Archiv der Mathematik **107** (2016) 173–184

QUASIAFFINITY AND INVARIANT SUBSPACES

A. MELLO AND C.S. KUBRUSLY

Abstract. Special classes of intertwining transformations between Hilbert spaces are introduced and investigated, whose purpose is to provide partial answers to some classical questions on the existence of nontrivial invariant subspaces for operators acting on separable Hilbert spaces. The main result ensures that if an operator is $\mathcal{D}$-intertwined to normal operator, then it has a nontrivial invariant subspace.

1. Introduction

The motivation for this paper lies in the following well-known open question on quasiaffinity and invariant subspaces (see, e.g., in [11, p.68] and [16, p.194]).

Question 1. Does a quasiaffine transform of a normal operator have a nontrivial invariant subspace?

A more general version reads as follows.

Question 2. Does a quasiaffine transform of a reducible operator have a nontrivial invariant subspace?

In order to investigate special instances of these questions we introduce two classes of intertwining transformations which, for lack of a better name, are called class $\mathcal{C}$ and class $\mathcal{D}$. We show the existence of nontrivial invariant subspaces for operators intertwined with normal operators through intertwining transformations of classes $\mathcal{C}$ and $\mathcal{D}$. The paper is organized as follows. Section 2 deals with notation and terminology, and Section 3 presents the definition of class $\mathcal{C}$ transformations and the auxiliary results that will be required in the sequel. Section 4 introduces the notion of class $\mathcal{D}$ and proves the main result of the paper, namely, Theorem 4.1, whose principal consequence in Corollary 4.1 ensures that if an operator is $\mathcal{D}$-intertwined to normal operator, then it has a nontrivial invariant subspace. Concluding remarks are considered in Section 5.

Further discussions along these lines (mainly considering quasisimilarity) can be found, for instance, in [8, 9, 10, 12, 13, 15]. In particular, we will focus on quasiaffine transforms of normal operators (Question 1), whose particularization to unitary operators is a well-know classical result for Hilbert-space contractions, namely, *every C_{11}-contraction is quasisimilar to a unitary operator* (see e.g., [19, p.79], [4, p.104], [16, p.109], [6, p.388], and [11, p.70]) — related problems dealing with quasisimilarity to isometries were considered in [20, 21], and restrictions of operators to quasisimilar subspaces in, for example, [2, 3]). For wide range surveys taking up similar questions linked to the invariant subspace problem the reader is referred to [1, 7, 14, 17, 18].

Date: October 30, 2015; revised March 20, 2016; April 27, 2016.
2010 *Mathematics Subject Classification.* Primary 47A15; Secondary 47B15.
Keywords. Invariant subspaces, intertwining transformations, quasiaffinity.

2. Notation and Terminology

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ will stand for complex, separable, and infinite-dimensional Hilbert spaces. Let $\mathcal{B}[\mathcal{H},\mathcal{K}]$ be the Banach space of all bounded linear transformations from $\mathcal{H}$ into $\mathcal{K}$. Set $\mathcal{B}[\mathcal{H}]=\mathcal{B}[\mathcal{H},\mathcal{H}]$ for short, which is the Banach algebra of all operators on $\mathcal{H}$ (i.e., of all bounded linear transformation of $\mathcal{H}$ into itself). By a subspace of $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$. For any set $M\subseteq\mathcal{H}$ let $M^\perp$ denote the orthogonal complement of M in $\mathcal{H}$ (which is a subspace of $\mathcal{H}$). The closure $\mathcal{M}^-$ of a linear manifold $\mathcal{M}$ is a subspace. Let $\mathcal{R}(X)=X(\mathcal{H})$ be the range of $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$, which is a linear manifold of $\mathcal{K}$, and let $\mathcal{N}(X)=X^{-1}(\{0\})$ be the kernel of X, which is a subspace of $\mathcal{H}$. Let $T^*\in\mathcal{B}[\mathcal{H}]$ stand for the adjoint of $T\in\mathcal{B}[\mathcal{H}]$. An operator $T\in\mathcal{B}[\mathcal{H}]$ is normal if $TT^*=T^*T$, and unitary if it is a normal isometry (i.e., if $TT^*=T^*T=I$, where I stands for the identity operator in $\mathcal{B}[\mathcal{H}]$). If $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$, $T\in\mathcal{B}[\mathcal{H}]$, $L\in\mathcal{B}[\mathcal{K}]$, and $XT=LX$, then we say that X intertwines T to L (and so T is intertwined to L). A transformation $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ is quasiinvertible or a quasiaffinity if it is injective with dense range (i.e., $\mathcal{N}(X)=\{0\}$ and $\mathcal{R}(X)^-=\mathcal{K}$). An operator $T\in\mathcal{B}[\mathcal{H}]$ is a quasiaffine transform of an operator $L\in\mathcal{B}[\mathcal{K}]$ if there exists a quasiinvertible transformation $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ intertwining T to L. A subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant for $T\in\mathcal{B}[\mathcal{H}]$ (or T-invariant) if $T(\mathcal{M})\subseteq\mathcal{M}$, and $\mathcal{M}$ reduces T (or $\mathcal{M}$ is a reducing subspace for T) if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for T, and $\mathcal{M}$ is nontrivial if $\{0\}\neq\mathcal{M}\neq\mathcal{H}$. An operator is reducible if it has a nontrivial reducing subspace.

