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DENSENESS OF SETS OF SUPERCYCLIC VECTORS

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ABSTRACT. The sets of strongly supercyclic, weakly l-sequentially supercyclic, weakly sequentially supercyclic, and weakly supercyclic vectors for an arbitrary normed-space operator are all dense in the normed space, regardless the notion of denseness one is considering, provided they are nonempty.

1. Introduction

Strong supercyclicity is an important topic in operator theory for some decades already (see, e.g., [1, 4, 5, 9, 10, 18]). Several forms of weak supercyclicity have recently been investigated as well (see, e.g., [2, 3, 7, 12, 13, 14, 16, 19, 20, 22]).

Supercyclicity for normed-space operators means denseness of a projective orbit. Associated with the notion of denseness one is considering there corresponds the concepts of strongly supercyclic, weakly l-sequentially supercyclic, weakly sequentially supercyclic, and weakly supercyclic vectors.

This paper focus on another question on denseness, viz., denseness of sets of supercyclic vectors. Besides common notions of denseness in the weak and norm topologies, intermediate notions of weak sequential and weak l-sequential denseness are considered. The main result appears in Theorem 5.1 which leads in Corollary 6.1 to the following consequence: if any of the above sets of supercyclic vectors is nonempty, then it is dense with respect to any notion of denseness. In particular, the set of weakly l-sequentially supercyclic vectors is norm dense (i.e., strongly dense), which is a useful improvement over previously known results along this line.

The paper gather results on the above mentioned four forms of denseness for sets of *supercyclic* vectors (regarding all four forms of supercyclicity) into a concise statement, with emphasis on the set of weakly l-sequentially supercyclic vectors.

2. Preliminary Notions

Let $\mathbb{F}$ stand either for the complex field $\mathbb{C}$ or for the real field $\mathbb{R}$, and let $\mathcal{X}$ be an infinite-dimensional normed space over $\mathbb{F}$. An $\mathcal{X}$-valued sequence $\{x_n\}$ is *strongly convergent* if there is an $x \in \mathcal{X}$ such that $\|x_n - x\| \to 0$ (notation: $x_n \xrightarrow{s} x$ or $x = s\text{-}\lim x_n$), and it is *weakly convergent* if there is an $x \in \mathcal{X}$ such that $|f(x_n - x)| \to 0$ for every f in the dual space $\mathcal{X}^*$ of $\mathcal{X}$ (notation: $x_n \xrightarrow{w} x$ or $x = w\text{-}\lim x_n$). Strong convergence trivially implies weak convergence (to the same limit).

Subsets of $\mathcal{X}$ are *strongly closed* or *weakly closed* if they are closed in the norm or weak topologies of $\mathcal{X}$. *Strong closure* or *weak closure* of a set A is the smallest strongly or weakly closed set that includes A (i.e., the intersection of all strongly or weakly closed sets including A — notation: A^- or A^{-w}). Thus A is strongly closed or weakly closed if and only if $A = A^-$ or $A = A^{-w}$. A set A is *strongly dense* or *weakly dense* in $\mathcal{X}$ if $A^- = \mathcal{X}$ or $A^{-w} = \mathcal{X}$.

Definition 2.1. Let A be a subset of $\mathcal{X}$.

(a) The set A is *weakly sequentially closed* if every A-valued weakly convergent sequence has its limit in A (i.e., if $\{x = w\text{-}\lim x_n$ with $x_n \in A \Longrightarrow x \in A\}$).

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(b) The *weak sequential closure* of A is the smallest weakly sequentially closed set that incudes A (i.e., is the intersection of all weakly sequentially closed sets including A) — notation: A^{-ws}.

(c) The set A is *weakly sequentially dense* in $\mathcal{X}$ if $A^{-ws} = \mathcal{X}$.

(d) The *weak limit set* A^{-wl} of A is the set of all weak limits of weakly convergent A-valued sequences (i.e., $A^{-wl} = \{x \in \mathcal{X} : x = w\text{-}\lim x_n \text{ with } x_n \in A\}$.

(e) The set A is *weakly l-sequentially dense* in $\mathcal{X}$ if $A^{-wl} = \mathcal{X}$.

A collection of basic results required in the sequel is given below. Most are either straightforward or well-known and standard. We prove item (e) only.

Proposition 2.1. *Consider the setup of Definition* 2.1.

(a) *A is weakly closed $\Longrightarrow$ A is weakly sequentially closed $\Longrightarrow$ A is strongly closed.*

(b) $A \subseteq A^{-} \subseteq A^{-wl} \subseteq A^{-ws} \subseteq A^{-w} \subseteq \mathcal{X}$.

(c) *Now consider the following assertions.*

(c_0) *A is strongly dense (equivalently, for every $x \in \mathcal{X}$ there exists an A-valued sequence $\{x_n\}$ such that $x_n \xrightarrow{s} x$),*

(c_1) *A is weakly l-sequentially dense (equivalently, for every $x \in \mathcal{X}$ there exists an A-valued sequence $\{x_n\}$ such that $x_n \xrightarrow{w} x$),*

(c_2) *A is weakly sequentially dense,*

(c_3) *A is weakly dense.*

They are related by this chain of implications.

$$(c_0) \Longrightarrow (c_1) \Longrightarrow (c_2) \Longrightarrow (c_3).$$

(d) *A is convex* $\Longrightarrow A^{-} = A^{-wl} = A^{-ws} = A^{-w}$.

