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BOUNDEDLY SPACED SUBSEQUENCES AND WEAK DYNAMICS

C.S. KUBRUSLY AND P.C.M. VIEIRA

Abstract. Weak supercyclicity is related to weak stability, which leads to the question that asks whether every weakly supercyclic power bounded operator is weakly stable. This is approached here by investigating weak l-sequential supercyclicity for Hilbert-space contractions through Nagy–Foiaş–Langer decomposition, thus reducing the problem to the quest of conditions for a weakly l-sequentially supercyclic unitary operator to be weakly stable, and this is done in light of boundedly spaced subsequences.

1. Introduction

The purpose of this paper is to characterize weak supercyclicity for Hilbert-space contractions, which is shown to be equivalent to characterizing weak supercyclicity for unitary operators. This is naturally motivated by an open question that asks whether every weakly supercyclic power bounded operator is weakly stable (which in turn is naturally motivated by a result that asserts that every supercyclic power bounded operator is strongly stable). Precisely, weakly supercyclicity is investigated in light of boundedly spaced subsequences as discussed in Lemma 3.1. The main result in Theorem 4.1 characterizes weakly l-sequentially supercyclic unitary operators U that are weakly unstable in terms of boundedly spaced subsequences of the power sequence $\{U^n\}$. Remark 4.1 shows that characterizing any form of weak supercyclicity for weakly unstable unitary operators is equivalent to characterizing any form of weak supercyclicity for weakly unstable contractions after the Nagy–Foiaş–Langer decomposition.

2. Notation and Terminology

Throughout this paper $\mathcal{H}$ denotes a complex (infinite-dimensional) Hilbert space, and $\mathcal{B}[\mathcal{H}]$ denotes the Banach algebra of all operators on $\mathcal{H}$ (i.e., of all bounded linear transformations of $\mathcal{H}$ into itself). An operator $T \in \mathcal{B}[\mathcal{H}]$ is power bounded if $\sup_n \|T^n\| < \infty$ (same notation for norm in $\mathcal{H}$ and for the induced uniform norm in $\mathcal{B}[\mathcal{H}]$). An operator $U \in \mathcal{B}[\mathcal{H}]$ is unitary if $UU^* = U^*U = I$, where I stands for the identity in $\mathcal{B}[\mathcal{H}]$ and $T^* \in \mathcal{B}[\mathcal{H}]$ denotes the adjoint of an operator $T \in \mathcal{B}[\mathcal{H}]$. An operator $T \in \mathcal{B}[\mathcal{H}]$ is strongly stable or weakly stable if the $\mathcal{H}$-valued power sequence $\{T^n x\}_{n\geq 0}$ converges strongly (i.e., in the norm topology) or weakly to zero for every $x \in \mathcal{H}$. In other words, if

$$T^n x \longrightarrow 0 \quad \text{or} \quad T^n x \xrightarrow{\;w\;} 0$$

for every $x \in \mathcal{H}$, which means $\|T^n x\| \to 0$ for every $x \in \mathcal{H}$ or $\langle T^n x\,; y\rangle \to 0$ for every $x, y \in \mathcal{H}$, respectively (clearly, strong stability implies weak stability). Let

$$\mathcal{O}_T(y) = \bigcup\nolimits_{n\geq 0} T^n y = \big\{T^n y \in \mathcal{H}\colon\; n \geq 0\big\}$$

