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SINGULAR-CONTINUOUS UNITARIES AND WEAK DYNAMICS

CARLOS S. KUBRUSLY

Abstract. Weak supercyclicity is linked to weak stability. This paper answers a pair of open questions along this line. It is shown the existence of singular-continuous unitary operators that are weakly stable, and also of singular-continuous unitary operators that are weakly unstable. The main result shows that every weakly l-sequentially supercyclic unitary operator is singular-continuous. A condition for weak l-sequential supercyclicity to imply weak stability is given.

1. Introduction

The purpose of this paper is to investigate weak stability and weak supercyclicity for unitary operators on Hilbert spaces. First it is answered a question on weak stability for unitary operators: is every weakly stable unitary operator absolutely continuous? Equivalently, is every singular-continuous unitary operator weakly unstable? It is shown the existence of singular-continuous unitary operators that are (i) weakly unstable, and also that are (ii) weakly stable (Propositions 3.2 and 3.3). Then it is shown when every weakly supercyclic unitary operator is singular-continuous (Theorem 4.2 and Question 5.1). Is every weakly supercyclic unitary operator weakly stable? It is given a sufficient condition for a weakly l-sequentially supercyclic unitary operator to be weakly stable (Theorem 5.1).

2. Notation and Terminology

Throughout this paper $\mathcal{H}$ will denote an infinite-dimensional complex Hilbert space. By an operator on $\mathcal{H}$ we mean a linear bounded (i.e., continuous) transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ denote the Banach algebra of all operators on $\mathcal{H}$, and let $T^* \in \mathcal{B}[\mathcal{H}]$ stand for the adjoint of $T \in \mathcal{B}[\mathcal{H}]$. An operator $U \in \mathcal{B}[\mathcal{H}]$ is unitary if $UU^* = U^*U = I$, where I stands for the identity in $\mathcal{B}[\mathcal{H}]$. An operator $T \in \mathcal{B}[\mathcal{H}]$ is weakly stable (notation: $T^n \xrightarrow{w} O$) if the $\mathcal{H}$-valued power sequence $\{T^n x\}_{n\geq 0}$ converges weakly to zero for every $x \in \mathcal{H}$. In other words, if

$$T^n x \xrightarrow{w} 0,$$

which means that $\langle T^n x\,; y\rangle \to 0$ for every $x, y \in \mathcal{H}$; equivalently (since $\mathcal{H}$ is complex), $\langle T^n x\,; x\rangle \to 0$ for every $x \in \mathcal{H}$. Let

$$\mathcal{O}_T(y) = \bigcup\nolimits_{n\geq 0} T^n y$$

be the orbit of a vector $y \in \mathcal{H}$ under an operator $T \in \mathcal{B}[\mathcal{H}]$ (i.e., $\mathcal{O}_T(y)$ is the set $\{T^n y \in \mathcal{H}\colon n \in \mathbb{N}_0\}$, $\mathbb{N}_0$ denoting the nonnegative integers). The orbit $\mathcal{O}_T(A)$ of a set $A \subseteq \mathcal{H}$ under T is likewise defined: $\mathcal{O}_T(A) = \bigcup_{n\geq 0} T^n(A) = \bigcup_{y\in A} \mathcal{O}_T(y)$. Let $\operatorname{span} A$ stand for the linear span of a set $A \subseteq \mathcal{H}$, and consider the projective orbit of a vector $y \in \mathcal{H}$ under an operator $T \in \mathcal{B}[\mathcal{H}]$,

$$\mathcal{O}_T(\operatorname{span}\{y\}) = \bigcup\nolimits_{n\geq 0} T^n(\operatorname{span}\{y\})$$

(i.e., $\mathcal{O}_T(\operatorname{span}\{y\})$ is the set $\{\alpha T^n y \in \mathcal{H}\colon \alpha \in \mathbb{C},\, n \in \mathbb{N}_0\}$). Let A^- and A^{-w} stand for the closure and the weak closure of a set $A \subseteq \mathcal{H}$: the closure of A in the norm

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topology and in the weak topology. A nonzero vector $y \in \mathcal{H}$ is a *weakly supercyclic vector* for an operator $T \in \mathcal{B}[\mathcal{H}]$ if

$$\mathcal{O}_T(\mathrm{span}\{y\})^{-w} = \mathcal{H}.$$

A set $A \subseteq \mathcal{H}$ is weakly sequentially closed if every A-valued weakly convergent sequence has its limit in A, and the weak sequential closure A^{-sw} of A is the smallest weakly sequentially closed subset of $\mathcal{H}$ including A, and A is weakly sequentially dense if $A^{-sw} = \mathcal{H}$. The weak limit set A^{-lw} of a set $A \subseteq \mathcal{H}$ is the set of all weak limits of weakly convergent A-valued sequences, and a set A is weakly l-sequentially dense if $A^{-lw} = \mathcal{H}$. In general the inclusions $A^{-lw} \subseteq A^{-sw} \subseteq A^{-w}$ may be proper. A nonzero vector $y \in \mathcal{H}$ is a *weakly sequentially supercyclic* or a *weakly l-sequentially supercyclic* for an operator $T \in \mathcal{B}[\mathcal{H}]$ if