(e) *The following assertions are pairwise equivalent.*

(e_1) *A is weakly sequentially closed.*

(e_2) $A = A^{-ws}$.

(e_3) $A = A^{-wl}$.

Proof. (e) Assertions (e_1) and (e_2) are trivially equivalent by Definition 2.1(b). If A is weakly sequentially closed, then $A^{-wl} = A$ by Definitions 2.1(a,d) and so (e_1) implies (e_3). Conversely, if $A^{-wl} = A$, equivalently, if $A^{-wl} = \{x \in \mathcal{X} : x = w\text{-}\lim x_n$ with $x_n \in A\} \subseteq A$, then $\{x = w\text{-}\lim x_n$ with $x_n \in A \Longrightarrow x \in A\}$, and hence A is weakly sequentially closed by Definition 2.1(a). Thus (e_3) implies (e_1). □

Although it may happen $A^{-wl} \subset A^{-ws}$ (proper inclusion) in Proposition 2.1(b), this is not the case if A is weakly sequentially closed by Proposition 2.1(e).

3. Supercyclic Vectors

Let $\mathcal{B}[\mathcal{X}]$ be the normed algebra of all bounded linear operators of a normed space $\mathcal{X}$ into itself. Given an operator $T \in \mathcal{B}[\mathcal{X}]$ consider its power sequence $\{T^n\}_{n\geq 0}$. The orbit $\mathcal{O}_T(y)$ or $\mathrm{Orb}(T, y)$ of a vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the set

$$\mathcal{O}_T(y) = \bigcup_{n\geq 0} T^n y = \{T^n y \in \mathcal{X} : n \in \mathbb{N}_0\}$$

where $\mathbb{N}_0$ denotes the set of nonnegative integers, and we write $\bigcup_{n\geq 0} T^n y$ for the set $\bigcup_{n\geq 0} T^n(\{y\}) = \bigcup_{n\geq 0}\{T^n y\}$. The orbit $\mathcal{O}_T(A)$ of a set $A \subseteq \mathcal{X}$ under T is likewise defined: $\mathcal{O}_T(A) = \bigcup_{n\geq 0} T^n(A) = \bigcup_{z\in A} \mathcal{O}_T(z)$. Let $[x] = \operatorname{span}\{x\}$ stand for the subspace of $\mathcal{X}$ spanned by a singleton $\{x\}$ at a vector $x \in \mathcal{X}$, which is a one-dimensional subspace of $\mathcal{X}$ whenever x is nonzero. The projective orbit of a vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the orbit of span of $\{y\}$; that is, the orbit $\mathcal{O}_T([y])$ of $[y]$:

$$\mathcal{O}_T([y]) = \mathcal{O}_T(\operatorname{span}\{y\}) = \bigcup_{z\in[y]} \mathcal{O}_T(z) = \big\{\alpha T^n y \in \mathcal{X} \colon\; \alpha \in \mathbb{F},\; n \in \mathbb{N}_0\big\}.$$

Supercyclicity means denseness of projective orbits (if denseness holds).

Note. Clearly, $\mathcal{O}_T(\operatorname{span}\{y\})$ and $\operatorname{span}\mathcal{O}_T(y)$ are different sets. Denseness of the latter is referred to as cyclicity, and denseness of the orbit itself is referred to as hypercyclicity. These will not be addressed in this paper. (For a brief discussion on them see, e.g., [13, Sections 2 and 3]; for a thorough view see, e.g., [4] and [8].)

Definition 3.1. Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a normed space $\mathcal{X}$.

(a) A vector $y \in \mathcal{X}$ is *strongly supercyclic* (or *supercyclic*) for T if $\mathcal{O}_T([y])^- = \mathcal{X}$.

(b) A vector $y \in \mathcal{X}$ is *weakly l-sequentially supercyclic* for T if $\mathcal{O}_T([y])^{-wl} = \mathcal{X}$.

(c) A vector $y \in \mathcal{X}$ is *weakly sequentially supercyclic* for T if $\mathcal{O}_T([y])^{-ws} = \mathcal{X}$.

(d) A vector $y \in \mathcal{X}$ is *weakly supercyclic* for T if $\mathcal{O}_T([y])^{-w} = \mathcal{X}$.

An operator T is *strongly supercyclic* (or simply *supercyclic*), *weakly l-sequentially supercyclic*, *weakly sequentially supercyclic*, or *weakly supercyclic* if there exists a strongly, weakly l-sequentially, weakly sequentially, or weakly supercyclic vector y for it. Any form of cyclicity for an operator T implies it acts on a separable space $\mathcal{X}$. According to Proposition 2.1(c) and Definition 3.1,

$$\begin{smallmatrix}\text{STRONG}\\ \text{SUPERCYCLICITY}\end{smallmatrix} \Longrightarrow \begin{smallmatrix}\text{WEAK L-SEQUENTIAL}\\ \text{SUPERCYCLICITY}\end{smallmatrix} \Longrightarrow \begin{smallmatrix}\text{WEAK SEQUENTIAL}\\ \text{SUPERCYCLICITY}\end{smallmatrix} \Longrightarrow \begin{smallmatrix}\text{WEAK}\\ \text{SUPERCYCLICITY}\end{smallmatrix}.$$

The above implications are nonreversible (see, e.g., [22, pp.38,39], [4, pp.259,260]).