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denote the orbit of a vector $y \in \mathcal{H}$ under an operator $T \in \mathcal{B}[\mathcal{H}]$ — we write $\bigcup_{n\geq 0}T^n y$ for $\bigcup_{n\geq 0}T^n(\{y\}) = \bigcup_{n\geq 0}\{T^n y\}$. The orbit $\mathcal{O}_T(A)$ of a set $A \subseteq \mathcal{H}$ under T is likewise defined: $\mathcal{O}_T(A) = \bigcup_{n\geq 0} T^n(A)$. In particular, the orbit of the one-dimensional space spanned by y,

$$\mathcal{O}_T(\operatorname{span}\{y\}) = \bigcup_{n\geq 0}T^n(\operatorname{span}\{y\}) = \big\{\alpha T^n y \in \mathcal{H}\colon\ \alpha \in \mathbb{C},\, n \geq 0\big\},$$

is referred to as the projective orbit of a vector $y \in \mathcal{H}$ under an operator $T \in \mathcal{B}[\mathcal{H}]$. A nonzero vector $y \in \mathcal{H}$ is a *supercyclic vector* for an operator $T \in \mathcal{B}[\mathcal{H}]$ if the projective orbit of y is dense in $\mathcal{H}$ in the norm topology; that is, if

$$\mathcal{O}_T(\operatorname{span}\{y\})^- = \mathcal{H},$$

where the upper bar $^-$ stands for closure in the norm topology. Thus a nonzero $y \in \mathcal{H}$ is a supercyclic vector for T if and only if for every $x \in \mathcal{H}$ there exists a $\mathbb{C}$-valued sequence $\{\alpha_i\}_{i\geq 0}$ (which depends on x and y and consists of nonzero numbers) such that

$$\alpha_i T^{n_i} y \longrightarrow x$$

for some subsequence $\{T^{n_i}\}_{i\geq 0}$ of $\{T^n\}_{n\geq 0}$. If $T \in \mathcal{B}[\mathcal{H}]$ has a supercyclic vector, then it is a *supercyclic operator*. The weak counterpart of the above convergence criterion reads as follows. A nonzero vector $y \in \mathcal{H}$ is a *weakly l-sequentially supercyclic vector* for an operator $T \in \mathcal{B}[\mathcal{H}]$ if for every $x \in \mathcal{H}$ there exists a $\mathbb{C}$-valued sequence $\{\alpha_i\}_{i\geq 0}$ (which depends on x and y and consists of nonzero numbers) such that, for some subsequence $\{T^{n_i}\}_{i\geq 0}$ of $\{T^n\}_{n\geq 0}$,

$$\alpha_i T^{n_i} y \xrightarrow{\,w\,} x.$$

An operator T in $\mathcal{B}[\mathcal{H}]$ is *weakly l-sequentially supercyclic* if it has a weakly l-sequentially supercyclic vector. An equivalent definition reads as follows. The weak limit set of $\mathcal{O}_T(\operatorname{span}\{y\})$ is the set consisting of all weak limits of weakly convergent $\mathcal{O}_T(\operatorname{span}\{y\})$-valued sequences, and a operator $T \in \mathcal{B}[\mathcal{H}]$ is weakly l-sequentially supercyclic if there exists a vector $y \in \mathcal{H}$ (called a weakly l-sequentially supercyclic vector for T) for which the weak limit set of $\mathcal{O}_T(\operatorname{span}\{y\})$ is equal to $\mathcal{H}$.

Several forms of weak supercyclicity, including weak l-sequential supercyclicity, have recently been investigated in [13, 14, 2, 12, 15, 8, 9, 10]. An operator $T \in \mathcal{B}[\mathcal{H}]$ is *weakly supercyclic* if there exists a vector $y \in \mathcal{H}$ (called a *weakly supercyclic vector* for T) such that the projective orbit $\mathcal{O}_T(\operatorname{span}\{y\})$ is weakly dense in $\mathcal{H}$ (i.e., dense in the weak topology of $\mathcal{H}$). A set $A \subseteq \mathcal{H}$ is weakly sequentially closed if every A-valued weakly convergent sequence has its limit in it; and the weak sequential closure of A is the smallest weakly sequentially closed set (i.e., the intersection of all weakly sequentially closed sets) including A. An operator $T \in \mathcal{B}[\mathcal{H}]$ is *weakly sequentially supercyclic* if there exists a vector $y \in \mathcal{H}$ (called a *weakly sequentially supercyclic vector* for T) for which the weak sequential closure of $\mathcal{O}_T(\operatorname{span}\{y\})$ is equal to $\mathcal{H}$. Observe that

