is not power bounded – a contradiction. Hence A is a contraction. Evidently, $A^{-1} \in \mathcal{Q}$ is power bounded, and the preceding argument applies equally to A^{-1}. Hence A^{-1} also is a contraction. □

The hypothesis A is normaloid on its own is not enough to guarantee the unitarily of A (as observed by DePrima – see [5], also [7], for an example demonstrating this). The following theorem goes some way towards explaining why.

Theorem 3.2. *Suppose that $\triangle_{A,A^*}(S) = 0$ for operators $A, S \in B(\mathcal{H})$, where $0 \notin \overline{W(S)}$. If A is normaloid and has SVEP at 0, then either $\lambda \notin \sigma_p(A)$ for all $\lambda \in \sigma(A)$* (i.e., $\sigma_p(A) = \emptyset$), *or, A is unitary.*

Proof. Assume that A has SVEP at 0 and is normaloid. Then, as seen above, the hypotheses $\triangle_{A,A^*}(S) = 0$, $0 \notin \overline{W(S)}$ and A has SVEP at 0 imply (A is invertible and that) there exist a positive invertible operator P and a unitary V such that $AP^{1/2} = P^{1/2}(\alpha V^*)$ for all complex α such that $|\alpha| = 1$. (Thus A is similar to a unitary operator.) Since the peripheral spectrum $\sigma_\pi(A)$ of A equals the spectrum $\sigma(A) \subseteq \partial\mathcal{D}$, it follows from an application of [17, Proposition 54.3] that

$$\operatorname{asc}(A - \lambda) \le 1 \text{ and } \beta(A - \lambda) = \dim(\mathcal{H} \setminus (A - \lambda)\mathcal{H}) > 0$$

for all $\lambda \in \sigma(A)$. Again, since

$$\alpha(A-\lambda)^* = \dim(\mathcal{H} \setminus \overline{(A-\lambda)\mathcal{H}}) \leq \beta(A-\lambda),$$

we have one of two possibilities:

$$\text{Either } \alpha(A-\lambda)^* = \alpha(A^*-\overline{\lambda}) = 0, \text{ or, } \alpha(A^*-\overline{\lambda}) > 0.$$

Since the hypothesis A is normaloid implies $||A|| = 1$, if $x \in (A-\lambda)^{-1}(0)$, then

$$||(A^*-\overline{\lambda})x||^2 = ||A^*x||^2 - (A^*x, \overline{\lambda}x) - (\overline{\lambda}x, A^*x) + ||\overline{\lambda}x||^2 \leq 0.$$

Thus

$$x \in (A-\lambda)^{-1}(0) \Longleftrightarrow x \in (A^*-\overline{\lambda})^{-1}(0).$$

Hence A has normal eigenvalues (i.e., the eigenspaces corresponding to the eigenvalues of the operator reduce the operator), and

$$\alpha(A-\lambda) = 0 \Longleftrightarrow \alpha(A^*-\overline{\lambda}) = 0.$$

Consequently, $\alpha(A^*-\overline{\lambda}) = 0$ implies ($\lambda \notin \sigma(A)$, in particular) $\lambda \notin \sigma_p(A)$. Consider thus the case in which $\alpha(A^*-\overline{\lambda}) > 0$. Then

$$\begin{aligned} & \lambda x = Ax = P^{1/2}(\alpha V^*)P^{-1/2}x \\ \Longrightarrow \quad & (\lambda\overline{\alpha})P^{-1/2}x = V^*P^{-1/2}x \\ \Longleftrightarrow \quad & VP^{-1/2}x = (\overline{\lambda}\alpha)P^{-1/2}x \end{aligned}$$

for every α such that $|\alpha| = 1$. But then

$$\overline{\lambda}\alpha \in \sigma_p(V) \text{ for every } \alpha \in \partial\mathcal{D},$$

and hence

$$\mu \in \sigma_p(A) \text{ for all } \mu \in \sigma(A).$$

Consequently, A (also A^{-1}) is a normal operator such that

$$\begin{aligned} & AP^{1/2} = P^{1/2}(\alpha V^*) \Longleftrightarrow A^*P^{1/2} = P^{1/2}(\overline{\alpha}V) \\ \Longrightarrow \quad & [A, P] = 0 = [A^*, P^{1/2}], P^{-1/2}A = P^{-1/2}A^{*-1}, \end{aligned}$$