3. Auxiliary Results

Before introducing the notion of transformation of class $\mathcal{C}$, recall that every normal operator (on a Hilbert space of dimension greater than 1) has a nontrivial reducing subspace. Then Question 1 and 2 have the following partial answers.

Proposition 3.1. [11, Corollary4.4]. *Take $T\in\mathcal{B}[\mathcal{H}]$, $L\in\mathcal{B}[\mathcal{K}]$, and $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $XT=LX$. Let $\mathcal{M}\subset\mathcal{K}$ be a nontrivial finite-dimensional reducing subspace for L. If $\mathcal{R}(X)^-=\mathcal{K}$, then $X^{-1}(\mathcal{M}^\perp)$ is a nontrivial invariant subspace for T.*

In particular, this applies to quasiaffine transforms of reducible operators.

Proposition 3.2. [11, Corollary 4.5]. *If an operator $T\in\mathcal{B}[\mathcal{H}]$ is a quasiaffine transform of another operator $L\in\mathcal{B}[\mathcal{K}]$ that has a nontrivial finite-dimensional reducing subspace, then T has a nontrivial invariant subspace.*

These results are based on the following proposition.

Proposition 3.3. [11, Lemma 4.1]. *Let $T\in\mathcal{B}[\mathcal{H}]$, $L\in\mathcal{B}[\mathcal{K}]$, and $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ be such that $XT=LX$. Suppose $\mathcal{M}\subset\mathcal{K}$ is a nontrivial invariant subspace for L. If $\mathcal{R}(X)^-=\mathcal{K}$ and $\mathcal{R}(X)\cap\mathcal{M}\neq\{0\}$, then $X^{-1}(\mathcal{M})$ is a nontrivial invariant subspace for T.*

Proposition 3.1, 3.2, and 3.3 motivate the definition of class $\mathcal{C}$ transformations.

Definition 1. A transformation $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ is of *class* $\mathcal{C}$ if

$$\mathcal{R}(X)\cap\mathcal{M}\neq\{0\}$$

for every infinite-dimensional subspace $\mathcal{M}$ of $\mathcal{K}$. In this case we say that $X\in\mathcal{C}$.

Let $T \in \mathcal{B}[\mathcal{H}]$, $L \in \mathcal{B}[\mathcal{K}]$, and $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ be such that X intertwines T to L (i.e., such that $XT = LX$). If in addition $X \in \mathcal{C}$, then we say that T *is* $\mathcal{C}$*-intertwined to* L, or that X is a transformation that $\mathcal{C}$*-intertwines* T *to* L. It can be verified that transformations of class $\mathcal{C}$ satisfy the following properties. (Proofs are omitted here since we prove these results for a more general class in Section 4; cf. Remark 4.1.)

Proposition 3.4. *If* $X \in \mathcal{C}$*, then* $\dim \mathcal{R}(X)^{\perp} < \infty$*, so that* $\dim \mathcal{R}(X) = \infty$*.*

Considering the definition of transformations of class $\mathcal{C}$, and according to Questions 1 and 2, we pose the following further question.

Question 3. Does every operator $\mathcal{C}$-intertwined to a normal operator have a nontrivial invariant subspace?

This question will be answered in Corollary 4.1. Meanwhile, transformations of class $\mathcal{C}$ yield the following result, which is naturally linked to Proposition 3.3

Proposition 3.5. *If* $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ *is a transformation of class* $\mathcal{C}$*, and if* $N \in \mathcal{B}[\mathcal{K}]$ *is a normal operator, then there exists a nontrivial infinite-dimensional reducing subspace for* N*, say* $\mathcal{M}$*, such that* $\mathcal{R}(X) \not\subseteq \mathcal{M}$*.*

Proof. Let $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ be an arbitrary transformation of class $\mathcal{C}$, and let $N \in \mathcal{B}[\mathcal{K}]$ be an arbitrary normal operator. The proof goes by contradiction. That is, assume that for every nontrivial reducing subspace $\mathcal{M}$ for N with $\dim \mathcal{M} = \infty$ we have

$$\mathcal{R}(X) \subseteq \mathcal{M}. \qquad (*)$$

This will lead to a contradiction. Since N is normal and $\mathcal{K}$ is infinite-dimensional, it follows that there exists a nontrivial reducing subspace $\mathcal{M}_1$ for N with $\dim \mathcal{M}_1 = \infty$ (reason: if $\mathcal{N}$ and $\mathcal{N}^{\perp}$ are reducing and nontrivial for N, then at least one is infinite-dimensional). Thus the inclusion in $(*)$ ensures that

$$\mathcal{R}(X) \subseteq \mathcal{M}_1,$$

and hence

$$\mathcal{R}(X) \cap \mathcal{M}_1^{\perp} = \{0\}.$$

Since $X \in \mathcal{C}$, it follows that $\dim \mathcal{M}_1^{\perp} = n_1 < \infty$. Moreover, since $\mathcal{M}_1$ is a nontrivial reducing subspace for N, it is clear that $(N|_{\mathcal{M}_1})^* = N^*|_{\mathcal{M}_1}\colon \mathcal{M}_1 \to \mathcal{M}_1$ so that $N_1 = N|_{\mathcal{M}_1}\colon \mathcal{M}_1 \to \mathcal{M}_1$ is normal, and hence there is a nontrivial reducing subspace $\mathcal{M}_2 \subset \mathcal{M}_1$ for N_1 (thus a nontrivial reducing subspace for N) with $\dim \mathcal{M}_2 = \infty$. Again by the inclusion in $(*)$ we get