$$\text{SUPERCYCLICITY} \Longrightarrow \begin{array}{c}\text{WEAK L-SEQUENTIAL}\\ \text{SUPERCYCLICITY}\end{array} \Longrightarrow \begin{array}{c}\text{WEAK SEQUENTIAL}\\ \text{SUPERCYCLICITY}\end{array} \Longrightarrow \begin{array}{c}\text{WEAK}\\ \text{SUPERCYCLICITY}\end{array},$$

and the reverse implications fail (see, e.g., [15, pp.38,39], [3, pp.259,260]).

The notion of weak l-sequential supercyclicity was introduced explictly in [4] and implicitly in [2], and investigated in [15] who introduced a terminology similar to the one adopted here (we use the letter "l" for "limit" in stead of the numeral "1" used in [15]). Supercyclicity for an operator T implies weak supercyclicity, which in turn implies the operator T acts on a separable space (see, e.g., [9, Section 3]), and so separability for $\mathcal{H}$ is a consequence of any form of supercyclicity for T, including weak l-sequential supercyclicity.

3. An Auxiliary Result

Let $\{n\}_{n\geq 0}$ denote the self-indexing of the set of all nonnegative integers $\mathbb{N}_0$ e-quipped with the natural order. A subsequence (in fact, a subset) $\{n_k\}_{k\geq 0}$ of $\{n\}_{n\geq 0}$ is of *bounded increments* (or has *bounded gaps*) if $\sup_{k\geq 0}(n_{k+1}-n_k)<\infty$. (Integer sequences of bounded increments have been used in [11] towards weak stability.) We say a subsequence $\{A_{n_k}\}$ of any sequence $\{A_n\}$ is *boundedly spaced* if it is indexed by a subsequence of bounded increments (i.e., $\{A_{n_k}\}$ is boundedly spaced if $\sup_k\,(n_{k+1}-n_k)<\infty$).

A vector $y\in\mathcal{H}$ is a *collapsing vector* for an operator $T\in\mathcal{B}[\mathcal{H}]$ if the orbit of y under T meets the origin (i.e, if $T^ny=0$ for some $n\geq 1$). Otherwise we say y is a *noncollapsing vector* for T (i.e., if $T^ny\neq 0$ for every $n\geq 0$). If there exists a collapsing vector for an operator, then we say the operator *has a collapsing orbit*. If there exists a noncollapsing vector for an operator, then we say the operator has a *noncollapsing orbit*. It is clear that a collapsing orbit is a finite set, and so a weakly supercyclic operator has a noncollapsing orbit. Moreover, T has a non-collapsing orbit if and only if its adjoint T^* has. (Indeed, every vector is collapsing for T if and only if every vector is collapsing for T^*; that is, for every $y\in\mathcal{H}$ there exists an $n\geq 1$ such that $T^ny=0$ if and only if $\langle T^ny\,;z\rangle=0$ for every $y,z\in\mathcal{H}$ for some $n\geq 1$, which means $\langle y\,;T^{*n}z\rangle=0$ for every $y,z\in\mathcal{H}$ for some $n\geq 1$, which is equivalent to saying $T^{*n}z=0$ for every $z\in\mathcal{H}$ for some $n\geq 1$.)

Lemma 3.1. *Take an arbitrary operator $T\in\mathcal{B}[\mathcal{H}]$ and an arbitrary nonzero vector $x\in\mathcal{H}$. Let $\{T^{n_k}\}$ be any subsequence of the power sequence $\{T^n\}$ and let P_i and P_i' for $i=1,2$ stand for the following properties.*

P_1: $T^nx\xrightarrow{w}0$,
P_1': $T^{n_k}x\xrightarrow{w}0$,
P_2: $\liminf_n|\langle T^nx\,;z\rangle|>0$ *for some* $z\in\mathcal{H}$,
P_2': $\liminf_k|\langle T^{n_k}x\,;z\rangle|>0$ *for some* $z\in\mathcal{H}$.