A word on terminology. Weak l-sequential supercyclicity was considered in [6] (and implicitly in [3]), and it was referred to as weak 1-sequential supercyclicity in [22]. Although there are reasons for such a terminology we have changed it here to weak l-sequential supercyclicity, replacing the numeral "1" with the letter "l" for "limit" which better describes the way this notion has been introduced here so far.

4. Auxiliary Notation, Terminology, and Auxiliary Results

Take an arbitrary subset A of the normed space $\mathcal{X}$ and set

$A^{-0} = A^-$, the strong closure of A (or the *0-closure* of A),
$A^{-1} = A^{-wl}$, the weak limit set of A (or the *1-closure* of A),
$A^{-2} = A^{-ws}$, the weak sequential closure of A (or the *2-closure* of A),
$A^{-3} = A^{-w}$, the weak closure of A (or the *3-closure* of A).

Thus, according to Proposition 2.1(b),

$$A \subseteq A^{-0} \subseteq A^{-1} \subseteq A^{-2} \subseteq A^{-3} \subseteq \mathcal{X},$$

and hence, following the denseness chain of Proposition 2.1(c),

$$A^{-0} = \mathcal{X} \implies A^{-1} = \mathcal{X} \implies A^{-2} = \mathcal{X} \implies A^{-3} = \mathcal{X},$$

where the notions of k-*denseness* ($A^{-k} = \mathcal{X}$) are in general distinct for each $k = 0, 1, 2, 3$. Accordingly, a set A is *0-closed*, *2-closed* or *3-closed* if it is strongly closed, weakly sequentially closed or weakly closed, respectively. From Proposition 2.1(a)

$$A \text{ is 3-closed} \implies A \text{ is 2-closed} \implies A \text{ is 0-closed},$$

and from Proposition 2.1(e)

$$A \text{ is 2-closed} \iff A = A^{-1} \iff A = A^{-2}.$$

Strongly open (or *0-open*) and *weakly open* (or *3-open*) are sets which are open in the norm or in the weak topologies of $\mathcal{X}$ (complements of strongly and weakly closed sets). A set A is *weakly sequentially open* (or *2-open*) if its complement is weakly sequentially closed (i.e., $A \subseteq \mathcal{X}$ is 2-open if $\mathcal{X} \backslash A$ is 2-closed). By Proposition 2.1(a)

$$A \text{ is 3-open} \implies A \text{ is 2-open} \implies A \text{ is 0-open}.$$

The norm topology and the weak topology are precisely the collections of all 0-open and 3-open sets, respectively. Consider the collection of all 2-open (i.e., of all weakly sequentially open) subsets of $\mathcal{X}$. This is a topology on $\mathcal{X}$ as well.

Proposition 4.1. *The collection of all* 2*-open sets is a topology on* $\mathcal{X}$*.*

Proof. From Definition 2.1(a)

$$\text{A is 2-closed} \iff \big\{x = w\text{-}\lim x_n \text{ with } x_n \in A \implies x \in A\big\}.$$

(a) *The empty set and the whole set are* 2*-open* (since they are trivially 2-closed).

(b) Take a nonempty intersection $\bigcap_\gamma A_\gamma$ of 2-closed subsets A_γ of $\mathcal{X}$. Take any $\bigcap_\gamma A_\gamma$-valued weakly convergent sequence $\{c_n\}$. Since A_γ are all 2-closed, the weak limit of $\{c_n\}$ lies in each A_γ, and so in $\bigcap_\gamma A_\gamma$. So $\bigcap_\gamma A_\gamma$ is 2-closed. Thus $\bigcup_\gamma A_\gamma = \mathcal{X} \backslash \bigcap_\gamma A_\gamma$ is 2-open. Outcome: *an arbitrary union of* 2*-open sets is* 2*-open.*

(c) Consider the union $A \cup B$ of two nonempty 2-closed subsets A and B of $\mathcal{X}$. Take any $A \cup B$-valued weakly convergent (infinite) sequence $\{c_n\}$, say

$$c_n \xrightarrow{w} c \in \mathcal{X}$$

(i.e., the $\mathbb{F}$-valued sequence $\{f(c_n)\}$ converges in the metric space $(\mathbb{F}, |\cdot|)$ to $f(c)$ in $\mathbb{F}$ for every $f \in \mathcal{X}^*$). If $\{c_n\}$ is eventually in one of the sets A or B, then c lies in such a set (because both sets are 2-closed) and so $c \in A \cup B$. If $\{c_n\}$ is not eventually in one of the sets, then it has infinitely many entries in A and infinitely many entries in B. Let $\{a_m\}$ and $\{b_m\}$ be subsequences of $\{c_n\}$ whose entries are all in A and all in B, respectively. Since the above displayed convergence takes place in the metric space $(\mathbb{F}, |\cdot|)$, every subsequence of $\{f(c_n)\}$ converges to the same limit $f(c)$ for every $f \in \mathcal{X}^*$. Then $\{a_m\}$ and $\{b_m\}$ converge weakly to c:

$$a_m \xrightarrow{w} c \in \mathcal{X} \quad \text{and} \quad b_m \xrightarrow{w} c \in \mathcal{X}.$$