i.e., A is unitary. □

Remark 3.3. (i) An important class of operators, which subsumes the class of hyponormal operators, for which both the operator and its inverse (whenever it exists) are normaloid is that of paranormal operators (i.e., operators $A \in B(\mathcal{H})$ for which $||Ax||^2 \leq ||A^2x||$ for all unit vectors $x \in \mathcal{H}$). Thus: *If a paranormal operator $A \in B(\mathcal{H})$ satisfies $\triangle_{A,A^*}(S)$ for an operator $S \in B(\mathcal{H})$ such that $0 \notin \overline{W(S)}$, then A is unitary.* Paranormal operators are class $\mathcal{Q}$ operators. However, there exist class $\mathcal{Q}$ operators which are not normaloid [14]. As we have observed above, a necessary and sufficient condition for the operator A of Theorem 3.1 to be unitary is that $\delta_{A,\alpha V^*}(P^{1/2}) = 0 \Longrightarrow \delta_{A^*,\overline{\alpha}V}(P^{1/2}) = 0$. Indeed, if an operator $A \in B(\mathcal{H})$ is such that $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ for isometries $B \in B(\mathcal{H})$, then $\triangle_{A,A^*}(S) = 0$, $0 \notin \overline{W(S)}$, implies

$$\delta_{A,\alpha V^*}(P^{\frac{1}{2}}) = 0 \Longrightarrow \delta_{A^*,\overline{\alpha}V}(P^{\frac{1}{2}}) = 0 \Longrightarrow A \text{ is unitary.}$$

Example of non-normaloid operators $A \in B(\mathcal{H})$ which satisfy the property that $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$ for normal $B^* \in B(\mathcal{H})$ (indeed, hyponormal, even M-hyponormal, B^*) are provided by

- M-hyponormal operators: there exists a scalar $M > 0$ such that $|(A-\lambda)^*|^2 \leq M|A-\lambda|^2$ for all complex λ;
- dominant operators: there exists a scalar $M(\lambda) > 0$ such that $|(A-\lambda)^*|^2 \leq M(\lambda)|A-\lambda|^2$ for every complex λ;
- class $\mathcal{Y} = \bigcup_{\alpha \geq 1} \mathcal{Y}_\alpha$ operators: $A \in \mathcal{Y}_\alpha$ if there exists a scalar $M_\alpha > 0$ such that $|A^*A - AA^*|^\alpha \leq M_\alpha^2 |A-\lambda|^2$ for every complex λ [11, 18, 19, 27].

Letting $\mathcal{D}_0$ denote the operators belonging to these classes, we note that operators $A \in \mathcal{D}_0$ have SVEP and $\triangle_{A,A^*}(S) = 0$, $S \in B(\mathcal{H})$ such that $0 \notin \overline{W(S)}$, for M-hyponormal or class $\mathcal{Y})_\alpha$ operators $A \in \mathcal{D}_0$ implies A is unitary .

(ii) Invertible operators $A \in B(\mathcal{H})$ such that $\triangle_{A,A^*}(S) = 0$ for an operator $S \in B(\mathcal{H})$, $0 \notin \overline{W(S)}$, of Theorem 3.2 being power bounded are generalized scalar [20, Theorem 1.5.13]. Letting L_A and $R_A \in B(B(\mathcal{H}))$ denote the operators $L_A(X) = AX$ and $R_A(X) = XA$ (of left and right multiplication by A, respectively), it is seen that the operator $L_A R_{A^*}$ is a doubly bounded invertible operator with spectrum in the unit circle. Evidently, $L_A R_{A^*}$ is a generalized scalar operator with spectrum in the unit circle such that $1 \in \sigma_p(L_A R_{A^*})$. It is clear that if the invertible operator A and its inverse A^{-1} are normaloid, then the Banach space operators $L_A R_{A^*}$ and $(L_A R_{A^*})^{-1} = L_{A^{-1}} R_{A^{*-1}}$ are normaloid, hence invertible contractions. Consequently, the operator $L_A R_{A^*}$ is (then) an invertible Banach space isometry.

Every part of a contraction $A \in B(\mathcal{H})$ (i.e., every restriction of A to a closed invariant subspace) is a contraction. If such a contraction is similar to a unitary operator (hence a C_{11}-contraction), then every of its parts is normaloid, i.e., (in the terminology of [12, Proposition 1]) A is *hereditarily normaloid*, or, $A \in (\mathcal{HN})$. An operator $A \in B(\mathcal{H})$ is *totally hereditarily normaloid*, or $A \in (\mathcal{THN})$, if, along with $A \in \mathcal{HN}$, every invertible part of A is normaloid [12]. The following corollary says that if a normaloid operators A satisfying $\triangle_{A,A^*}(S) = 0$ for some operator S with $0 \notin \overline{W(S)}$ fails to be unitary, then it is an operator in $(\mathcal{HN}) \setminus (\mathcal{THN})$ (*cf.* [12, Corollary 1]).