Moreover, if T has a noncollapsing orbit, then consider the following additional properties which hold whenever T is a power bounded operator.

P_3: $\limsup_n|\langle T^nx\,;z\rangle|<\gamma\,\|x\|\,\|z\|$ *for some* $\gamma>0$ *for every noncollapsing vector z for T^*,*
P_3': $\limsup_k|\langle T^{n_k}x\,;z\rangle|<\gamma'\|x\|\,\|z\|$ *for some* $\gamma'>0$ *for every noncollapsing vector z for T^*,*

(*where γ and γ' may depend on x*) *and $\gamma'=\gamma$ if T is a contraction.*

Claim: *For each $i=1,2,3$ the assertions bellow are pairwise equivalent.*

(a) *P_i holds.*
(b) *P_i' holds for some boundedly spaced subsequence $\{T^{n_k}\}$ of $\{T^n\}$.*
(c) *P_i' holds for every boundedly spaced subsequence $\{T^{n_k}\}$ of $\{T^n\}$.*

Proof. We split the proof into 2 parts.

Part 1. *Let $\{\alpha_{n_k}\}$ be a boundedly spaced* (infinite) *subsequence of an* (infinite) *sequence of complex numbers $\{\alpha_n\}$, let α be an arbitrary complex number, and let β and γ be arbitrary nonnegative and positive real numbers, respectively.*

(a$_1$) $\alpha_{n_k+j} \to \alpha$ *as* $k \to \infty$ *for every* $j \ge 0$ *if and only if* $\alpha_n \to \alpha$ *as* $n \to \infty$.

(b$_1$) $\liminf_k |\alpha_{n_k+j} - \alpha| > \beta$ *for every* $j \ge 0$ *if and only if* $\liminf_n |\alpha_n - \alpha| > \beta$.

(c$_1$) $\limsup_k |\alpha_{n_k+j} - \alpha| < \gamma$ *for every* $j \ge 0$ *if and only if* $\limsup_n |\alpha_n - \alpha| < \gamma$.

Proof of Part 1. This is an easy consequence of the elementary fact that if $\{n_k\}_{k\ge0}$ is of bounded increments with $M = \sup_k(n_{k+1} - n_k)$, then $\bigcup_{j\in[0,M]}\{n_k + j\}_{k\ge0} = \{n\}_{n\ge0} = \mathbb{N}_0$. $\square$

Part 2. *Take any $x \in \mathcal{H}$ and let $\{T^{n_k}\}$ be a boundedly spaced subsequence of $\{T^n\}$.*

(a$_2$) $\langle T^{n_k}x\,;z\rangle \to 0$ *for every* $z \in \mathcal{H}$ *if and only if* $\langle T^n x\,;z\rangle \to 0$ *for every* $z \in \mathcal{H}$ (i.e., $T^{n_k}x \xrightarrow{w} 0$ *if and only if* $T^n x \xrightarrow{w} 0$).

(b$_2$) $\liminf_k |\langle T^{n_k}x\,;z\rangle| > 0$ *for some* $z \in \mathcal{H}$ *if and only if* $\liminf_n |\langle T^n x\,;z\rangle| > 0$ *for some* $z \in \mathcal{H}$.

If T is power bounded and has a noncollapsing orbit, then

(c$_2$) $\limsup_k |\langle T^{n_k}x\,;z\rangle| < \gamma\|x\|\|z\|$ *for every noncollapsing vector z for T^* if and only if* $\limsup_n |\langle T^n x\,;z\rangle| < \gamma'\|x\|\|z\|$ *for every noncollapsing vector z for T^**,

for some $\gamma, \gamma' > 0$ where $\gamma' = \gamma$ if T is a contraction.