$$\mathcal{R}(X) \subseteq \mathcal{M}_2,$$

and therefore,

$$\mathcal{R}(X) \cap \mathcal{M}_2^{\perp} = \{0\}.$$

Since $X \in \mathcal{C}$, it follows that $\dim \mathcal{M}_2^{\perp} = n_2 < \infty$. Then, since $\mathcal{M}_2 \subset \mathcal{M}_1$ if and only if $\mathcal{M}_1^{\perp} \subset \mathcal{M}_2^{\perp}$, it also follows that $n_1 < n_2$. Hence, by a trivial induction, for each integer $k \geq 1$ there is a nontrivial reducing subspace $\mathcal{M}_k$ for N such that

$\dim \mathcal{M}_k = \infty$,
$\mathcal{R}(X) \subseteq \mathcal{M}_{k+1} \subset \mathcal{M}_k$,
$\mathcal{R}(X) \cap \mathcal{M}_k^{\perp} = \{0\}$,
$\dim \mathcal{M}_k^{\perp} = n_k < n_{k+1}$.

Since the sequence $\{n_k\}_{k \geq 1}$ of positive integers $n_k = \dim \mathcal{M}_k^{\perp}$ is increasing and

$$\mathcal{R}(X) \subseteq \mathcal{M}_k \implies \mathcal{M}_k^\perp \subseteq \mathcal{R}(X)^\perp,$$

it follows that $\dim \mathcal{R}(X)^\perp = \infty$, which contradicts Proposition 3.4, thus completing the proof. □

4. Class $\mathcal{D}$ and Main Results

Now we introduce the notion of transformations of class $\mathcal{D}$, and we show how this class provides an affirmative answer to the existence of nontrivial invariant subspaces as in the forthcoming Question 4.

Definition 2. A transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is of *class* $\mathcal{D}$ if

$$\mathcal{R}(X) \cap \mathcal{M} \neq \{0\} \quad \text{or} \quad \mathcal{R}(X) \cap \mathcal{M}^\perp \neq \{0\}$$

for every subspace $\mathcal{M}$ of $\mathcal{K}$. In this case we say that $X \in \mathcal{D}$.

Let $T \in \mathcal{B}[\mathcal{H}]$, $L \in \mathcal{B}[\mathcal{K}]$, and $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ be such that X intertwines T to L (i.e., such that $XT = LX$). If in addition $X \in \mathcal{D}$, then we say that T *is* $\mathcal{D}$*-intertwined to* L, or that X is a transformation that $\mathcal{D}$*-intertwines* T *to* L. Since $\mathcal{K} = \mathcal{M} + \mathcal{M}^\perp$ and $\dim \mathcal{K} = \infty$, it is clear that if $X \in \mathcal{C}$, then $X \in \mathcal{D}$; that is,

$$\mathcal{C} \subseteq \mathcal{D}.$$

Transformations of class $\mathcal{D}$ satisfy the following properties.

Proposition 4.1. *If* $X \in \mathcal{D}$ *, then* $\dim \mathcal{R}(X)^\perp < \infty$*, so that* $\dim \mathcal{R}(X) = \infty$*.*

Proof. Let $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ be of class $\mathcal{D}$. We split the proof into two parts.

Part 1. $\dim \mathcal{R}(X) = \infty$.

Take an arbitrary $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. Suppose $\dim \mathcal{R}(X) = n < \infty$. This implies that there exists a finite orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathcal{R}(X)$ so that

$$\mathcal{R}(X) = \operatorname{span}\{e_i\}_{i=1}^n.$$

Since $\mathcal{R}(X)$ is finite-dimensional, it is closed, and so $\mathcal{K} = \mathcal{R}(X) + \mathcal{R}(X)^\perp$. Since $\mathcal{K}$ is infinite-dimensional and separable, then there exists a countably infinite orthonormal basis $\{f_i\}_{i=1}^\infty$ for $\mathcal{R}(X)^\perp$. Thus

$$\mathcal{R}(X)^\perp = \big(\operatorname{span}\{f_i\}_{i=1}^\infty\big)^-.$$

Consider the finite-dimensional subspace

$$\mathcal{M} = \operatorname{span}\{e_i + f_i\}_{i=1}^n = \operatorname{span}\big\{\tfrac{e_i + f_i}{\sqrt{2}}\big\}_{i=1}^n$$

of $\mathcal{K} = \mathcal{R}(X) + \mathcal{R}(X)^\perp$ spanned by the orthonormal set $\{\frac{e_i+f_i}{\sqrt{2}}\}_{i=1}^n$, which is an orthonormal basis for $\mathcal{M}$. Since every nonzero vector in the finite-dimensional space $\mathcal{M}$ has a component in $\mathcal{R}(X)$ and a component in $\mathcal{R}(X)^\perp$, we get

$$\mathcal{R}(X) \cap \mathcal{M} = \{0\} \quad \text{and} \quad \mathcal{R}(X) \cap \mathcal{M}^\perp = \{0\},$$

and hence $X \notin \mathcal{D}$. Therefore,

$$X \in \mathcal{D} \quad \Longrightarrow \quad \dim \mathcal{R}(X) = \infty.$$

which completes the proof of the claimed result in Part 1.