Because A and B are 2-closed, we get $c \in A \cap B$. Hence $c \in A \cup B$ again. Outcome: the union of a pair of 2-closed sets is 2-closed. Thus (by induction) a finite union of 2-closed sets is 2-closed, and so *a finite intersection of* 2*-open sets is* 2*-open.* □

Since a set A is 0-closed, 2-closed, or 3-closed if and only if $A = A^{-0}$, $A = A^{-2}$, or $A = A^{-3}$, we say a set A is *1-closed* if $A = A^{-1}$ (and *1-open* if its complement is 1-closed). Such a definition of 1-closedness collapses to the definition of 2-closedness since $A = A^{-1} \iff A = A^{-2}$ (Proposition 2.1(e)). Even though $A^{-1} \subseteq A^{-2}$, if A is not 1 or not 2-closed, then A^{-1} may be properly included in A^{-2}, and the notion of 1-denseness implies (but is not implied by) the notion of 2-denseness. We refer to the weak limit set A^{-1} of A as the 1-closure of A. This is an abuse of terminology since the map $A \mapsto A^{-1}$ is not a topological closure operation. Strong, weak sequential, and weak topologies are referred to as 0, 2, and 3-topologies, respectively. There is no 1-topology (and so 1-closure and 1-denseness are not topological terminologies).

The *k-interior* of a set A, denoted $A^{\circ k}$, is the interior of it regarding the respective notion of k-openness: the largest k-open set included in A (i.e., the union of all k-open subsets of A). Since 1-closedness coincides with 2-closedness, the notions of 1-interior and 2-interior coincide as well: $A^{\circ 1} = A^{\circ 2}$. Also, $(\mathcal{X}\backslash A)^{-k} = \mathcal{X}\backslash A^{\circ k}$ and $(\mathcal{X}\backslash A)^{\circ k} = \mathcal{X}\backslash A^{-k}$ for $k = 0, 2, 3$ since these are bona fide closures on different topologies. For $k = 1$ these identities survive as inclusions only. Indeed, $(\mathcal{X}\backslash A)^{-1} \subseteq (\mathcal{X}\backslash A)^{-2} = \mathcal{X}\backslash A^{\circ 2} = \mathcal{X}\backslash A^{\circ 1}$ and $(\mathcal{X}\backslash A)^{\circ 1} = (\mathcal{X}\backslash A)^{\circ 2} = \mathcal{X}\backslash A^{-2} \subseteq \mathcal{X}\backslash A^{-1}$. A set is *k-nowhere dense* if its k-closure has empty k-interior (i.e., $(A^{-k})^{\circ k} = \varnothing$). The notions of 1-open and 2-open coincide, but since A^{-1} may be properly included in A^{-2}, it may happen $\varnothing = (A^{-1})^{\circ 1} \subset (A^{-2})^{\circ 2}$. The next results will be required later.

Proposition 4.2. *Suppose $B \subseteq A$ are nonempty subsets of a normed space $\mathcal{X}$. If the difference $A\backslash B$ lies in a finite union of one-dimensional subspaces of $\mathcal{X}$, then*

$$A^{-k} = \mathcal{X} \implies B^{-k} = \mathcal{X} \quad \textit{for every } k = 0, 1, 2, 3.$$

Proof. If $B = A$ the result is tautological. First suppose $\varnothing \neq B \subset A \subseteq \mathcal{X}$ are such that $A\backslash B \subseteq [u]$, where $[u]$ is an arbitrary one-dimensional subspace of $\mathcal{X}$ (spanned by a singleton $\{u\}$ at a nonzero vector $u \in \mathcal{X}$). We split the proof into two parts.

(a) Consider the k-topologies for $k = 0, 2, 3$. The identity $B^{-k} = A^{-k}$ holds if and only if the difference $A\backslash B$ is k-nowhere dense, that is, if and only if $((A\backslash B)^{-k})^{\circ k} = \varnothing$. Since $((A\backslash B)^{-k})^{\circ k} \subseteq ([u]^{-k})^{\circ k}$, then $([u]^{-k})^{\circ k} = \varnothing$ implies $B^{-k} = A^{-k}$. If $k = 0$ (norm topology), then $([u]^{-0})^{\circ 0} = \varnothing$ because $[u]$ is 0-closed and $[u]^{\circ 0} = \varnothing$. If $k = 2, 3$, then $([u]^{-k})^{\circ k} = \varnothing$ as well, since in a finite-dimensional subspace weak and strong (and the intermediate weak sequential) topologies coincide (see, e.g., [15, Proposition 2.5.13, 2.5.22]), and so $[u]$ 0-closed means $[u]$ k-closed and $[u]^{\circ k} = [u]^{\circ 0}$ for $k = 2, 3$ on the one-dimensional subspace $[u]$. Then $B^{-k} = A^{-k}$. Hence $A^{-k} = \mathcal{X}$ implies $B^{-k} = \mathcal{X}$ for $k = 0, 2, 3$ whenever $A\backslash B$ lies in a one-dimensional space.