Proof of Part 2. Let $\{T^{n_k}\}$ be a boundedly spaced subsequence of $\{T^n\}$ so that $\{\langle T^{n_k}x\,;z\rangle\}$ is a boundedly spaced subsequence of $\{\langle T^n x\,;z\rangle\}$ for $x, z\in\mathcal{H}$. Recall: $T^{m+n} = T^m T^n$ for every $m, n \ge 0$. Take $x, y \in \mathcal{H}$ arbitrary so that for each $j \ge 0$

$$\langle T^{n_k+j}x\,;y\rangle = \langle T^{n_k}x\,;T^{*j}y\rangle.$$

Let x be an arbitrary vector in $\mathcal{H}$.

(a$_2$) Suppose $\langle T^{n_k}x\,;z\rangle \to 0$ for every $z \in \mathcal{H}$. In particular, $\langle T^{n_k}x\,;z\rangle \to 0$ for every $z \in \mathcal{O}_{T^*}(y)$ for every $y \in \mathcal{H}$. This means $\langle T^{n_k}x\,;T^{*j}y\rangle \to 0$ for every $j \ge 0$ and every $y \in \mathcal{H}$; that is, $\langle T^{n_k+j}x\,;y\rangle \to 0$ for every $j \ge 0$ and every $y \in \mathcal{H}$. This is equivalent to $\langle T^n x\,;y\rangle \to 0$ for every $y \in \mathcal{H}$ by (a$_1$) in Part 1. The converse is trivial.

(b$_2$) We prove (b$_2$) contrapositively. Suppose $\liminf_k |\langle T^{n_k}x\,;z\rangle| = 0$ for every z in $\mathcal{H}$. Then $\liminf_k |\langle T^{n_k}x\,;z\rangle| = 0$ for every z in $\mathcal{O}_{T^*}(y)$ and every y in $\mathcal{H}$, which means $\liminf_k |\langle T^{n_k+j}x\,;y\rangle| = 0$ for every $j \ge 0$ and every $y \in \mathcal{H}$. By (b$_1$) in Part 1 this is equivalent to $\liminf_n |\langle T^n x\,;y\rangle| = 0$ for every $y \in \mathcal{H}$. The converse is trivial.

(c$_2$) Let $T \in \mathcal{B}[\mathcal{H}]$ be a power bounded operator with a noncollapsing orbit (and so its adjoint has a noncollapsing orbit as well). Take an arbitrary positive number γ. Suppose $\limsup_k |\langle T^{n_k}x\,;y\rangle| < \gamma\|x\|\|y\|$ for every noncollapsing vector $y \in \mathcal{H}$ for T^* (i.e., for every vector y such that $T^{*j}y \ne 0$ for every $j \ge 0$ — such vectors do exist since T^* has a noncollapsing orbit). Take an arbitrary noncollapsing vector y for T^* and an arbitrary $z \in \mathcal{O}_{T^*}(y)$ so that $0 \ne z = T^{*j}y$ for some $j \ge 0$ and hence $T^{*i}z = T^{*(j+i)}y \ne 0$ for every $i \ge 0$, which means z is a noncollapsing vector for T^* as well. Thus $\limsup_k |\langle T^{n_k}x\,;z\rangle| < \gamma\|x\|\|z\|$ for every $z \in \mathcal{O}_{T^*}(y)$ and every noncollapsing vector y for T^*. This implies $\limsup_k |\langle T^{n_k+j}x\,;y\rangle| = \limsup_k |\langle T^{n_k}x\,;T^{*j}y\rangle| < \gamma\|x\|\|T^{*j}y\| \le \gamma'\|x\|\|y\|$ for every $j \ge 0$ whenever y is an arbitrary noncollapsing vector for T^* (since $\|T^{*j}y\| \ne 0$ every $j \ge 0$) with $\gamma' \ge \gamma \sup_n \|T^n\|$ (since T is power bounded — if T is a contraction so that $\sup_n \|T^n\| \leq 1$ then we may take $\gamma' = \gamma$). Therefore $\limsup_n |\langle T^n x\,; y\rangle| < \gamma'\|x\|\|y\|$ for every noncollapsing vector y for T^* by (c_1) in Part 1. Again the converse is trivial. $\square$