Part 2. $\dim \mathcal{R}(X)^\perp < \infty$.

The proof goes as follows. Since $\dim \mathcal{R}(X) = \infty$ by Part 1, and since $\mathcal{R}(X)^-$ is separable, let $\{e_i\}_{i=1}^\infty$ be a countably infinite orthonormal basis for $\mathcal{R}(X)^-$, so that

$$\mathcal{R}(X)^- = \big(\operatorname{span}\{e_i\}_{i=1}^\infty\big)^-.$$

Suppose $\dim \mathcal{R}(X)^\perp = \infty$. Thus there exists a countably infinite orthonormal basis $\{f_i\}_{i=1}^\infty$ for $\mathcal{R}(X)^\perp$ such that

$$\mathcal{R}(X)^\perp = \big(\operatorname{span}\{f_i\}_{i=1}^\infty\big)^-.$$

Consider the countably infinite orthonormal set $\{\frac{e_i+f_i}{\sqrt{2}}\}_{i=1}^\infty$ and take the subspace $\mathcal{M}$ spanned by it (so that $\{\frac{e_i+f_i}{\sqrt{2}}\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{M}$),

$$\mathcal{M} = \big(\operatorname{span}\big\{\tfrac{e_i+f_i}{\sqrt{2}}\big\}_{i=1}^\infty\big)^- = \big(\operatorname{span}\{e_i+f_i\}_{i=1}^\infty\big)^-,$$

which is an infinite-dimensional subspace of $\mathcal{K} = \mathcal{R}(X)^- + \mathcal{R}(X)^\perp$. Since every nonzero vector in the infinite-dimensional space $\mathcal{M}$ has a component in $\mathcal{R}(X)^-$ and a component in $\mathcal{R}(X)^\perp$, we get

$$\mathcal{R}(X)^- \cap \mathcal{M} = \{0\} \quad \text{and} \quad \mathcal{R}(X)^- \cap \mathcal{M}^\perp = \{0\}.$$

Thus

$$\mathcal{R}(X) \cap \mathcal{M} = \{0\} \quad \text{and} \quad \mathcal{R}(X) \cap \mathcal{M}^\perp = \{0\},$$

and hence $X \notin \mathcal{D}$. Therefore,

$$X \in \mathcal{D} \quad \Longrightarrow \quad \dim \mathcal{R}(X)^\perp < \infty.$$

which completes the proof of the claimed result in Part 2. □

Remark 4.1. Observe that Proposition 3.4 is a particular case of Proposition 4.1, once class $\mathcal{C}$ is included in class $\mathcal{D}$ (although Proposition 3.4 can be independently proved by using similar arguments). Also note that the proof of Part 2 in Proposition 4.1 ensures the following assertion.

If $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ is such that $\dim \mathcal{R}(X) = \infty$ and $\dim \mathcal{R}(X)^\perp = \infty$, then there is an infinite-dimensional subspace $\mathcal{M} \subset \mathcal{K}$ such that $\mathcal{R}(X) \cap \mathcal{M} = \{0\}$ and $\mathcal{R}(X) \cap \mathcal{M}^\perp = \{0\}$.

Actually, since class $\mathcal{D}$ includes class $\mathcal{C}$, and based on Question 3, we pose the following central question.

Question 4. Does every operator $\mathcal{D}$-intertwined to a normal operator have a nontrivial invariant subspace?

This has an affirmative answer whose proof depends on the following result.

Theorem 4.1. *If $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ is a transformation of class $\mathcal{D}$, and if $N \in \mathcal{B}[\mathcal{K}]$ is a normal operator, then there is a nontrivial reducing subspace $\mathcal{M}$ for N such that $\mathcal{R}(X) \nsubseteq \mathcal{M}$ and $\mathcal{R}(X) \cap \mathcal{M} \neq \{0\}$.*

Proof. Let $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ be a transformation of class $\mathcal{D}$ and let $N \in \mathcal{B}[\mathcal{K}]$ be a normal operator. Suppose that every nontrivial reducing subspace $\mathcal{M}$ for N is such that

$$\mathcal{R}(X) \cap \mathcal{M} = \{0\} \quad \text{or} \quad \mathcal{R}(X) \subseteq \mathcal{M}. \tag{$*$}$$

Since N is normal, there is a nontrivial reducing subspace $\mathcal{M}_1$ for N. Since in addition $X \in \mathcal{D}$, it also follows by Definition 2 that

$$\mathcal{R}(X) \cap \mathcal{M}_1 \neq \{0\} \quad \text{or} \quad \mathcal{R}(X) \cap \mathcal{M}_1^\perp \neq \{0\}.$$