(b) For $k = 1$ proceed as follows. Suppose $A^{-1} = \mathcal{X}$. Take an arbitrary $x \in \mathcal{X}$. Thus there is an A-valued sequence $\{a_n\}$ such that $a_n \xrightarrow{w} x$. If $a_n \xrightarrow{s} x$, then $x \in A^{-0}$. But $B^{-0} = A^{-0}$ by item (a). Hence $x \in B^{-0}$ and so $x \in B^{-1}$ (since $B^{-0} \subseteq B^{-1}$). Therefore $B^{-1} = \mathcal{X}$ (i.e., $\mathcal{X} \subseteq B^{-1}$). On the other hand, if $a_n \overset{s}{\not\longrightarrow} x$, then $\{a_n\}$ is not eventually in $A\backslash B \subseteq [u]$ (where weak and strong convergence coincide as we saw above). Then there is a subsequence $\{b_n\}$ of $\{a_n\}$ for which $b_n \notin A\backslash B \subseteq [u]$ for every n, and so $\{b_n\}$ is a B-valued sequence such that $b_n \xrightarrow{w} x$. Therefore $B^{-1} = \mathcal{X}$.

Thus $\varnothing \neq B \subset A \subseteq \mathcal{X}$ and $A^{-k} = \mathcal{X}$ imply $B^{-k} = \mathcal{X}$ for every $k = 0, 1, 2, 3$ if $A\backslash B$ lies in a one-dimensional space, and the same line of reasoning holds if $A\backslash B$ lies in a finite union of one-dimensional spaces (properly included in $\mathcal{X}$). □

Remark. Proposition 4.2 still holds if the difference $A\backslash B$ lies in a finite union of proper finite-dimensional subspaces of $\mathcal{X}$.

Proposition 4.3. *If A is a set in a normed space $\mathcal{X}$ and $L \in \mathcal{B}[\mathcal{X}]$, then*

$$L(A^{-k}) \subseteq L(A)^{-k} \quad \textit{for every} \;\; k = 0, 1, 2, 3.$$

Proof. The inclusion holds for a continuous map L between topological spaces (see, e.g., [11, Problem 3.46]). If L is a linear continuous map between normed spaces, then weak and strong (and the intermediate weak sequential) continuities coincide (see, e.g., [15, Theorem 2.5.11]). Thus the inclusion holds for $k = 0, 2, 3$ whenever $L \in \mathcal{B}[\mathcal{X}]$ (i.e., L is continuous in the (norm) 0-topology). The case of $k = 1$ requires a separate proof. If $x \in L(A^{-1})$, then $x = La$ for some $a \in A^{-1}$. But $a \in A^{-1}$ if and only if $f(a) = \lim_j f(a_j)$ with $a_j \in A$ for every $f \in \mathcal{X}^*$. Since $f \circ L = L^*f \in \mathcal{X}^*$ for the normed-space adjoint L^* (see, e.g., [21, Section 3.2]) we get with $g = L^*f \in \mathcal{X}^*$

$$f(x) = f(La) = (L^*f)(a) = g(a) = \lim_j g(a_j) = \lim_j (L^*f)(a_j) = \lim_j f(La_j)$$

for every $f \in \mathcal{X}^*$. Since $L(a_j) \in L(A)$, then $x \in L(A)^{-1}$. Thus $L(A^{-1}) \subseteq L(A)^{-1}$. □

For each k a vector $y \in \mathcal{X}$ is *k-supercyclic* for an operator $T \in \mathcal{B}[\mathcal{X}]$ (and T is a *k-supercyclic* operator) if the projective orbit $\mathcal{O}_T([y])$ is k-dense in $\mathcal{X}$. Thus let

Y_0 be the collection of all strongly supercyclic vectors for T,

Y_1 be the collection of all weakly l-sequentially supercyclic vectors for T,

Y_2 be the collection of all weakly sequentially supercyclic vectors for T,

Y_3 be the collection of all weakly supercyclic vectors of T.

So *T is k-supercyclic if and only if $Y_k \neq \varnothing$*. According to Proposition 2.1(b,c),

$$Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq Y_3, \tag{1}$$

$$Y_k^{-0} \subseteq Y_k^{-1} \subseteq Y_k^{-2} \subseteq Y_k^{-3} \quad \text{for every} \;\; k = 0, 1, 2, 3. \tag{2}$$

5. Denseness of Supercyclic Vectors

The punctured projective orbit of a vector y in a normed space $\mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the projective orbit of y excluding the origin,

$$\mathcal{O}_T([y])\backslash\{0\} = \big\{\alpha T^n y \in \mathcal{X} \colon \; \alpha \in \mathbb{F}\backslash\{0\}, \; n \in \mathbb{N}_0\big\}\backslash\{0\}.$$

By definition of k-supercyclicity, for each $k = 0, 1, 2, 3$

$$y \in Y_k \quad \Longleftrightarrow \quad (\mathcal{O}_T([y])\backslash\{0\})^{-k} = \mathcal{X}. \tag{3}$$

Lemma 5.1. *For every $k = 0, 1, 2, 3$*

$$y \in Y_k \;\; \Longrightarrow \;\; (\mathcal{O}_T([y])\backslash\{0\}) \subseteq Y_k.$$