$$\mathcal{O}_T(\mathrm{span}\{y\})^{-sw} = \mathcal{H} \qquad \text{or} \qquad \mathcal{O}_T(\mathrm{span}\{y\})^{-lw} = \mathcal{H},$$

respectively. An operator $T \in \mathcal{B}[\mathcal{H}]$ *is weakly supercyclic*, or *weakly sequentially supercyclic*, or *weakly l-sequentially supercyclic* if it has a weakly supercyclic, or a weakly sequentially supercyclic, or a weakly l-sequentially supercyclic vector, respectively, so that (see, e.g., [19, pp.38,39], [3, pp.259,260])

$$\begin{matrix}\text{weak l-sequential} \\ \text{supercyclicity}\end{matrix} \implies \begin{matrix}\text{weak sequential} \\ \text{supercyclicity}\end{matrix} \implies \begin{matrix}\text{weak} \\ \text{supercyclicity}\end{matrix}.$$

Thus a vector $y \in \mathcal{H}$ is weakly l-sequentially supercyclic for an operator $T \in \mathcal{B}[\mathcal{H}]$ (i.e., $\mathcal{O}_T(\mathrm{span}\{y\})^{-lw} = \mathcal{H}$) if and only if for every $x \in \mathcal{H}$ there exists a $\mathbb{C}$-valued sequence $\{\alpha_i\}_{i\geq 0}$ (that depends on x and y, and consists of nonzero numbers) such that, for some subsequence $\{T^{n_i}\}_{i\geq 0}$ of $\{T^n\}_{n\geq 0}$,

$$\alpha_i T^{n_i} y \xrightarrow{w} x.$$

Weak l-sequential supercyclicity was introduced [4], implicitly explored [2], and detailedly examined in [19], form which we have borrowed the terminology.

It is worth noticing that if T has a weakly supercyclic vector, then $\mathcal{H}$ is separable (so that separability is not an assumption, but a consequence of cyclicity). Indeed, the orbit $\mathcal{O}_T(y)$ is a countable set, and so if it spans $\mathcal{H}$, that is, if $\big(\mathrm{span}\,\mathcal{O}_T(y)\big)^- = \big(\mathrm{span}\bigcup_n T^n y\big)^- = \bigvee_n\{T^n y\} = \mathcal{H}$, then $\mathcal{H}$ is separable. In this case y is said to be a *cyclic vector* for T, and T is a *cyclic operator*. If $\big(\mathrm{span}\,\mathcal{O}_T(y)\big)^{-w} = \mathcal{H}$, then y is said to be a *weakly cyclic vector* for T, and T is a *weakly cyclic operator*. If a set A is convex, then $A^- = A^{-w}$ (see, e.g., [6, Theorem V.1.4]), and so cyclicity coincides with weak cyclicity as span is convex. Outcome: weak supercyclicity implies weak cyclicity, which is equivalent to (plain) cyclicity, which implies that $\mathcal{H}$ is separable.

3. Singular-Continuous Unitaries and Weak Stability

Let λ be a σ-finite measure on the σ-algebra $\mathcal{A}_{\mathbb{T}}$ of Borel subsets of the unit circle $\mathbb{T}$ (about the origin in the complex plane). If μ is a σ-finite measure on $\mathcal{A}_{\mathbb{T}}$, then it has a unique decomposition $\mu = \mu_a + \mu_s$ and a unique decomposition $\mu = \mu_c + \mu_d$, where the σ-finite measures μ_a, μ_s, μ_c, and μ_d are absolutely continuous, singular, continuous, and discrete, respectively, with respect to λ. Thus $\mu_s = \mu_{sc} + \mu_{sd}$, where μ_{sc} and μ_{sd} are singular-continuous and singular-discrete, respectively, with respect to λ. Therefore,

$$\mu = \mu_a + \mu_{sc} + \mu_{sd}.$$

Since discrete implies singular, singular-discrete coincides with discrete (i.e., $\mu_d = \mu_{sd}$). This is the well-known Lebesgue Decomposition Theorem. A measure μ is pure if either $\mu = \mu_a$, or $\mu = \mu_{sc}$, or $\mu = \mu_{sd}$.