Corollary 3.4. *If an invertible normaloid operator* $A \in B(\mathcal{H})$ *satisfying* $\triangle_{A,A^*}(S) = 0$ *for an operator* $S \in B(\mathcal{H})$ *with* $0 \notin \overline{W(S)}$ *is not unitary, then* $A \in (\mathcal{HN}) \setminus (\mathcal{THN})$.

Proof. The hypotheses imply the existence of an operator $P > 0$ and a unitary V such that $AP^{1/2} = P^{1/2}V^*$, hence A is a power bounded C_{11}-operator (with spectrum in $\partial\mathcal{D}$). Consequently, if A is normaloid, then A is a C_{11}-contraction with $\sigma(A) \subseteq \partial\mathcal{D}$. This, by [12, Proposition 1], implies $A \in (\mathcal{HN})$. Assume now that A is not unitary, and that there exists a closed invariant subspace M $(\neq \{0\})$ of A such that A_1 is an invertible cnu $C_{1.}$-contraction. We assert that A_1^{-1} is not normaloid. For suppose A_1^{-1} is normaloid. Then, since $APA^* = P$ is equivalent to $A^*P^{-1}A = P^{-1}$, it follows upon letting $P^{-1}|_M = P_1^{-1}$ that $A_1^* P_1^{-1} A_1 = P_1^{-1}$. Equivalently, $A_1 P_1 A_1^* = P_1$, $P_1 > 0$, where both A_1 and A_1^{-1} are normaloid. This, by Theorem 3.1, implies that A_1 is unitary – a contradiction. Hence $A \notin (\mathcal{THN})$. □

Recall from [5, Theorem 4] that if an $A \in B(\mathcal{H})$ is an invertible normal operator which satisfies $\triangle_{A,A^*}(S) = 0$ for an invertible operator $S \in B(\mathcal{H})$ such that $0 \notin W(S)$, then A is unitary. (Observe the weakened condition $0 \notin W(S)$. We remark here that the hypothesis A is invertible is superfluous: The invertibility of S implies the left invertibility of A^*, hence A is surjective, and normal operators have SVEP, hence A is invertible.) This result is the normal operators version of a more general result. Consider operators A, B and $S \in B(\mathcal{H})$ such that S is a quasi-affinity (i.e., S is injective with a dense range) and $\triangle^{-1}_{A,B^*}(0) \subseteq \triangle^{-1}_{A^*,B}(0)$. Then

$$\triangle_{A,B^*}(S) = ASB^* - S = 0 \Longrightarrow A^*SB - S = 0,$$

and it follows from Lemma 2.2 that if S has the polar decomposition $S = U|S|$, U unitary, then

$$AUB^*U^* = I = UB^*U^*A, [A, |S^*|] = 0 = [B, |S|]$$

and A, ${B^*}^{-1}$ are unitarily equivalent normal operator. Thus if we let $B = A$, let S be an invertible operator and set $ASA^{-1}S^{-1} = T$, then $AS - S{A^*}^{-1} = 0 = A^*S - SA^{-1}$ implies $T = A(SA^{-1})S^{-1} = A(A^*S)S^{-1} = AA^* = A^*A$ is a normal operator which satisfies $[T, A] = 0$. Hence, see [6, Theorem 1], that either $[A, S] = 0$ (which then implies A is unitary) or $0 \in W(S)$.

The inclusion $\triangle^{-1}_{A,B^*}(0) \subseteq \triangle^{-1}_{A^*,B}(0)$ is satisfied by a number of classes of operators more general than the class of normal operators. Let $\mathcal{E}_0$ denote the class of operators $A = U|A| \in B(\mathcal{H})$ which are either

- hyponormal: $|A^*|^2 \leq |A|^2$, or,
- p-hyponormal, $0 < p \leq 1$: $|A^*|^{2p} \leq |A|^{2p}$;

let $\mathcal{E}_1$ denote operators $A = U|A| \in B(\mathcal{H})$ which are either

- w-hyponormal: $(|A|^{\frac{1}{2}}U|A|U^*|A|^{\frac{1}{2}})^{\frac{1}{2}} \leq |A| \leq (|A|^{\frac{1}{2}}U^*|A|U|A|^{\frac{1}{2}})^{\frac{1}{2}}$, or,
- $\mathcal{A}(s,t)$, $0 < s, t \leq 1$: $|A^*|^{2t} \leq (|A^*|^t|A|^{2s}|A^*|^t)^{\frac{t}{s+t}}$.