Thus the claimed equivalences in (a), (b) and (c) follow from Part 2 since $\{T^{n_k}\}$ was taken to be an arbitrary boundedly spaced subsequence of $\{T^n\}$. $\square$

Lemma 3.1 is naturally extended to normed spaces by replacing inner product with dual pairs, and Hilbert-space adjoints with normed-space adjoints.

4. Weak Supercyclicity and Weak stability

It was shown in [1, Theorem 2.2] that *if a power bounded operator on a Banach space is supercyclic, then it is strongly stable*. Such a result naturally prompts the question: *does weak supercyclicity imply weak stability for power bounded operators*? The question was posed and investigated in [9], and remains unanswered even if Banach-space power bounded operators are restricted to Hilbert-space contractions, where the problem is equivalently stated for unitary operators (see Remark 4.1 below), and also if weak supercyclicity is strengthened to weak l-sequential supercyclicity: *does there exist a weakly unstable and weakly l-sequentially supercyclic unitary operator*?

No unitary operator is supercyclic (reason: no Banach-space isometry is supercyclic [1, Proof of Theorem 2.1]) and, in addition to this, no Hilbert-space hyponormal operator is supercyclic [5, Theorem 3.1]. But the existence of weakly supercyclic (weakly l-sequentially supercyclic, actually) unitary operators was shown in [2, Example 3.6, pp.10,12] (also see [15, Question 1]), and the existence of weakly supercyclic unitary operators that are not l-sequentially supercyclic was shown in [15, Proposition 1.1 and Theorem 1.2]. Next we consider in Theorem 4.1 the case of weakly l-sequentially supercyclic unitary operators U that are not weakly stable in light of boundedly spaced subsequences of the power sequence $\{U^n\}$, whose proof applies [9, Theorem 6.2] and Lemma 3.1.

Theorem 4.1. *If a unitary operator U on a Hilbert space $\mathcal{H}$ is weakly l-sequentially supercyclic but not weakly stable, then there exists a weakly l-sequentially supercyclic vector $y \in \mathcal{H}$ for U such that $U^n y \overset{w}{\not\longrightarrow} 0$. Moreover, for every weakly l-sequentially supercyclic vector $y \in \mathcal{H}$ for U either*

(a) $\liminf_n |\langle U^{n_k} y\,; z\rangle| = 0$ *for every (equivalently, for some) boundedly spaced subsequence $\{U^{n_k}\}$ of $\{U^n\}$, or*

(b) $\limsup_k |\langle U^{n_k} y\,; z\rangle| < \|z\|\,\|y\|$ *for some subsequence $\{U^{n_k}\}$ of $\{U^n\}$.*

Furthermore, if (a) *fails and if the subsequence in* (b) *is boundedly spaced, then*

$$0 < \liminf_n |\langle U^n y\,; z\rangle| < \limsup_n |\langle U^n y\,; z\rangle| < \|z\|\,\|y\|.$$

Proof. We begin with a definition. A normed space $\mathcal{X}$ is said to be of *type 1* if strong convergence (i.e., convergence in the norm topology) for an arbitrary $\mathcal{X}$-valued sequence $\{x_k\}$ coincides with weak convergence plus convergence of the norm sequence $\{\|x_k\|\}$ (i.e., $x_k \longrightarrow x \iff \big\{x_k \xrightarrow{w} x$ and $\|x_k\| \to \|x\|\big\}$ — also called *Radon–Riesz space* and the *Radon–Riesz property*, respectively). Every Hilbert space is a Banach space of type 1.