A unitary operator is *absolutely continuous, singular, continuous, discrete, singular-continuous, or singular-discrete* if its scalar spectral measure is absolutely continuous, singular, continuous, discrete, singular-continuous, or singular-discrete, respectively, with respect to the normalized Lebesgue measure on $\mathcal{A}_\mathbb{T}$ (i.e., with respect to the Lebesgue measure on $\mathcal{A}_\mathbb{T}$ such that $\lambda(\mathbb{T}) = 1$). Thus by the Lebesgue Decomposition Theorem and by the Spectral Theorem every unitary operator U on a Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_s$ is uniquely decomposed as the direct sum $U = U_a \oplus U_s$ of an absolutely continuous unitary U_a on $\mathcal{H}_a$ and a singular unitary U_s on $\mathcal{H}_s$, which is also uniquely decomposed as the direct sum $U = U_c \oplus U_d$ (on the same Hilbert space, now decomposed as $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_d$) of a continuous unitary U_c on $\mathcal{H}_c$ and a discrete unitary U_d on $\mathcal{H}_d$. Thus $U_s = U_{sc} \oplus U_{sd}$ on $\mathcal{H}_s = \mathcal{H}_{sc} \oplus \mathcal{H}_{sd}$, where U_{sc} on $\mathcal{H}_{sc}$ is a singular-continuous unitary and U_{sd} on $\mathcal{H}_{sd}$ is a singular-discrete unitary. Therefore, a unitary operator U is the direct sum of an absolutely continuous unitary U_a, a singular-continuous unitary U_{sc}, and a singular-discrete unitary U_{sd} (where any direct summand may be missing),

$$U = U_a \oplus U_{sc} \oplus U_{sd},$$

with respect to the decomposition of the Hilbert space $\mathcal{H}$ into $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{sd}$. Again, since discrete implies singular, a singular-discrete unitary coincides with a discrete unitary (i.e., $U_d = U_{sd}$ on $\mathcal{H}_d = \mathcal{H}_{sd}$). A unitary operator U is pure if either $U = U_a$, or $U = U_{sc}$, or $U = U_{sd}$. On a finite-dimensional space unitaries are discrete (i.e., singular-discrete — reason: on a finite-dimensional space spectra are finite).

Consider a bilateral shift (of any multiplicity) or a direct summand of a bilateral shift, acting on any Hilbert space. A unitary operator is absolutely continuous if and only if it is a bilateral shift or a direct summand of a bilateral shift (see, e.g., [8, pp.55,56]). It can be readily verified that a bilateral shift is a weakly stable unitary operator, and so is any direct summand of it. Hence U_a is weakly stable:

$$U_a^n \xrightarrow{w} O.$$

Another way to see this goes as follows. An absolutely continuous unitary operator is similar to a completely nonunitary C_{11}-contraction [12, Lemma 2]. Thus every direct summand of a bilateral shift (itself included) is similar to a completely nonunitary contraction and hence weakly stable (reason: similarity preserves stability, and a completely nonunitary contraction is weakly stable by the Foguel decomposition; see, e.g., [14, Corollary 7.4]). On the other hand, a singular-discrete unitary operator is weakly unstable. Indeed, if the scalar spectral measure of a unitary U is discrete, then it has a countable support, and so U has a countable spectrum, and hence (by the Spectral Theorem) U is unitarily equivalent to a unitary diagonal, and therefore it has eigenvalues (in the unit circle) which ensures that U is not weakly stable:

$$U_{sd}^n \not\xrightarrow{w} O.$$

Thus a weakly stable unitary operator is either absolutely continuous or singular-continuous (or a direct sum of them). This is summarized in the next proposition.

Proposition 3.1. *If a unitary operator is weakly stable, then it is continuous.*

Remark 3.1. Every unitary operator U can be decomposed as $U = S \oplus W$, where S is a bilateral shift and W is a reductive unitary operator (see, e.g., [8, p.18] — an operator is reducible if it has a nontrivial reducing subspace, and reductive if all its invariant subspaces are reducing). Since a bilateral shift is nonreductive, a unitary operator is nonreductive if and only if it has a bilateral shift as a direct summand. (Equivalently, a unitary U is nonreductive if and only if there exist an orthonormal family $\{e_k\}$ indexed by the integers $\mathbb{Z}$ which is shifted by U — i.e., $Ue_k = e_{k+1}$ for every $k \in \mathbb{Z}$ — see, e.g., [7, Theorem 13.14].) Outcome: if a unitary U is nonreductive, then its spectrum is the unit circle: $\sigma(U) = \mathbb{T}$ (since $\mathbb{T} = \sigma(S) \subseteq \sigma(U) \subseteq \mathbb{T}$). The converse however does not hold: there exist reductive unitary operators whose spectra are the whole unit circle $\mathbb{T}$. The classical example comprises a unitary diagonal on ℓ^2_+. Set $W = \operatorname{diag}(\{\gamma_j\}_{j\geq 0})$ in $\mathcal{B}[\ell^2_+]$, where $\gamma_j = e^{2\pi i\alpha_j}$ for each j and $\{\alpha_j\}$ is a distinct enumeration of all rationals in $(0,1]$, so that $\sigma(W) = \mathbb{T}$ (since $\{\alpha_j\}$ is dense in $[0,1]$) but the unitary W is reductive [7, Example 13.5]. This also gives an example of a singular-discrete unitary whose spectrum is the whole unit circle $\mathbb{T}$.