Proof. Take an operator $T \in \mathcal{B}[\mathcal{X}]$ and an arbitrary $k = 0, 1, 2, 3$. Suppose $y \in Y_k$ (i.e., $\mathcal{O}_T([y])^{-k} = \mathcal{X}$) and take an arbitrary $z \in \mathcal{O}_T([y])\backslash\{0\}$. Thus $z = \gamma T^m y$ for some nonzero scalar $\gamma \in \mathbb{F}$ and some nonnegative integer $m \in \mathbb{N}_0$, and hence

$$\mathcal{O}_T([z]) = \mathcal{O}_T([T^m y]) = \bigcup\nolimits_{n \geq 0} [T^n T^m y] = \bigcup\nolimits_{n \geq 0} [T^{n+m} y] = \bigcup\nolimits_{n \geq m} [T^n y].$$

So $\varnothing \neq \mathcal{O}_T([z]) \subseteq \mathcal{O}_T([y])$ and the difference $\mathcal{O}_T([y])\backslash\mathcal{O}_T([z])$ lies in a finite union $\bigcup_{n \in [0, m-1]} [T^n y]$ of one-dimensional subspaces of $\mathcal{X}$. Since $\mathcal{O}_T([y])^{-k} = \mathcal{X}$, then $\mathcal{O}_T([z])^{-k} = \mathcal{X}$ by Proposition 4.2. Thus $z \in Y_k$. Therefore $(\mathcal{O}_T([y])\backslash\{0\}) \subseteq Y_k$. □

Remark 5.1. Take an arbitrary index $k=0,1,2,3$. By Lemma 5.1 and (3),

$$Y_k \neq \varnothing \implies Y_k^{-k} = \mathcal{X}.$$

Moreover, using (1) and (2) this can be readily extended to

$$Y_k \neq \varnothing \implies Y_i^{-j} = \mathcal{X} \quad \text{for every } \ i,j \in [k,3].$$

In particular (for $k=0$), $Y_0 \neq \varnothing$ implies $Y_1^{-0} = \mathcal{X}$. This, however, is not enough to answer the question whether, for instance,

$$\varnothing = Y_0 \subset Y_1 \neq \varnothing \ \overset{?}{\implies} \ Y_1^{-0} = \mathcal{X}.$$

Along this line it was proved in [19, Proposition 2.1] that $Y_3 \neq \varnothing \Longrightarrow Y_3^{-0} = \mathcal{X}$. So

$$Y_3 \neq \varnothing \implies Y_3^{-j} = \mathcal{X} \quad \text{for all } \ j$$

by (2), which still does not answer the above question. This is extended in the next theorem (using an argument similar to the one in [19, Proposition 2.1]) to show that

$$Y_k \neq \varnothing \implies Y_k^{-0} = \mathcal{X} \quad \text{for every } \ k=0,1,2,3.$$

In particular, for $k=1$ this represents a real and useful gain over the previously known results along this line, answering the above question, and leading to a general case for any nonempty set of supercyclic vectors with respect to any notion of denseness (including the nontopological 1-denseness).

Regarding the above remark and the next theorem, the condition $Y_k \neq \varnothing$ is fulfilled whenever $Y_\ell \neq \varnothing$ for some $\ell \in [0,k]$ by (1).

Theorem 5.1. *Take an operator T on a complex normed space $\mathcal{X}$. For $k=0,1,2,3$*

$$Y_k \neq \varnothing \implies Y_i^{-j} = \mathcal{X} \ \text{ for every } i \in [k,3] \text{ and all } j=0,1,2,3.$$

Proof. Take $T \in \mathcal{B}[\mathcal{X}]$.

Claim. $Y_k \neq \varnothing \implies Y_k^{-0} = \mathcal{X}$ for every $k=0,1,2,3$.

Proof. Let k be an arbitrary index in $[0,3]$. Suppose $Y_k \neq \varnothing$ and take any $y \in Y_k$. Let p be an arbitrary nonzero polynomial. Then

$$p(T)(\mathcal{X}) = p(T)(\mathcal{O}_T([y])^{-k}) \subseteq \big(p(T)(\mathcal{O}_T([y]))\big)^{-k} = \mathcal{O}_T([p(T)y])^{-k},$$

where the above inclusion holds for each k by Proposition 4.3. Thus if $p(T)(\mathcal{X})^{-k} = \mathcal{X}$, then $\mathcal{O}_T([p(T)y])^{-k} = \mathcal{X}$ and so $p(T)y \in Y_k$ by (3). In other words, if the range of $p(T)$ is k-dense in $\mathcal{X}$, then the vector $p(T)y$ is k-supercyclic whenever y is:

$$y \in Y_k \quad \text{and} \quad p(T)(\mathcal{X})^{-k} = \mathcal{X} \quad \implies \quad p(T)y \in Y_k. \tag{$*$}$$