With no loss of generality assume that $\mathcal{R}(X) \cap \mathcal{M}_1 \neq \{0\}$. By $(*)$ we get

$$\mathcal{R}(X) \subseteq \mathcal{M}_1, \tag{$**$}$$

which implies that

$$\mathcal{M}_1^\perp \subseteq \mathcal{R}(X)^\perp.$$

By Proposition 4.1, Part 2, $\dim \mathcal{M}_1^\perp = n_1 < \infty$, so that $\dim \mathcal{M}_1 = \infty$. Since N is normal and $\mathcal{M}_1$ is a nontrivial reducing subspace for N, it follows that the restriction $N_1 = N|_{\mathcal{M}_1} \colon \mathcal{M}_1 \to \mathcal{M}_1$ is normal, so that there is a nontrivial reducing subspace $\mathcal{M}_2 \subset \mathcal{M}_1$ for N_1 (which is nontrivial and reducing for N). Let $\mathcal{M}_{12}$ denote the orthogonal complement of $\mathcal{M}_2$ in $\mathcal{M}_1$, and write

$$\mathcal{M}_1 = \mathcal{M}_2 + \mathcal{M}_{12},$$

where $\mathcal{M}_{12} \perp \mathcal{M}_2$, and $\mathcal{M}_{12}$ is nontrivial and reducing for N_1, and hence nontrivial and reducing for N. Since $X \in \mathcal{D}$, it follows by Definition 2 that

$$\mathcal{R}(X) \cap \mathcal{M}_2 \neq \{0\} \quad \text{or} \quad \mathcal{R}(X) \cap \mathcal{M}_2^\perp \neq 0.$$

If $\mathcal{R}(X) \cap \mathcal{M}_2 \neq \{0\}$, then by $(*)$

$$\mathcal{R}(X) \subseteq \mathcal{M}_2 \subset \mathcal{M}_1.$$

If $\mathcal{R}(X) \cap \mathcal{M}_2^\perp \neq \{0\}$, then by $(*)$ and $(**)$

$$\mathcal{R}(X) \subseteq \mathcal{M}_{12} \subset \mathcal{M}_1.$$

Again, with no loss of generality assume that

$$\mathcal{R}(X) \subseteq \mathcal{M}_2 \subset \mathcal{M}_1,$$

which implies that

$$\mathcal{M}_1^\perp \subset \mathcal{M}_2^\perp \subseteq \mathcal{R}(X)^\perp,$$

and so (by Proposition 4.1, Part 2) $\dim \mathcal{M}_2^\perp = n_2 < \infty$, so that $\dim \mathcal{M}_2 = \infty$, where $n_1 < n_2$. Repeating this argument we construct (by induction) a sequence $\{\mathcal{M}_k\}_{k\geq 1}$ of infinite-dimensional nontrivial reducing subspaces for N such that, for each $k \geq 1$,

$\mathcal{R}(X) \subseteq \mathcal{M}_k$,
$\mathcal{M}_k^\perp \subseteq \mathcal{R}(X)^\perp$,
$\dim \mathcal{M}_k^\perp = n_k < n_{k+1}$.

Since the sequence $\{n_k\}_{k\geq 1}$ of positive integers $n_k = \dim \mathcal{M}_k^\perp$ is increasing, it follows that $\dim \mathcal{R}(X)^\perp = \infty$, which contradicts Proposition 4.1, Part 2. Therefore, there is a nontrivial reducing subspace $\mathcal{M}$ for N such that

$$\mathcal{R}(X) \cap \mathcal{M} \neq \{0\} \quad \text{and} \quad \mathcal{R}(X) \not\subseteq \mathcal{M}. \qquad \square$$

An important consequence of Theorem 4.1 is the answer to Question 4.

Corollary 4.1. *Let $T \in \mathcal{B}[\mathcal{H}]$, $N \in \mathcal{B}[\mathcal{K}]$, and $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ be such that $XT = NX$. If $X \in \mathcal{D}$ and N is normal, then T has nontrivial invariant subspace.*

Proof. Since the intertwining transformation X is of class $\mathcal{D}$, Theorem 4.1 says that N has a nontrivial reducing subspace $\mathcal{M}$ for which

$$\mathcal{R}(X) \not\subseteq \mathcal{M} \quad \text{and} \quad \mathcal{R}(X) \cap \mathcal{M} \neq \{0\}. \tag{$*$}$$

(a) Since X is linear and continuous, $X^{-1}(\mathcal{M})$ is a subspace of $\mathcal{H}$.

Actually, the inverse linear image of a linear manifold is a linear manifold, and the inverse continuous image of a closed set is closed.

(b) Since $\mathcal{M}$ is invariant for N, $X^{-1}(\mathcal{M})$ is an invariant for T.

Indeed, since $X[X^{-1}(\mathcal{M})] \subseteq \mathcal{M}$, it follows that $NXX^{-1}(\mathcal{M}) \subseteq N(\mathcal{M}) \subseteq \mathcal{M}$ (because $\mathcal{M}$ is N-invariant). Thus, since $XT = NX$, we get $XTX^{-1}(\mathcal{M}) \subseteq \mathcal{M}$, so that $[TX^{-1}(\mathcal{M})] \subseteq X^{-1}(\mathcal{M})$, and hence $X^{-1}(\mathcal{M})$ is invariant for T.

(c) Since $(*)$ holds, $X^{-1}(\mathcal{M})$ is nontrivial

In fact, the condition $\mathcal{R}(X) \cap \mathcal{M} \neq \{0\}$ in $(*)$ ensures that $X^{-1}(\mathcal{M}) \neq \{0\}$. On the other hand, if $X^{-1}(\mathcal{M}) = \mathcal{H}$, then $\mathcal{R}(X) = X(\mathcal{H}) = XX^{-1}(\mathcal{M}) \subseteq \mathcal{M}$, which contradicts the condition $\mathcal{R}(X) \not\subseteq \mathcal{M}$ in $(*)$, and therefore $X^{-1}(\mathcal{M}) \neq \mathcal{H}$. □

Corollary 4.1 supplies an affirmative answer to Question 4.