Let $L^2(\mathbb{T},\mu) = L^2(\mathbb{T},\mu;\mathbb{C})$ denote the Hilbert space of all (equivalence classes of) scalar-valued functions $f\colon \mathbb{T} \to \mathbb{C}$ on the unit circle that are square-integrable with respect to a measure μ on $\mathcal{A}_\mathbb{T}$. Suppose μ is singular-discrete (i.e., discrete) with respect to the Lebesgue measure on $\mathcal{A}_\mathbb{T}$ so that μ is concentrated on a countable set, and therefore on a set of Lebesgue measure zero. In this case, $L^2(\mathbb{T},\mu)$ is identified with (i.e., it is unitarily equivalent to) $\ell^2_+ = L^2(\mathbb{N},\mu') = L^2(\mathbb{N},\mu';\mathbb{C})$, where μ' is the counting measure on the power set $\wp(\mathbb{N})$ and ℓ^2_+ is the Hilbert space of all square-summable scalar-valued sequences $x\colon \mathbb{N} \to \mathbb{C}$ (denoted by $\{x_n\}_{n\geq 1}$).

Proposition 3.2. *There exist weakly unstable singular-continuous unitary operators.*

Proof. Let λ be the normalized Lebesgue measure on $\mathcal{A}_\mathbb{T}$ and take the unitary operator U on the Hilbert space $L^2(\mathbb{T},\lambda)$ given by

$$(Uf)(z) = z^q f(\gamma z) \quad \lambda\text{-a.e. for } z \in \mathbb{T}$$

for every $f \in L^2(\mathbb{T},\lambda)$, where q is a sufficiently small nonzero rational (for instance, $0 < |q| \leq 1/12$) and γ is an irrational in $\mathbb{T}$ (i.e., $\gamma = e^{2\pi i\alpha}$ with $\alpha \in (0,1]$ irrational). It has been verified in [5] that U is not singular-discrete and that $\{U^n\}$ has a subsequence, say $\{U^{n_k}\}$, such that $0 < \inf_k |\langle U^{n_k}1\,;1\rangle|$. Thus U is not weakly stable, and so U is not absolutely continuous as well. Nevertheless, the spectral measure of U is pure (see, e.g., [10]), so that U must be a singular-continuous unitary. □

Take the unit circle $\mathbb{T}$, an arbitrary $z \in \mathbb{T}$, and an arbitrary integer $k \in \mathbb{Z}$, so that $z^k = e^{ik\theta} = e^{2\pi ikt}$ for $\theta \in (0,2\pi]$ and $t \in (0,1]$. Let μ be a (positive) measure on the σ-algebra $\mathcal{A}_\mathbb{T}$ of Borel subsets of $\mathbb{T}$, and consider the Hilbert space $L^2(\mathbb{T},\mu)$. Let $U_\mu\colon L^2(\mathbb{T},\mu) \to L^2(\mathbb{T},\mu)$ be the multiplication operator induced by the identity function, also called the position operator,

$$(U_\mu f)(z) = zf(z) \quad \mu\text{-a.e. for } z \in \mathbb{T},$$

for every $f \in L^2(\mathbb{T},\mu)$, which is unitary (in fact, $U_\mu^*\colon L^2(\mathbb{T},\mu) \to L^2(\mathbb{T},\mu)$ is such that $(U_\mu^* f)(z) = \overline{z}f(z)$, and hence $U_\mu^* U_\mu = U_\mu U_\mu^* = I$, the identity operator on $L^2(\mathbb{T},\mu)$), and henceforward refereed to as *the multiplication operator*. If the measure μ is finite, then it is a spectral measure for U_μ (see, e.g., [15, Remark to the proof of Lemma 4.7]) and so, in this case, the unitary U_μ is absolutely continuous, singular-continuous, or singular-discrete if and only if the finite measure μ is absolutely continuous, singular-continuous, or singular-discrete (with respect to the Lebesgue measure on $A_{\mathbb{T}}$). Let λ be any absolutely continuous finite measure on $\mathcal{A}_{\mathbb{T}}$, so that the multiplication operator $U_\lambda\colon L^2(\mathbb{T},\lambda)\to L^2(\mathbb{T},\lambda)$ is (purely) absolutely continuous, thus a bilateral shift or a direct summand of it. If λ is the normalized Lebesgue measure on $\mathcal{A}_{\mathbb{T}}$ (so that $\lambda(\mathbb{T})=1$, and hence λ is a probability measure on $\mathcal{A}_{\mathbb{T}}$), then the Hilbert space $L^2(\mathbb{T},\lambda)$ is separable and $\{z^k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for it, and U_λ shifts the orthonormal basis $\{z^k\}_{k\in\mathbb{Z}}$.