It is well known [18] that $\mathcal{A}(s,t)$ operators, indeed operators in $\mathcal{E}_0 \cup \mathcal{E}_1$, are $\mathcal{A}(1,1)$ operators, $A \in \mathcal{A}(1,1)$ implies A^2 is w-hyponormal, $\mathcal{A}(1,1)$ operators are paranormal and $A \in \mathcal{A}(1,1) \Longrightarrow A^{-1} \in \mathcal{A}(1,1)$ (whenever A^{-1} exists). Let $\mathcal{D}_1$ denote operators $A \in \mathcal{D}_0$ such that $A^2 \in \mathcal{D}_0$.

Theorem 3.5. *Given operators A, B and $S \in B(\mathcal{H})$ such that S is invertible and $\triangle_{A,B^*}(S) = 0$, either of the conditions* (i), (ii) and (iii) *below implies ($\triangle_{A^*,B}(S) = 0$, hence) A and ${B^*}^{-1}$ are unitarily equivalent normal operators.*

(i) *$A, B \in \mathcal{E}_0 \cup \mathcal{E}_1$.*

(ii) *$A \in \mathcal{D}_0$ and $B \in \mathcal{E}_0$, or B is w-hyponormal,or B is M-hyponormal.*

(iii) *$A \in \mathcal{D}_1$ and $B \in \mathcal{A}(1,1)$.*

Furthermore:

(a) *If, in the above, $S > 0$, then A and B are normal operators such that $AB^* = B^*A = I$;*

(b) *if $\triangle_{A,A^*}(S) = 0$, where $A \in \mathcal{E}_0 \cup \mathcal{E}_1$ (or A is M-hyponormal) and S is invertible with $0 \notin W(S)$, then A is unitary.*

Proof. The hypotheses S is invertible and $\triangle_{A,B^*}(S) = 0$ imply B is surjective. Since B has SVEP, B is invertible (and this then implies that A is also invertible). Hence

$$\triangle_{A,B^*}(S) = 0 \Longleftrightarrow \delta_{A,{B^*}^{-1}}(S) = 0.$$

Recall now that $B^{-1} \in \mathcal{E}_0 \cup \mathcal{E}_1$ for operators $B \in \mathcal{E}_0 \cup \mathcal{E}_1$ [15], B^{-1} is M-hyponormal for M-hyponormal B and M-hyponormal operators are $\mathcal{Y}_2$ operators. Hence [11, Proposition

2.5] and [27, Theorems 1 and 2] apply, and we have

$$\delta_{A,B^{*-1}}(S) = 0 \Longrightarrow \delta_{A^*,B^{-1}}(S) = 0 \Longleftrightarrow \triangle_{A^*,B}(S) = 0.$$

This implies A and ${B^*}^{-1}$ are unitarily equivalent normal operators. Suppose now that $S > 0$. Define the normal operator $T = A \oplus B$ and the (self-adjoint) invertible operator Q by

$$T = A \oplus B,\ Q = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix}.$$

Then

$$TQT^* = Q \Longleftrightarrow T^*QT = Q,\ [T, Q^2] = 0 \Longrightarrow [A, S] = 0 = [B, S].$$

Consequently, A and B are normal operators such that $AB^* = B^*A = I$. To complete the proof consider now the case $\triangle_{A,A^*}(S) = 0$ and $0 \notin W(S)$. If $A \in \mathcal{E}_0 \cup \mathcal{E}_1$, or if A is M-hyponormal, then ($\triangle_{A,A^*}(S) = 0 \Longleftrightarrow \delta_{A,A^{*-1}}(S) = 0 \Longrightarrow \delta_{A^*,A^{-1}}(S) = 0 \Longleftrightarrow$) $\triangle_{A^*,A}(S) = 0$ and the normal operator $T = ASA^{-1}S^{-1}$ satisfies $[T, A] = 0$. Since $0 \notin W(S)$, A is unitary. □

Similarities. The similarity analogue "if $\delta_{A,B^*}(S) = 0$ for an invertible operator S, with A, B satisfying one of the conditions (i) to (iii) of Theorem 3.5, implies $\delta_{A^*,B}(S) = 0$" holds under the additional hypothesis that $T^{-1}(0) \subseteq {T^*}^{-1}(0)$ for operators T which are w-hyponormal or $\mathcal{A}(1,1)$ (see [11, Theorem 2.6] and [27, Theorems 1 and 2]). Assuming $S > 0$ this then implies that A and B are normal operators which satisfy $A = B^*$. In the following we prove a more general version of the result that if $\delta_{A,A^*}(S) = 0$, where $A \in \mathcal{A}(1,1)$ and S is invertible, then either of the hypotheses $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and $0 \notin W(A)$, or $0 \notin \overline{W(A)}$, implies A is self adjoint. We start with some complementary results.