> *If an operator T is $\mathcal{D}$-intertwined to normal operator, then T has a nontrivial invariant subspace.*

Since $\mathcal{C} \subseteq \mathcal{D}$, Corollary 4.1 trivially holds if class $\mathcal{D}$ is replaced by class $\mathcal{C}$, supplying an affirmative answer to Question 3, which is stated as above with class $\mathcal{D}$ is replaced by class $\mathcal{C}$. Motivated by the affirmative answer to Question 3 we pose the question.

Question 5. Is every bounded linear transformation with dense range of class $\mathcal{C}$?

Question 5 is important because, according to Corollary 4.1, an affirmative answer to it would imply an affirmative answer to Question 1. However, we will see next that Question 5 has a negative answer.

Proposition 4.2. *There exists $X \in \mathcal{B}[\mathcal{H},\mathcal{K}] \setminus \mathcal{C}$ with dense range.*

Proof. Set $\mathcal{K} = \mathcal{H}$ so that $\mathcal{B}[\mathcal{H},\mathcal{K}] = \mathcal{B}[\mathcal{H}]$. Take an arbitrary noncompact operator $X \in \mathcal{B}[\mathcal{H}]$ with $\mathcal{R}(X) \neq \mathcal{R}(X)^- = \mathcal{H}$. Set $\mathcal{R} = \mathcal{R}(X)$ for short. Since X is not compact, it follows that $\mathcal{M} \subseteq \mathcal{R}$ for some infinite-dimensional subspace $\mathcal{M}$ of $\mathcal{H}$ [5, pp.265–266]. Moreover, since $\mathcal{R}$ is nonclosed and $\mathcal{H}$ is separable, there exists a unitary operator $U \in \mathcal{B}[\mathcal{H}]$ such that $\mathcal{R} \cap U(\mathcal{R}) = \{0\}$ [5, Theorem 3.6]. Therefore, $U(\mathcal{M})$ is an infinite-dimensional subspace of $\mathcal{H}$ (because U is an isomorphism and an isometry and $\dim \mathcal{M} = \infty$) such that $\mathcal{R} \cap U(\mathcal{M}) = \{0\}$, which means that X is not of class $\mathcal{C}$ according to Definition 1. □

Proposition 4.2 not only supplies a negative answer to Question 5 but in fact it shows that *every noncompact operator on a separable Hilbert space with nonclosed range is not of class $\mathcal{C}$.*

5. Concluding Remarks

Observe that the same argument in the proof of Corollary 4.1 ensures the following result (compare with Proposition 3.3).

Proposition 5.1. *Let $T \in \mathcal{B}[\mathcal{H}]$, $L \in \mathcal{B}[\mathcal{K}]$, and $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ be such that $XT = LX$. If L has a nontrivial invariant subspace $\mathcal{M}$ such that $\mathcal{R}(X) \not\subseteq \mathcal{M}$ and $\mathcal{R}(X) \cap \mathcal{M} \neq \{0\}$, then $X^{-1}(\mathcal{M})$ is a nontrivial invariant subspace for T.*

Proposition 5.1 can be thought of as a generalization of Proposition 3.3, which motivates the following generalizations of Questions 4 and 5.

Question 6. Does every operator $\mathcal{D}$-intertwined to reducible operator have a nontrivial invariant subspace?

Question 7. Is every bounded linear transformation with dense range of class $\mathcal{D}$?

We close the paper with a partial answer to Question 6 in Proposition 5.2, and with examples of noninvertible transformations of class $\mathcal{D}$ in Proposition 5.3.

Proposition 5.2. *Let* $T\in\mathcal{B}[\mathcal{H}]$, $L\in\mathcal{B}[\mathcal{K}]$, *and* $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ *be such that* $XT = LX$. *If* $X\in\mathcal{D}$ *and* L *has a nontrivial reducing subspace* $\mathcal{M}$ *such that* $\dim\mathcal{M}=\infty$ *and* $\dim\mathcal{R}(X)^\perp<\dim\mathcal{M}^\perp$, *then either* $X^{-1}(\mathcal{M})$ *or* $X^{-1}(\mathcal{M}^\perp)$ *is a nontrivial invariant subspace for* T.

Proof. Suppose $X\in\mathcal{D}$ so that $\mathcal{R}(X)\cap\mathcal{M}\neq\{0\}$ or $\mathcal{R}(X)\cap\mathcal{M}^\perp\neq\{0\}$ by Definition 2. If $\mathcal{R}(X)\cap\mathcal{M}\neq\{0\}$ and $\dim\mathcal{R}(X)^\perp<\dim\mathcal{M}^\perp$, then $\mathcal{R}(X)\not\subseteq\mathcal{M}$. (Reason: $\mathcal{R}(X)\subseteq\mathcal{M}\Rightarrow\mathcal{M}^\perp\subseteq\mathcal{R}(X)^\perp\Rightarrow\dim\mathcal{M}^\perp\leq\dim\mathcal{R}(X)^\perp$.) Thus, by Proposition 5.1, $X^{-1}(\mathcal{M})$ is a nontrivial invariant subspace for T. On the other hand, if $\mathcal{R}(X)\cap\mathcal{M}^\perp\neq\{0\}$ and $\dim\mathcal{M}=\infty$, then $\mathcal{R}(X)\not\subseteq\mathcal{M}^\perp$. (Reason: since $\mathcal{M}=\mathcal{M}^-=(\mathcal{M}^\perp)^\perp$, we get $\mathcal{R}(X)\subseteq\mathcal{M}^\perp\Rightarrow\mathcal{M}\subseteq\mathcal{R}(X)^\perp\Rightarrow\dim\mathcal{M}\leq\dim\mathcal{R}(X)^\perp\Rightarrow\dim\mathcal{R}(X)^\perp=\infty$.) Again, by Proposition 5.1, $X^{-1}(\mathcal{M}^\perp)$ is a nontrivial invariant subspace for T. Outcome: either