In light of Proposition 3.2 it is reasonable to ask whether the absolutely continuous unitaries (i.e., bilateral shifts and their direct summands) are the only weakly stable unitary operators. This has been raised in [13] and leads to the question: is every singular-continuous unitary operator weakly unstable? In other words, are the weakly stable unitary operators precisely the absolutely continuous ones? Equivalently, is a unitary operator weakly stable if and only if it is a bilateral shift or a direct summand of a bilateral shift? The answer is 'no', as we will see in Proposition 3.3 below, and it seems to have been folklore for a long time. Observe that questions about unitary operators can be reduced to questions about multiplication operators (after the Spectral Theorem), which in turn boils down to the behavior of coefficients of the Fourier transform of functions in L^1.

Proposition 3.3. *There exist weakly stable singular-continuous unitary operators.*

Proof. Let absolutely continuous, singular, singular-continuous and singular-discrete mean with respect to the Lebesgue measure on $\mathcal{A}_{\mathbb{T}}$. Consider the above setup where U_μ is the multiplication operator on $L^2(\mathbb{T},\mu)$, and suppose the measure μ is finite (so that it is a spectral measure for U_μ). A measure μ on $\mathcal{A}_{\mathbb{T}}$ is a Rajchman measure if and only if $\widehat{\mu}(k)=\int_{\mathbb{T}} z^k\,d\mu\to 0$ as $|k|\to\infty$.

(a) Every absolutely continuous measure is Rajchman, and the converse fails.

(b) Every Rajchman measure is continuous, and the converse also fails.

(See, e.g., [17, Section 1].) Thus there are Rajchman measures that are not absolutely continuous, and so (this is Menshov's Theorem — see, e.g., [17, Section 3]):

(c) There exist singular Rajchman measures.

Also, it is readily verified that μ is Rajchman if and only if $U_\mu^n\xrightarrow{w}0$ (see, e.g., [1]). (In fact, (i) $\langle U_\mu^n f;f\rangle=\int_{\mathbb{T}} z^n|f(z)|^2d\mu$, (ii) $\langle U_\mu^{*n}f;f\rangle=\langle f;U_\mu^n f\rangle=\int_{\mathbb{T}} z^{-n}|f(z)|^2d\mu$, (iii) U_μ and U_μ^* are weakly stable together, and (iv) $L^\infty(\mathbb{T},\mu)^-=L^2(\mathbb{T},\mu)$). Hence,

(d) μ is a Rajchman measure if and only if $U_\mu^n\xrightarrow{w}0$.

Then there is no singular-discrete Rajchman measure (cf. Proposition 3.1):

(e) Every singular Rajchman measure is singular-continuous.

Thus if a finite measure μ is a singular Rajchman measure (so that it is singular-continuous), then the multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ is a weakly stable singular-continuous unitary operator. □

Summing up: an absolutely continuous unitary is always weakly stable, a singular-discrete unitary is never weakly stable, and there exist singular-continuous unitaries that are either weakly stable or weakly unstable.

4. Singular-Continuous Unitaries and Weak Supercyclicity

A closed set $\Gamma \subseteq \mathbb{T}$ is a Kronecker set if for every continuous function $f\colon \mathbb{T} \to \mathbb{T}$ and every $\varepsilon > 0$ there is an integer $k \in \mathbb{Z}$ such that $\sup_{z\in\Gamma} |f(z) - z^k| < \varepsilon$. A set $\Gamma \subseteq \mathbb{T}$ is independent if for every positive integer $n \in \mathbb{N}$, every finite sequence of integers $\{k_i\}_{i=1}^n \subseteq \mathbb{Z}$, and every finite sequence $\{z_i\}_{i=1}^n \subseteq \Gamma$, the identity $\prod_{i=1}^n z_i^{k_i} = 1$ implies $k_i = 0$ for every $i = 1, \ldots, n$. Every finite independent subset of $\mathbb{T}$ is a Kronecker set, there exist perfect Kronecker sets, and perfect Kronecker sets are topologically homeomorphic to a Cantor discontinuum (i.e., to a perfect nowhere dense set; equivalently, to a closed set with no isolated points and with empty interior; see, e.g., [11, Section VI.9.4], [18, Section 5.2]). The next lemmas were proved in [2]. Lemma 4.1 shows the existence of weakly supercyclic unitary operators.