Lemma 3.6. *Operators $A \in \mathcal{A}(1,1) \cap B(\mathcal{H})$ with $\sigma(A) \subset \mathbb{R}$ are self-adjoint.*

Proof. Given an $A \in \mathcal{A}(1,1)$, $|A|A$ is a $\frac{1}{2}$-hyponormal operator (i.e., $(|A||A^*|^2|A|)^{1/2} \leq (A^*|A|^2A)^{1/2}$) such that $\sigma(|A|A) = \{\lambda : \lambda = r^2e^{i\theta}, re^{i\theta} \in \sigma(A)\}$ [3]. Hence, since $\sigma(A) \subset \mathbb{R}$, the Putnam Lebesgue area measure inequality for p-hyponormal operators implies that

$$A^*|A|^2A - |A||A^*|^2|A|| \leq m(\sigma(|A|A)) = 0.$$

(Here $m(.)$ denotes the Lebesgue area measure.) Thus $A^*|A|^2A = |A||A^*|^2|A|$. Since $A \in \mathcal{A}(1,1)$ if and only if $|A^*|^2 \leq (|A^*||A|^2|A^*|)^{1/2}$, and since [18]

$$|A^*|^2 \leq (|A^*||A|^2|A^*|)^{1/2} \Longrightarrow (|A||A^*|^2|A|)^{1/2} \leq |A|^2,$$

$$|A|^2 \leq |A^2| = (|A||A^*|^2|A|)^{1/2} \leq |A|^2 \Longrightarrow |A|^2 = |A^2| \Longleftrightarrow A^*(A^*A - AA^*)A = 0.$$

But then A is a quasihyponormal operator such that $m(\sigma(A)) = 0$. Consequently, A is normal [2], hence self-adjoint. □

An operator $A \in B(\mathcal{H})$ is k-quasi $\mathcal{A}(1,1)$, $A \in k - \mathcal{A}(1,1)$, for some positive integer $k \geq 1$, if $A^{*k}(|A^2| - |A|^2)A^k \geq 0$. Every $A \in k - \mathcal{A}(1,1)$ has an upper triangular matrix representation

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & N \end{pmatrix} \in B(\overline{A^k(\mathcal{H})} \oplus {A^{*k}}^{-1}(0)),$$

where $A_1 \in \mathcal{A}(1,1)$ and N is a k-nilpotent operator. The following lemma proves that if the operator A_1 (in the above representation of A) is normal, then A is the direct sum of an injective normal operator and a $(k+1)$-nilpotent operator.

Lemma 3.7. *Given an operator $A \in k - \mathcal{A}(1,1)$ such that $A_1 = A|_{\overline{A^k(\mathcal{H})}}$ is normal, A is the direct sum of an injective normal operator and a $(k+1)$-nilpotent operator.*

Proof. Letting E denote the orthogonal projection of $\mathcal{H}$ onto $\overline{A^k(\mathcal{H})}$,

$$|A_1|^2 \oplus 0 = E|A|^2E \leq E|A^2|E \leq (EA^{*2}A^2E)^{\frac{1}{2}} = |A_1^2| \oplus 0 = |A_1|^2 \oplus 0,$$

which implies that $|A|^2$ has a representation

$$|A|^2 = \begin{pmatrix} |A_1| & C \\ C^* & D \end{pmatrix}^2 = \begin{pmatrix} |A_1|^2 + CC^* & |A_1|C + CD \\ C^*|A_1| + DC^* & C^*C + D^2 \end{pmatrix}.$$