$$\mathcal{R}(X)\cap\mathcal{M}\neq\{0\}\text{ and }\mathcal{R}(X)\not\subseteq\mathcal{M}\quad\text{or}\quad\mathcal{R}(X)\cap\mathcal{M}^\perp\neq\{0\}\text{ and }\mathcal{R}(X)\not\subseteq\mathcal{M}^\perp.$$

Since $\mathcal{M}$ and $\mathcal{M}^\perp$ are nontrivial and invariant for L, it follows by Proposition 5.1 that either $X^{-1}(\mathcal{M})$ or $X^{-1}(\mathcal{M}^\perp)$ is a nontrivial invariant subspace for T. □

The next proposition ensures the existence of examples of noninvertible transformation of class $\mathcal{C}$ (and therefore of class $\mathcal{D}$). For instance, let $\mathcal{R}$ be an infinite-dimensional Hilbert space, consider the one-dimensional space $\mathbb{C}$, let I be the identity operator on $\mathcal{R}$, and set $X=0\oplus I$ on $\mathcal{H}=\mathbb{C}\oplus\mathcal{R}$. Thus $\mathcal{R}(X)=\mathcal{R}$ and $\mathcal{R}(X)^\perp=\mathbb{C}$, and so X is of class $\mathcal{C}$ (and hence it is of class $\mathcal{D}$) according to Proposition 5.3 below.

Proposition 5.3. *Take* $X\in\mathcal{B}[\mathcal{H},\mathcal{K}]$. *If* $\mathcal{R}(X)^-=\mathcal{R}(X)$ *and* $\dim\mathcal{R}(X)^\perp<\infty$, *then* $X\in\mathcal{C}$.

Proof. Suppose $\mathcal{R}(X)$ is closed and $\dim\mathcal{R}(X)^\perp<\infty$. If $\dim\mathcal{R}(X)^\perp=0$ (i.e., if $\mathcal{R}(X)^\perp=\{0\}$, then $\mathcal{R}(X)^-=\mathcal{R}(X)=\mathcal{K}$, and $X\in\mathcal{C}$ trivially. Thus suppose $\dim\mathcal{R}(X)^\perp=n$ for some positive integer n. Let $\mathcal{M}$ be an arbitrary subspace of $\mathcal{K}$ with infinite dimension. We show that

$$\mathcal{R}(X)\cap\mathcal{M}\neq\{0\}.$$

Take $n+1$ linearly independent vectors $z_1,\dots,z_{n+1}$ in $\mathcal{M}$. Since $\mathcal{K}$ is a Hilbert space and $\mathcal{R}(X)$ is closed, it follows that $\mathcal{M}\subseteq\mathcal{K}=\mathcal{R}(X)+\mathcal{R}(X)^\perp$, and so there exist $x_1,\dots,x_{n+1}$ in $\mathcal{R}(X)$ and $y_1,\dots,y_{n+1}$ in $\mathcal{R}(X)^\perp$ such that

$$z_i=x_i+y_i$$

for each integer $i\in[1,n+1]$. Since $\dim\mathcal{R}(X)^\perp=n$, the finite sequence $\{y_i\}_{i=1}^{n+1}$ is made up of linearly dependent vectors in $\mathcal{R}(X)^\perp$, so that

$$\sum\nolimits_{i=1}^{n+1}\alpha_iy_i=0$$

for some finite sequence of $\{\alpha_i\}_{i=1}^{n+1}$ of scalars, not all null. Moreover, since the finite sequence $\{z_i\}_{i=1}^{n+1}$ is made up of linearly independent vectors in $\mathcal{M}$, it follows that

$$\sum\nolimits_{i=1}^{n+1}\alpha_iz_i=\sum\nolimits_{i=1}^{n+1}\alpha_ix_i+\sum\nolimits_{i=1}^{n+1}\alpha_iy_i=\sum\nolimits_{i=1}^{n+1}\alpha_ix_i\ \in\ \mathcal{R}(X)$$

by the above two displayed identities, and also

$$0 \neq \sum\nolimits_{i=1}^{n+1} \alpha_i z_i \;\in\; \mathcal{M}.$$

Therefore

$$0 \neq \sum\nolimits_{i=1}^{n+1} \alpha_i z_i \;\in\; \mathcal{R}(X) \cap \mathcal{M}.$$

Thus $X \in \mathcal{C}$. □

As a final remark we include an example yielding a negative answer to Question 7, which has been communicated to us by an anonymous referee.