Lemma 4.1. *If μ is a continuous probability measure on $\mathcal{A}_\mathbb{T}$ whose support is a perfect Kronecker set, then the multiplication operator U_μ on $L^2(\mathbb{T}, \mu)$ is weakly supercyclic.*

Proof. [2, p.10, Corollary to Example 3.6]. □

Lemma 4.2. *Let μ be a probability measure on $\mathcal{A}_\mathbb{T}$. If the multiplication operator U_μ on $L^2(\mathbb{T}, \mu)$ is weakly supercyclic, then μ is singular.*

Proof. This is an immediate consequence of [2, Example 3.10]. □

Lemmas 4.1 and 4.2 show that *a continuous probability measure on $\mathcal{A}_\mathbb{T}$ whose support is a perfect Kronecker set is singular-continuous.* Theorem 4.1 below is complementary to Lemma 4.2 and can be viewed as a sort of converse to Lemma 4.1.

The next proposition says that *every discrete unitary multiplication operator U_{μ_d} on $L^2(\mathbb{T}, \mu_d)$ is not weakly l-sequentially supercyclic.*

Proposition 4.1. *Let μ be a finite measure on $\mathcal{A}_\mathbb{T}$. If μ is discrete, then the multiplication operator U_μ on $L^2(\mathbb{T}, \mu)$ is not weakly l-sequentially supercyclic.*

Proof. Let μ be a finite discrete measure on $\mathcal{A}_\mathbb{T}$ and take the multiplication operator U_μ on $L^2(\mathbb{T}, \mu)$, so that U_μ is a discrete unitary:

$$(U_\mu f)(z) = z f(z) \quad \mu\text{-a.e. for } \ z \in \mathbb{T}$$

for every $f \in L^2(\mathbb{T}, \mu)$. If U_μ is weakly l-sequentially supercyclic, then $L^2(\mathbb{T}, \mu)$ is separable. Since the measure μ is discrete (i.e., singular-discrete), it is concentrated on a countable set (and so on a set of Lebesgue measure zero). Thus the multiplication operator U_μ is identified with (i.e., it is unitarily equivalent to) a diagonal operator U on ℓ^2_+:

$$(Ux)_k = z_k\, x_k \quad \text{for } \ z_k \in \mathbb{T}$$

for every $x = \{x_k\}_{k\geq 1} \in \ell^2_+$. Write $U = \operatorname{diag}(\{z_k\})$, so that

$$U^n = \operatorname{diag}(\{z_k^n\})$$

for every integer $n \geq 0$. If U_μ on $L^2(\mathbb{T}, \mu)$ is weakly l-sequentially supercyclic, then so is U on ℓ^2_+. Let $y = \{y_k\}_{k\geq 1} \in \ell^2_+$ be a weakly l-sequentially supercyclic vector for U. Fix an arbitrary nonzero $x = \{x_k\}_{k\geq 1} \in \ell^2_+$. Then there exists a scalar-valued sequence $\{\alpha_i\}_{i\geq 0}$ of nonzero numbers (which depends on x and y) such that

$$\alpha_i U^{n_i} y \xrightarrow{\;w\;} x$$

for some subsequence $\{U^{n_i}\}_{i\geq 0}$ of $\{U^n\}_{n\geq 0}$, which means that

$$\alpha_i\langle U^{n_i}y\,;w\rangle \to \langle x\,;w\rangle$$

for every $w\in\ell^2_+$; equivalently,

$$\alpha_i\sum\nolimits_k z_k^{n_i}y_k\overline{w}_k \to \sum\nolimits_k x_k\overline{w}_k,$$

for every $w=\{w_k\}_{k\geq 1}\in\ell^2_+$. Let $\{e_j\}_{j\geq 1}$ be the canonical orthonormal basis for ℓ^2_+. Take an arbitrary positive integer j and set $w=e_j$ so that

$$\alpha_i z_j^{n_i}y_j \xrightarrow[i]{} x_j.$$

Note that $y_j\neq 0$ for every $j\geq 1$ because $y=\{y_j\}_{j\geq 1}\in\ell^2_+$ is a weakly supercyclic (thus cyclic) vector for $U=\operatorname{diag}(\{z_k\})$. Therefore, since $z_j\in\mathbb{T}$ for every $j\geq 1$,

$$|\alpha_i| \xrightarrow[i]{} \tfrac{|x_j|}{|y_j|}$$

for all $j\geq 1$, implying that $\frac{|x_j|}{|y_j|}$ is a constant for every $j\geq 1$, which is a contradiction, since x was taken to be an arbitrary nonzero vector in ℓ^2_+ (clearly independent of the vector $y\in\ell^2_+$). Thus U_μ cannot be weakly l-sequentially supercyclic. □

Theorem 4.1. *Let μ be a finite measure on $\mathcal{A}_{\mathbb{T}}$. If the multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ is weakly l-sequentially supercyclic, then μ is continuous.*