Now take the dual $\mathcal{X}^*$ of the complex normed space $\mathcal{X}$, let $T^* \in \mathcal{B}[\mathcal{X}^*]$ stand for the normed-space adjoint of $T \in \mathcal{B}[\mathcal{X}]$, and let $\sigma_P(T^*)$ be the point spectrum (i.e., the set of all eigenvalues) of T^*. According to [17, Lemma 2] the range of $p(T)$ is dense in a complex locally convex space if and only if all eigenvalues of T^* are not zeros of p. The strong (norm) topology of a normed space yields a locally convex space. Thus the range of $p(T)$ is strongly dense if and only if all eigenvalues of T^* are not zeros of p. But range is a linear manifold, thus a convex set, and hence all k-closures coincide (cf. Proposition 2.1(d)) so that

$$p(T)(\mathcal{X})^{-k} = \mathcal{X} \quad \iff \quad p(\lambda) \neq \{0\} \text{ for all } \lambda \in \sigma_P(T^*). \tag{$**$}$$

Next for any $y \in \mathcal{X}$ consider the sets

$$P_T(y) = \big\{p(T)y \in \mathcal{X}\colon\; p \text{ is a polynomial}\big\} = \operatorname{span}\mathcal{O}_T(y),$$
$$P'_T(y) = \big\{p(T)y \in P_T(y)\colon\; p(\lambda) \neq 0 \;\text{ for all }\; \lambda \in \sigma_P(T^*)\big\}.$$

According to $(**)$ and $(*)$,

$$y \in Y_k \quad\Longrightarrow\quad P'_T(y) = \big\{p(T)y \in P_T(y)\colon\; p(T)(\mathcal{X})^{-k} = \mathcal{X}\big\} \subseteq Y_k. \qquad \text{(i)}$$

We will show that $P'_T(y)$ is strongly dense in $\mathcal{X}$, and consequently Y_k is strongly dense in $\mathcal{X}$. First consider the following auxiliary result.

$$\text{If } T \text{ is 3-supercyclic, then } \#\sigma_P(T^*) \leq 1$$

where # stands for cardinality. This was verified for supercyclic operators on a Hilbert space in [9, Proposition 3.1], extended to supercyclic operators on a normed space in [1, Theorem 3.2], and further extended to supercyclic operators on a locally convex space in [17, Lemma 1, Theorem 4]. But the weak topology of a normed space is a locally convex subtopology of the locally convex norm topology — see e.g., [15, Theorems 2.5.2 and 2.2.3]. Then the latter extension holds in particular on a normed space under the weak topology, thus including weakly supercyclic operators on a normed space. Since $Y_k \neq \varnothing \Longrightarrow Y_3 \neq \varnothing$ for any k by (1) we get

$$Y_k \neq \varnothing \quad\Longrightarrow\quad \#\sigma_P(T^*) \leq 1. \qquad \text{(ii)}$$

Moreover, $\mathcal{O}_T([y]) = \mathcal{O}_T(\operatorname{span}\{y\}) \subseteq \operatorname{span}\mathcal{O}_T(y) = P_T(y)$. So $\big(\operatorname{span}\mathcal{O}_T(y)\big)^{-0} = \mathcal{X}$ whenever $\mathcal{O}_T([y])^{-k} = \mathcal{X}$ (i.e., whenever $y \in Y_k$) for an arbitrary k since $\operatorname{span}\mathcal{O}_T(y)$ is convex (cf. Proposition 2.1(d)). Hence

$$y \in Y_k \quad\Longrightarrow\quad P_T(y)^{-0} = \mathcal{X}. \qquad \text{(iii)}$$

If $\#\sigma_P(T^*) = 0$ and $y \in Y_k$, then $P'_T(y) = P_T(y)$ and so by (iii)

$$y \in Y_k \;\text{ and }\; \#\sigma_P(T^*) = 0 \quad\Longrightarrow\quad P'_T(y)^{-0} = \mathcal{X}.$$

On the other hand, if $\#\sigma_P(T^*) = 1$ (i.e., if $\sigma_P(T^*) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$), then $P'_T(y) = \{p(T)y \in P_T(y)\colon p(\lambda_0) \neq 0\}$ is dense in $P_T(y)$ in the norm topology, and $P_T(y)$ is dense in $\mathcal{X}$ in the norm topology by (iii) whenever $y \in Y_k$. Thus

$$y \in Y_k \;\text{ and }\; \#\sigma_P(T^*) = 1 \quad\Longrightarrow\quad P'_T(y)^{-0} = \mathcal{X}.$$

Hence by (i), (ii), and the preceding two implications we get the claimed result:

$$y \in Y_k \quad\Longrightarrow\quad P'_T(y)^{-0} = \mathcal{X} \quad\Longrightarrow\quad Y_k^{-0} = \mathcal{X}. \;\square$$

If $Y_k^{-0} = \mathcal{X}$, then $Y_k^{-j} = \mathcal{X}$ for $j = 0, 1, 2, 3$ by (2). So $Y_i^{-j} = \mathcal{X}$ for $i \geq k$ by (1). □