Remark 5.1. Let $\mathbb{T}$ denote the unit circle in the complex place, let μ stand for the normalized Lebesgue measure on Borel subsets of $\mathbb{T}$, and consider the Hilbert space $L^2(\mathbb{T}) = L^2(\mathbb{T}, \mu)$. Take the standard orthonormal basis $\{e_k\} = \{e_k(z) = z^k;\, z \in \mathbb{T},\, k \in \mathbb{Z}\}$ for $L^2(\mathbb{T})$, and let H^2 be the Hardy space spanned by the elements of $\{e_k\}$ with nonnegative indices, viz., $H^2 = \bigvee_{k\in\mathbb{N}} e_k$. Consider the arcs

$$\mathbb{T}^+ = \{z \in \mathbb{T} \colon z = e^{i\theta},\; \theta \in [0, \pi]\},$$
$$\mathbb{T}^+_1 = \{z \in \mathbb{T}^+ \colon z = e^{i\theta},\; \theta \in [0, \tfrac{\pi}{2}]\},$$
$$\mathbb{T}^+_2 = \{z \in \mathbb{T}^+ \colon z = e^{i\theta},\; \theta \in [\tfrac{\pi}{2}, \pi]\},$$

and let $X \colon H^2 \to L^2(\mathbb{T}^+)$ be the map that takes $f \in H^2$ into $f|_{\mathbb{T}^+} \in L^2(\mathbb{T}^+)$. This is a bounded linear transformation between the Hilbert spaces H^2 and $L^2(\mathbb{T}^+)$ that has a dense range. Recalling that the only function in H^2 that vanishes on a set of positive measure is the null function, we get

$$\mathcal{R}(X) \cap L^2(\mathbb{T}^+_1) = \mathcal{R}(X) \cap L^2(\mathbb{T}^+_2) = \{0\}.$$

Since $L^2(\mathbb{T}^+) = L^2(\mathbb{T}^+_1) \oplus L^2(\mathbb{T}^+_2)$, so that $L^2(\mathbb{T}^+_2) = L^2(\mathbb{T}^+_1)^\perp$, it follows that X is not of class $\mathcal{D}$.

Acknowledgment

We thank Nilson Bernardes for crucial suggestions on earlier drafts of his paper, and an anonymous referee for bringing our attention to sensible points.

References

1. H. Bercovici, *Notes on invariant subspaces*, Bull. Amer. Math. Soc. **23** (1990), 1–36.
2. H. Bercovici and T. Smotzer, *Quasisimilarity of invariant subspaces for uniform Jordan operators of infinite multiplicity*, J. Functional Anal. **140** (1996), 87–99.
3. R. Clouâtre, *Quasisimilarity of invariant subspaces for C_0 operators with multiplicity two*, J. Operator Theory **70** (2013), 495–511.
4. P.A. Fillmore, *Notes on Operator Theory*, Van Nostrand, New York, 1970.
5. P.A. Fillmore and J.P. Williams, *On operator ranges*, Adv. Math. **7** (1971), 254–281.
6. V.I. Istrăţescu, *Introduction to Linear Operator Theory*, Marcel Dekker, New York, 1981.
7. P.R. Halmos, *Invariant subspaces 1969*, 7º Colóquio Brasileiro de Matemática, Poços de Caldas, 1969 [also in P.R. Halmos, *Selecta Research Contributions* (Springer, New York, 1983), 249–273].
8. L. Kérchy, *On invariant subspace lattices of C_{11} -contractions*, Acta Sci. Math. (Szeged) **43** (1981), 281–293.
9. L. Kérchy, *Invariant subspaces of $C_{1\cdot}$-contractions with nonreductive unitary extensions*, Bull. London Math. Soc. **19** (1987), 161–166.
10. L. Kérchy, *Quasi-similarity invariance of reflexivity*, Linear and Complex Analysis Problem Book 3 – Part 1 (Lecture Notes in Math. Vol. 1573, Springer, Berlin, 1994), 401–404.

11. C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhuser, Boston, 1997.
12. C.S. Kubrusly, *Equivalent invariant subspace problems*, J. Operator Theory **38** (1997), 323–328.
13. C.S. Kubrusly, *Invariant subspaces and quasiaffine transforms of unitary operators*, Progr. Nonlinear Differential Equations Appl. **42** (2000), 167–173.
14. C.S. Kubrusly, *Contractions T for which A is a projection*, Acta Sci. Math. (Szeged) **80** (2014), 603–624.
15. V. Müller and Y. Tomilov, *Quasisimilarity of power bounded operators and Blum–Hanson property*, J. Functional Anal. **246** (2007), 385–399.
16. H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, New York, 1973.
17. H. Radjavi, and P. Rosenthal, *The invariant subspace problem*, Math. Intelligencer **4** (1982), 33–37.
18. P. Rosenthal, *Equivalents of the invariant subspace problem*, Paul Halmos: Celebrating 50 Years of Mathematics (Springer, New York, 1991), 179–188.
19. B. Sz.-Nagy, C. Foiaş, H. Bercovici and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, Springer, New York, 2010; enlarged 2nd edn. of B. Sz.-Nagy and C. Foiaş, North-Holland, Amsterdam, 1970.
20. P.Y. Wu, *When is a contraction quasi-similar to an isometry*?, Acta Sci. Math. (Szeged) **44** (1982), 151–155.
21. P.Y. Wu, *Contractions quasisimilar to an isometry*, Acta Sci. Math. (Szeged) **53** (1989), 139–145.

Rural Federal University of Rio de Janeiro, Department of Technologies and Languages, Nova Iguaçu, RJ, Brazil
E-mail address: `aleksandrodemello@yahoo.com.br`

Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: `carlos@ele.puc-rio.br`