Proof. Take a multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ for a finite measure μ on $\mathcal{A}_{\mathbb{T}}$ (so that μ is a scalar spectral measure for the unitary operator U_μ). Consider the decomposition $\mu=\mu_c+\mu_d$, where μ_c and μ_d are the continuous and discrete (with respect to the normalized Lebesgue measure on $\mathcal{A}_{\mathbb{T}}$) components of the finite measure μ, which are again finite measures on $\mathcal{A}_{\mathbb{T}}$. Consider the decomposition

$$U_\mu = U_{\mu_c}\oplus U_{\mu_d},$$

where the unitary operators U_{μ_c} and U_{μ_d} are the continuous and discrete direct summands of the multiplication operator U_μ on $L^2(\mathbb{T},\mu)=L^2(\mathbb{T},\mu_c)\oplus L^2(\mathbb{T},\mu_d)$, which are again multiplication operators on $L^2(\mathbb{T},\mu_c)$ and $L^2(\mathbb{T},\mu_d)$, respectively. Suppose the discrete measure μ_d is nonzero. This means that the discrete direct summand U_{μ_d} on $L^2(\mathbb{T},\mu_d)$ is not missing in the decomposition of the unitary U_μ. If U_μ on $L^2(\mathbb{T},\mu)$ is weakly l-sequentially supercyclic, then so are U_{μ_c} on $L^2(\mathbb{T},\mu_c)$ and U_{μ_d} on $L^2(\mathbb{T},\mu_d)$. However, Proposition 4.1 says that U_{μ_d} cannot be weakly l-sequentially supercyclic, which leads to a contradiction. Then the discrete direct summand U_{μ_d} on $L^2(\mathbb{T},\mu_d)$ is missing in the decomposition of the unitary U_μ, which means that the discrete measure μ_d must be zero, and so $\mu=\mu_c$; that is, the measure μ is continuous. □

Theorem 4.2. *Every weakly l-sequentially supercyclic unitary operator is singular-continuous.*

Proof. Let U be a unitary operator on a Hilbert space $\mathcal{H}$. Suppose U is weakly l-sequentially supercyclic, so that U is cyclic (and so it is star-cyclic — i.e., there exists a vector $y\in\mathcal{H}$ such that $\bigvee\{U^nU^{*n}y\}=\mathcal{H}$ — since it is normal). Thus $\mathcal{H}$ is separable and, by the Spectral Theorem, U is identified with (i.e., it is unitarily equivalent to) a unitary multiplication operator (induced by the identity function) U_μ on $L^2(\mathbb{T},\mu)$, where the measure μ on $\mathcal{A}_{\mathbb{T}}$ (supported on $\sigma(U)$) is finite, and so it coincides with a scalar spectral measure for U (see, e.g., [15, part (a), proof of Theorem 3.11, and the

Remark following the proof of Lemma 4.7]). Since U on $\mathcal{H}$ is weakly l-sequentially supercyclic, then so is U_μ on $L^2(\mathbb{T},\mu)$, and hence U_μ is continuous according to Theorem 4.1, and singular according to Lemma 4.2 (since a finite measure is a positive multiple of a probability measure, and since weak l-sequential supercyclicty implies weak supercyclicity). Thus the multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ is singular-continuous, and so the unitary U on $\mathcal{H}$ is singular-continuous since it is unitarily equivalent to U_μ. □

5. Concluding Remarks

Question 5.1. Is it true that if a *discrete* unitary operator is weakly supercyclic, then it is weakly l-sequentially supercyclic?

An affirmative answer to Question 5.1 ensures that Proposition 4.1 and Theorems 4.1 and 4.2 still hold if weakly l-sequentially supercyclic is replaced with weakly supercyclic (thus leading to the nonexistence of weakly supercyclic *discrete* unitary operators). Along the same line, it was asked in [2, Question 3.11] whether

> there exists a Rajchman probability measure μ on $\mathcal{A}_\mathbb{T}$ for which the multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ is weakly supercyclic.

Recall that μ is Rajchman if and only if U_μ is weakly stable (cf. part (d), proof of Proposition 3.3). Thus the above question is equivalently stated as follows.

> Does there exist a weakly stable multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ that is weakly supercyclic?

The affirmative answer was given in [19, Proposition 1.1 and Theorem 2.1]:

> there exists a Rajchman probability measure μ on $\mathcal{A}_\mathbb{T}$ for which the (unitary) multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ is weakly supercyclic but not weakly sequentially supercyclic

and, consequently,

> there exists a weakly supercyclic *continuous* unitary operator that is not weakly l-sequentially supercyclic.

(Being a continuous unitary, it is not singular, and therefore not singular-discrete, which means that it is not a *discrete* unitary, as asked in Question 5.1). As stated in Lemma 4.1, the existence of a weakly supercyclic (a weakly l-sequentially supercyclic, actually) unitary operator was shown in [2, Example 3.6, pp.10,12] (see [19, Question 1]), and therefore what has also been shown in [19, Proposition 1.1 and Theorem 1.2] was the first instance of a weakly supercyclic unitary operator that is not weakly l-sequentially supercyclic. In view of such an affirmative answer, it seems sensible to enquire in the opposite direction.