Part 1. *If a power bounded operator T on a type 1 normed space $\mathcal{X}$ is weakly l-sequentially supercyclic, then either*

(i) *T is weakly stable, or*

(ii) *the set*

$$M_T = \big\{y \in \mathcal{X}\colon\ y \text{ is a weakly l-sequentially supercyclic vector for } T \text{ such that } T^n y \overset{w}{\not\longrightarrow} 0\big\}$$

is nonempty, and if y is any vector in M_T, then for every nonzero $f \in \mathcal{X}^*$ such that $f(T^n y) \not\to 0$ either

(a$'$) $\liminf_n |f(T^n y)| = 0$, *or*

(b$'$) $\limsup_k |f(T^{n_k} y)| < \|f\| \limsup_k \|T^{n_k} y\|$ *for some subsequence $\{T^{n_k}\}$ of $\{T^n\}$.*

(Where $\mathcal{X}^*$ stands for the dual of $\mathcal{X}$.) This is Theorem 6.2 from [9]. If a vector $y \in \mathcal{X}$ is such that $T^n y \overset{w}{\longrightarrow} 0$, then (a$'$) is tautologically satisfied for every $f \in \mathcal{X}^*$, and hence alternative (ii) can be rewritten as

(ii) *if $y \in \mathcal{X}$ is any weakly l-sequentially supercyclic vector for T, then for an arbitrary $f \in \mathcal{X}^*$ either* (a$'$) *or* (b$'$) *holds true.*

If $\mathcal{X} = \mathcal{H}$ is a Hilbert space and if T is a contraction, then $T = C \oplus U$ is uniquely a direct sum of a completely nonunitary contraction C and a unitary operator U (where any of the these direct summands may be missing) by Nagy–Foiaş–Langer decomposition for Hilbert-space contractions (see, e.g., [16, p.8] or [7, p.76]). Since every completely nonunitary contraction is weakly stable (see, e.g., [6, p.55] or [7, p.106]), the above result when restricted to contractions is equivalent to the case of plain unitary operators, and this is stated as follows. *If a unitary operator U on a Hilbert space $\mathcal{H}$ is weakly l-sequentially supercyclic, then either*

(i) *U is weakly stable, or*

(ii) *if $y \in \mathcal{H}$ is a weakly l-sequentially supercyclic vector for U, then for an arbitrary $z \in \mathcal{H}$ either*

(a$''$) $\liminf_n |\langle U^n y\,;z\rangle| = 0$, *or*

(b) $\limsup_k |\langle U^{n_k} y\,;z\rangle| < \|z\|\,\|y\|$ *for some subsequence $\{U^{n_k}\}$ of $\{U^n\}$.*

(Because the completely nonunitary part of any contraction is necessarily weakly stable; and unitary operators are isometries.)

Part 2. Suppose a unitary operator U is weakly l-sequentially supercyclic and not weakly stable. Let y be a weakly l-sequentially supercyclic vector for U. Thus either (a$''$) or (b) holds. But property (a$''$) holds if and only if (for an arbitrary $z \in \mathcal{H}$)

(a) $\liminf_k |\langle U^{n_k} y\,;z\rangle| = 0$ *for every (equivalently, for some) boundedly spaced subsequence $\{U^{n_k}\}$* of $\{U^n\}$.

Indeed, by Lemma 3.1(P_2, P_2') $\liminf_n |\langle U^n y\,;z\rangle| = 0$ for every $z \in \mathcal{H}$ if and only if $\liminf_k |\langle U^{n_k} y\,;z\rangle| = 0$ for every $z \in \mathcal{H}$ for every (equivalently, for some) boundedly spaced subsequence $\{U^{n_k}\}$ of $\{U^n\}$. Hence (a$''$) holds if and only if (a) holds. On the other hand, if (a$''$) fails, then (b) holds. Thus suppose (a$''$) fails; equivalently, suppose (a) fails. Fix an arbitrary weakly l-sequentially supercyclic vector $y \in \mathcal{H}$ for U for which (a$''$) fails so that (b) holds. Since U is a weakly l-sequentially supercyclic operator, it has a noncollapsing vector, and so has U^*. Let $z \in \mathcal{H}$ be an arbitrary noncollapsing vector for U^*. Now suppose (b) holds for some *boundedly spaced* subsequence. Then Lemma 3.1(P_3, P_3') (with $\gamma = \gamma' = 1$) ensures

$$0 < \liminf_n |\langle U^n y\,;z\rangle| \le \limsup_n |\langle U^n y\,;z\rangle| < \|z\|\,\|y\|$$