Comparing this with

$$|A|^2 = \begin{pmatrix} |A_1|^2 & A_1^*A_2 \\ A_2^*A_1 & |A_2|^2 + |N|^2 \end{pmatrix}$$

we conclude that ($C^*C = 0 \Longleftrightarrow C = 0$, hence) $A_1^*A_2 = 0$ $(= A_1A_2)$ and $|A| = |A_1| \oplus D$. Let $A_1 = A_{11} \oplus 0 \in B(A^{-1}(0)^{\perp} \oplus A^{-1}(0)) = B(\overline{A^k(\mathcal{H})})$. Then A and $|A|$ have representations

$$A = \begin{pmatrix} A_{11} & 0 & A_{21} \\ 0 & 0 & A_{21} \\ 0 & 0 & N \end{pmatrix} = \begin{pmatrix} A_{11} & B_1 \\ 0 & N_1 \end{pmatrix} \text{ and } |A| = \begin{pmatrix} |A_{11}| & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D \end{pmatrix} = \begin{pmatrix} |A_{11}| & 0 \\ 0 & D_1 \end{pmatrix}$$

(in $B(A_1^{-1}(0)^{\perp} \oplus (A^{-1}(0) \oplus A^{*k^{-1}}(0))))$, where we have set $(0, A_{21}) = B_1$, $\begin{pmatrix} 0 & A_{22} \\ 0 & N \end{pmatrix} = N_1$ and $\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} = D_1$. Comparing ($A^*A$ with $|A|^2$) we have now that $A_{11}^*B_1 = 0$. Hence $B_1 = 0$ and $A = A_{11} \oplus N_1$. □

Part (b) of the following theorem generalizes a result of [19] to $k - \mathcal{A}(1,1)$ operators, using an argument which is simpler (than the one used in [19] to prove the result for $1 - \mathcal{A}(1,1)$ operators).

Theorem 3.8. *Given operators $A, S \in B(\mathcal{H})$ such that $A \in k - \mathcal{A}(1,1)$, S is invertible and $\delta_{A,A^*}(S) = 0$, if:*

(a) $A^{-1}(0) \subseteq A^{*-1}(0)$, *then a sufficient condition for $A = A^*$ is that $0 \notin W(S)$.*

(b) $0 \notin \overline{W(S)}$, *then A is the direct sum of a self-adjoint operator with a k-nilpotent operator. In particular, if $k = 1$, then A is self-adjoint.*

Proof. (a) If $A^{-1}(0) \subseteq A^{*-1}(0)$, then $A \in B(A^{-1}(0)^{\perp} \oplus A^{-1}(0))$ has a representation $A = A_1 \oplus 0$, where $A_1 = A|_{A^{-1}(0)^{\perp}} \in k - \mathcal{A}(1,1)$ is injective. If $\delta_{A,A^*}(S) = 0$ and $S \in B(A^{-1}(0)^{\perp} \oplus A^{-1}(0))$ has a representation $S = \begin{pmatrix} S_1 & S_3 \\ S_4 & S_2 \end{pmatrix}$, then $A_1S_3 = 0 = S_4A_1^*$ implies $S_3 = S_4 = 0$ (and, hence, both S_1 and S_2 are invertible). Again, since $A_1S_1 - S_1A_1^* = 0$, A_1 is injective with a dense range, hence (an $\mathcal{A}(1,1)$ operator which satisfies $A_1^*S_1 - S_1A_1 = 0$ and therefore is) normal. There exists a real number $\lambda \in \rho(A)$, the resolvent set of A, such that $A - \lambda$ is a normal invertible operator. Define the operator T by $T = (A-\lambda)S(A-\lambda)^{-1}S^{-1}$. Since $(A-\lambda)S - S(A-\lambda)^* = 0 = (A-\lambda)^*S - S(A-\lambda)$, $T = (A-\lambda)(A-\lambda)^* = (A-\lambda)(A^* - \lambda)$ is a normal operator which commutes with $A - \lambda$. This, since $0 \notin W(S)$, implies $A - \lambda = A^* - \lambda$ [6].

(b) Letting $A \in B(\overline{A^k(\mathcal{H})} \oplus A^{*k^{-1}}(0))$ have the representation of the proof of Lemma 3.7, and letting S have the corresponding representation $S = \begin{pmatrix} S_1 & S_3 \\ S_4 & S_2 \end{pmatrix}$, it is seen that the

$\mathcal{A}(1,1)$ operator A_1 satisfies $\delta_{A_1,A_1^*}(S_1)=0$, where $0\notin\overline{W(S_1)}$. Hence $\sigma(A_1)\subset\mathbb{R}$ [28], and it follows from Lemma 3.6 that A_1 is self-adjoint. By Lemma 3.7 this implies that $A=A_1\oplus N_1$, where (using the notation of the proof of Lemma 3.7) $N_1=\begin{pmatrix}0 & A_{22}\\ 0 & N\end{pmatrix}$, N is k-nilpotent, $N_1S_2=S_2N_1^*$ and $0\notin\overline{W(S_2)}$. Now if let $S_2=\begin{pmatrix}S_{21} & S_{23}\\ S_{24} & S_{22}\end{pmatrix}$, $0\notin\overline{W(S_{22})}$, then

$$N_1^kS_2=S_2N_1^{*k}\Longrightarrow A_{22}N^{k-1}S_{22}=0\Longleftrightarrow A_{22}N^{k-1}=0\Longrightarrow N_1^k=0.$$