> Does there exist a weakly stable multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ that is not weakly supercyclic?

In this case the answer is again in the affirmative, since

> if μ is a finite absolutely continuous measure on $\mathcal{A}_\mathbb{T}$, then μ is Rajchman and the multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ is not weakly supercyclic.

Indeed, let μ be a finite (or a probability) measure. According to Lemma 4.2, if U_μ is weakly supercyclic, then μ is singular. Therefore, since every absolutely continuous measure is Rajchman, it follows that an absolutely continuous measure is a Rajchman measure that is not singular, and therefore U_μ is not weakly supercyclic. However, the reverse question remains unanswered.

Question 5.2. Is every weakly supercyclic unitary operator weakly stable?

This is a particular case of a more general question which has, in its most general case, a negative answer: is every weakly supercyclic Hilbert-space operator weakly stable? No, since weak hypercyclicity implies weak supercyclicity, since there exist Hilbert-space weakly hypercyclic operators (see, e.g., [9, Exercise 12.2.1]), and since a weakly hypercyclic operator cannot be weakly stable. However, the question seems to remain answered when restricted to power bounded operators even for weak l-sequential supercyclicity: *is every weakly l-sequentially supercyclic power bounded operator weakly stable?*

The next result gives a condition for a weakly unstable unitary operator not to be weakly l-sequentially supercyclic, which brings to mind the so-called Angle Criterion for supercyclicity (see, e.g., [3, Theorem 9.1]).

Theorem 5.1. *If a unitary operator U on a Hilbert space $\mathcal{H}$ is not weakly stable, and if for every $x \in \mathcal{H}$ for which $U^n x \overset{w}{\not\longrightarrow} 0$ there is a nonzero $x' \in \mathcal{H}$ such that*

$$|\langle U^n x\,;x'\rangle| \to \|x\|\,\|x'\|,$$

then U is not weakly l-sequentially supercyclic.

Proof. We begin with a definition. A normed space $\mathcal{X}$ is said to be of *type 1* if strong convergence (i.e., convergence in the norm topology) for an arbitrary $\mathcal{X}$-valued sequence $\{x_k\}$ coincides with weak convergence plus convergence of the norm sequence $\{\|x_k\|\}$ (i.e., $x_k \longrightarrow x \iff \{x_k \overset{w}{\longrightarrow} x \text{ and } \|x_k\| \to \|x\|\}$ — also called *Radon–Riesz space* and the *Radon–Riesz property*, respectively). Every Hilbert space is a Banach space of type 1. Now suppose U is a unitary operator on a Hilbert space $\mathcal{H}$, so that dual pairs amount to inner products (after the Riesz Representation Theorem). Moreover, suppose U is not weakly stable. Since U is a power bounded operator on a Banach space of type 1, which is not weakly stable, it follows by [16, Theorem 6.2] that if U is weakly l-sequentially supercyclic, then the set $M_U =$ $\{y \in \mathcal{H}\colon\, y \text{ is a weakly l-sequentially supercyclic vector for } U \text{ such that } U^n y \overset{w}{\not\longrightarrow} 0\}$ is nonempty, and if y is any vector in M_U, then for every nonzero vector y' in $\mathcal{H}$ such that $\langle U^n y\,;y'\rangle \not\to 0$ either

(1) $\liminf_n |\langle U^n y\,;y'\rangle| = 0$, or

(2) $\limsup_k |\langle U^{n_k} y\,;y'\rangle| < \|y'\| \limsup_k \|U^{n_k} y\|$

for some subsequence $\{U^{n_k}\}$ of $\{U^n\}$. Since U is an isometry this means

(2) $\limsup_k |\langle U^{n_k} y\,;y'\rangle| < \|y\|\,\|y'\|$ for some subsequence $\{U^{n_k}\}$ of $\{U^n\}$.

Therefore, if for every vector x in $\mathcal{H}$ for which $U^n x \overset{w}{\not\longrightarrow} 0$ there exists a nonzero vector x' in $\mathcal{H}$ such that

$$|\langle U^n x\,;x'\rangle| \to \|x\|\,\|x'\|,$$

then

(2′) $\limsup_k |\langle U^{n_k} x\,;x'\rangle| = \|x\|\,\|x'\|$ for every subsequence $\{U^{n_k}\}$ of $\{U^n\}$,

and this implies that

(1′) $\liminf_n |\langle U^n x\,;x'\rangle| > 0$,

which contradict (1) and (2), and so U is not weakly l-sequentially supercyclic. □

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Catholic University of Rio de Janeiro, 2451-900, Rio de Janeiro, RJ, Brazil
E-mail address: `carlos@ele.puc-rio.br`