Remark 5.2. Propositions 4.1 to 4.3, besides being required for proving Theorem 5.1, may be seen as relevant by themselves in the following sense. Proposition 4.1 applies standard convergence techniques for building sequential topological spaces from topological spaces, which is needed to support the topological case of $k = 2$. Propositions 4.2 and 4.3 are also standard for the topological cases of $k = 0, 2, 3$, but they seem new, nontrivial, and relevant for the nontopological case of $k = 1$. In fact, the emphasis of the whole paper, especially the emphasis of the main result in Theorem 5.1, is on the nontopological case of $k = 1$, which is here brought together with the other topological cases. In particular, Theorem 5.1 answers the question posed in Remark 5.1: *the set of weakly l-essentially supercyclic vectors is dense in the norm topology if it is not empty, even if there is no supercyclic vector*,

$$\varnothing = Y_0 \subset Y_1 \neq \varnothing \quad \Longrightarrow \quad Y_1^{-0} = \mathcal{X}.$$

6. Weak Supercyclicity and Stability

An immediate consequence of Theorem 5.1 says, roughly speaking, that *if an arbitrary set of supercyclic vectors is not empty, then it is dense with respect to any notion of denseness*. This is properly stated as follows.

Corollary 6.1. *Consider the sets of strongly supercyclic, weakly l-sequentially supercyclic, weakly sequentially supercyclic, and weakly supercyclic vectors for an arbitrary normed-space operator. All these sets are dense in the normed space, regardless the notion of denseness one is considering, provided they are nonempty.*

An operator $T \in \mathcal{B}[\mathcal{X}]$ is *power bounded* if $\sup_{n\geq 0} \|T^n\| < \infty$. It is *strongly stable* if $T^n x \xrightarrow{s} 0$ for every $x \in \mathcal{X}$, and *weakly stable* if $T^n x \xrightarrow{w} 0$ for every $x \in \mathcal{X}$.

Remark 6.1. If $\{T_n\}$ is a bounded sequence of operators on a normed space $\mathcal{X}$ and $\{T_n y\}$ converges strongly to Ty for some $T \in \mathcal{B}[\mathcal{X}]$ for every y in a strongly dense subset Y of $\mathcal{X}$, then $\{T_n x\}$ converges strongly to Tx for every $x \in \mathcal{X}$. In particular, since $Y_k^{-0} = \mathcal{X}$ for each $k = 0, 1, 2, 3$ whenever $Y_k \neq \varnothing$ by Theorem 5.1, we get:

> *If $T \in \mathcal{B}[\mathcal{X}]$ is a power bounded k-supercyclic operator for an arbitrary $k = 0, 1, 2, 3$, and if $T^n y \xrightarrow{s} 0$ for every $y \in Y_k$, then T is strongly stable.*

It was proved in [1, Theorem 2.2] that *if a power bounded operator is strongly supercyclic, then it is strongly stable*. Thus the above strong stability result holds for $k = 0$ without any additional assumption. *Does any form of weak supercyclicity* (i.e., k-supercyclicity for $k = 1, 2, 3$) *imply weak stability for power bounded operators*? The question was posed and investigated in [13] and remains unanswered even if Banach-space power bounded operators are restricted to Hilbert-space contractions.

Here is the weak version of the above italicized displayed statement.

Corollary 6.2. *If $T \in \mathcal{B}[\mathcal{X}]$ is a power bounded k-supercyclic operator for an arbitrary $k = 0, 1, 2, 3$, and if $T^n y \xrightarrow{w} 0$ for every $y \in Y_k$, then T is weakly stable.*

Proof. This is a consequence of Theorem 5.1 and the following result.

Claim. If $\{T_n\}$ is a bounded sequence of operators on a normed space $\mathcal{X}$ and $\{T_n y\}$ converges weakly to Ty for some $T \in \mathcal{B}[\mathcal{X}]$ for every y in a strongly dense subset Y of $\mathcal{X}$, then $\{T_n x\}$ converges weakly to Tx for every $x \in \mathcal{X}$.

Proof. Take any $x \in \mathcal{X}$. If $Y^{-0} = \mathcal{X}$, then there exists a Y-valued sequence $\{y_m\}$ converging strongly to x, which means $\|y_m - x\| \to 0$. Suppose $T_n y \xrightarrow{w} Ty$ for every $y \in Y$, which means $f(T_n y) \to f(Ty)$ for every $f \in \mathcal{X}^*$ and every $y \in Y$, and so $|f(T_n y_m - T y_m)| \to 0$ for every m. Thus since for every $f \in \mathcal{X}^*$ and every $x \in \mathcal{X}$

$$\begin{aligned} |f\big((T_n - T)\,x)| &\leq |f((T_n - T)\,(y_m - x))| + |f((T_n - T)\,y_m)| \\ &\leq \|f\|\,(\sup_n \|T_n\| + \|T\|)\,\|y_m - x\| + |f(T_n y_m - T y_m)|, \end{aligned}$$

then we get the claimed assertion: $T_n x \xrightarrow{w} Tx$ for every $x \in \mathcal{X}$. □

Suppose the power sequence of a power bounded operator on a complex normed space is weakly stable over a set of k-supercyclic vectors Y_k. Since $Y_k^{-0} = \mathcal{X}$ for every $k = 0, 1, 2, 3$ if $Y_k \neq \varnothing$ by Theorem 5.1, the above claim ensures the stated result. $\square$

In particular, if T is power bounded, $Y_1 \neq \varnothing$, and $T^n y \xrightarrow{w} 0$ for every y in the set Y_1 of all weakly l-sequentially supercyclic vectors, then T is weakly stable.

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