Thus $A=A_1\oplus N_1$ is the direct sum of a self-adjoint operator and a k-nilpotent operator. In particular, if $k=1$, then A is self-adjoint. □

Remark 3.9. The argument of the proof of Theorem 3.8 applies to operators $A\in k-\mathcal{A}(1,1)$ satisfying $\triangle_{A,A^*}(S)=0$. Operators $A\in k-\mathcal{A}(1,1)$ have SVEP: This follows from the fact that $\mathcal{A}(1,1)$ operators are paranormal, paranormal operators have SVEP and operators $A\in k-\mathcal{A}(1,1)$ have a triangulation $A=\begin{pmatrix}A_1 & A_2\\ 0 & N\end{pmatrix}\in B(\overline{A^k(\mathcal{H})}\oplus {A^*}^{k^{-1}}(0))$ with $A_1\in\mathcal{A}(1,1)$ and N a k-nilpotent. Thus $\triangle_{A,A^*}(S)=0$, with $0\notin\overline{W(S)}$, implies (A is an invertible $\mathcal{A}(1,1)$ operator, indeed an invertible normal operator since $m(\sigma(A))=0$, which satisfies $\triangle_{A,A^*}(S)=0$, $0\notin\overline{W(S)}$, hence) A is unitary. Again, if S is invertible with $0\notin W(S)$, then $\triangle_{A,A^*}(S)=0\Longleftrightarrow\delta_{A,A^{*-1}}(S)=0$, where both A and A^{-1} are $\mathcal{A}(1,1)$ operators; hence A is unitary.

If an $A\in B(\mathcal{H})$ is a paranormal operator, then both A and A^{-1} (whenever it exists) are normaloid; hence, given a paranormal operator $A\in B(\mathcal{H})$, $\triangle_{A,A^*}(S)=0$ for an operator $S\in B(\mathcal{H})$ with $0\notin\overline{W(S)}$ implies A is unitary. Does $\delta_{A,A^*}(S)=0$ imply A is self-adjoint (for paranormal A and $0\notin\overline{W(S)}$)? Paranormal operators do not satisfy the property that $\delta^{-1}_{A,A^*}(0)\subseteq\delta^{-1}_{A^*,A}(0)$ [21]; additional hypotheses are required for the operator A above to be self-adjoint. One such hypothesis is given by the following theorem. Paranormal operators being normaloid, it is clear that if $\delta_{A,A^*}(S)=0$ then (upon diving by $||A||$ if need be) one may assume that A is a contraction. Recall from [9] that paranormal contractions operators have a $C_{.0}$ cnu part.

Theorem 3.10. *Let $A\in B(\mathcal{H})$ be a paranormal contraction with a Hilbert-Schmidt class defect operator $D_A=(1-A^*A)^{1/2}$ such that the normal subspaces of A reduce A. If $\delta_{A,A^*}(S)=0$ for an operator $S\in B(\mathcal{H})$ with $0\notin\overline{W(S)}$, then A is self-adjoint.*

Proof. Assume (without loss of generality) that $\delta_{A,A^*}(P)=0$ for some $P>0$. If the paranormal contraction A satisfies the properties that $(1-A^*A)^{1/2}$ is Hilbert-Schmidt class and the normal subspaces of A reduce A, then $A=A_n\oplus A_{10}\in B(\mathcal{H}_1\oplus\mathcal{H}_2)$ for a normal contraction A_n and a C_{10} cnu contraction A_{10} with a Hilbert-Schmidt defect operator $D_{A_{10}}=(1-A_{10}^*A_{10})^{1/2}$ (see [12, Theorem1]). Letting P have a corresponding representation $P=\begin{pmatrix}P_1 & P_3\\ P_3^* & P_2\end{pmatrix}$, it is then seen (from $\delta_{A,A^*}(P)=0$) that

$$A_nP_3=P_3A_{10}^*,\quad A_{10}P_2=P_2A_{10}^*.$$

Since the contraction A_{10} has a Hilbert-Schmidt defect operator and is of the class C_{10}, there exists a unilateral shift U and a quasi-affinity Z such that $ZA_{10}=UZ$ [26]. Hence

$$A_nP_3Z^*=P_3Z^*U^*,\quad A_n\ \text{normal and}\ U\ \text{hyponormal}.$$

The commutativity theorem for hyponormal operators applies and we conclude that (the restriction of U to the closure of the range of ZP_3^* is normal, implies $ZP_3^*=0$, hence)