(since (a$''$) fails). If

$$\liminf_n |\langle U^n y\,;z\rangle| = \limsup_n |\langle U^n y\,;z\rangle|,$$

then $\{|\langle U^n y\,;z\rangle|\}$ converges to a positive number $\alpha(y,z)$,

$$|\langle U^n y\,;z\rangle| \to \alpha(y,z).$$

Since y is an l-sequentially weakly supercyclic vector for U, for each $x \in \mathcal{H}$ there exists a scalar sequence $\{\alpha_i(y,x)\}$ such that $\alpha_i(y,x)\,U^{n_i}y \xrightarrow{w} x$ for some subsequence $\{U^{n_i}\}_{i\ge0}$ of $\{U^n\}_{n\ge0}$, which implies

$$|\alpha_i(y,x)|\,|\langle U^{n_i}y\,;w\rangle| \to |\langle x\,;w\rangle|$$

for every $w \in \mathcal{H}$. Hence, since $|\langle U^n y\,;z\rangle| \to \alpha(y,z) \ne 0$,

$$|\alpha_i(y,x)| \to \tfrac{|\langle x\,;z\rangle|}{\alpha(y,z)}$$

for every $x \in \mathcal{H}$, which is a contradiction because $\alpha_i(y,x)$ does not depend on z and $\alpha(y,z)$ does not depend on x. (Indeed, since z was taken to be an arbitrary noncollapsing vector for U, the above limit holds for every such a z and every $x \in \mathcal{H}$, and so take $x \in \mathcal{H}$ and a pair of noncollapsing vectors z, z' for U^* such that x is orthogonal to z but not to z'.) Outcome:

$$\liminf_n |\langle U^n y\,;z\rangle| < \limsup_n |\langle U^n y\,;z\rangle|.$$

Therefore, if the subsequence in (b) is boundedly spaced, then

$$0 < \liminf_n |\langle U^n y\,;z\rangle < \limsup_n |\langle U^n y\,;z\rangle| < \|z\|\,\|y\|. \qquad \square$$

Remark 4.1. As we saw in the proof of Theorem 4.1, the Nagy–Foiaş–Langer decomposition for contractions on a Hilbert space $\mathcal{H}$ says that $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H} = \mathcal{U}^\perp \oplus \mathcal{U}$, where a contraction $T = C \oplus U$ is uniquely a direct sum of a completely nonunitary contraction $C = T|_{\mathcal{U}^\perp} \in \mathcal{B}[\mathcal{U}]$ and a unitary operator $U = T|_{\mathcal{U}} \in \mathcal{B}[\mathcal{U}]$, where C is the completely nonunitary part of T and U is the unitary part of T (any of the these parcels may be missing). Moreover, a completely nonunitary contraction is weakly stable (see, e.g., [7, pp.76,106]). Thus a Hilbert-space contraction T is not weakly stable if and only if it has a (nontrivial) unitary part U which is not weakly stable (and so $\|T\| = \|U\| = 1$). Therefore the question "*does weak l-sequential supercyclicity for a Hilbert-space contraction implies weak stability*?" is equivalent to the question "*does weak l-sequential supercyclicity for a unitary operator implies weak stability*?"

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Applied Mathematics Department, Federal University, Rio de Janeiro, RJ, Brazil
E-mail address: carloskubrusly@gmail.com

National Laboratory for Scientific Computation, Petrópolis, RJ, Brazil
E-mail address: paulocm@lncc.br