$P_3 = 0$. The operator P_2 being invertible, it follows that for all $x \in \mathcal{H}_2$,

$$\begin{aligned} & A_{10}P_2x = P_2A_{10}^*x \Longrightarrow A_{10}^nP_2x = P_2{A_{10}^*}^nx \\ \Longrightarrow\ & \|A_{10}^nP_2x\| \leq \|P_2\|\|{A_{10}^*}^nx\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

This implies that the part A_{10} is missing. Hence A is normal, indeed self-adjoint. □

Unbounded operators. A version of Theorem 3.8(b) holds for (unbounded) closed densely defined (Hilbert space) operators. Recall that a densely defined closed operator A is said to be hyponormal if the domain of A is a subset of the domain of the adjoint A^*, dom(A) $\subset$ dom(A*), and $\|A^*x\| \leq \|Ax\|$ for all $x \in$ dom(A). Hyponormal operators A have SVEP, and satisfy the property that their translates $A - \lambda = A - \lambda I$ are also (densely defined closed) hyponormal operators. The following theorem generalises Theorems 8 and 9 of [4].

Theorem 3.11. *Let A be a densely defined closed operator with SVEP such that*

$$\delta^{-1}_{(A-\lambda)^{-1},(A^*-\lambda)^{-1}}(0) \subseteq \delta^{-1}_{(A-\lambda)^{*-1},(A^*-\lambda)^{*-1}}(0)$$

for some λ in the resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$. If $SA^* \subset AS$ for an operator $S \in B(\mathcal{H})$ with $0 \notin \overline{W(S)}$, then A is self-adjoint.

(Recall here that $SA^* \subset AS$ means $SA^*x = ASx$ for all $x \in$ dom(A*).)

Proof. We may assume without loss of generality that $\frac{S+S^*}{2} = P > 0$, and (then $SA^* \subset AS$ implies) $PA^* \subset AP$. Since A has SVEP, $\sigma(A^*) = \sigma_a(A^*)$ (i.e., the spectrum of A^* consists of the approximate point spectrum of A^*). Choose a $\lambda \in \sigma_a(A^*)$, and let $\{x_n\} \subset$ dom(A*) be a sequence of unit vectors such that $\|(A^* - \lambda)x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Then

$$\begin{aligned} \|(\overline{\lambda} - \lambda)(Px_n, x_n)\| &= |\{(PA^*P^{-1} - \lambda) - (A - \overline{\lambda}\}(Px_n, x_n)\| \\ &\leq \|P((A^* - \lambda)x_n, x_n)\| + \|P(x_n, (A^* - \lambda)x_n)\| \end{aligned}$$

implies

$$|\overline{\lambda} - \lambda|\|P^{\frac{1}{2}}x_n\|^2 \leq 2\|P\|\|(A^* - \lambda)x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $P > 0$, $\lambda = \overline{\lambda}$. Hence $\sigma(A) \subset \mathbb{R}$. There exists $\lambda \in \mathbb{C} \setminus \sigma(A)$ such that $(A - \lambda)^{-1}$ is a bounded operator which (since $PA^* \subset AP$) satisfies $\delta_{(A-\lambda)^{-1},(A^*-\lambda)^{-1}}(P) = 0$. Since $\delta^{-1}_{(A-\lambda)^{-1},(A^*-\lambda)^{-1}}(0) \subseteq \delta^{-1}_{(A-\lambda)^{*-1},(A^*-\lambda)^{*-1}}(0)$, we have

$$\delta_{(A-\lambda)^{-1},(A^*-\lambda)^{-1}}(P) = 0 = \delta_{(A-\lambda)^{*-1},(A^*-\lambda)^{*-1}}(P) = 0.$$

But then $[(A-\lambda)^{-1}, P] = 0$, and (consequently) $(A-\lambda)^{-1} = (A^*-\lambda)^{-1}$. Hence $A = A^*$. □

Theorem 3.11 applies to densely defined closed M-hyponormal operators, where we say that a densely defined closed (Hilbert space) operator is M-hyponormal if there exists an $M > 0$ such that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $\|(A - \lambda)^*x\|^2 \leq M\|(A - \lambda)x\|^2$ for all complex λ and $x \in$ dom(A).

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