

ON WEAK SUPERCYCLICITY I

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ABSTRACT. This paper provides conditions (i) to distinguish weak supercyclicity from supercyclicity for operators acting on normed and Banach spaces, and also (ii) to ensure when weak supercyclicity implies weak stability.

1. INTRODUCTION

The purpose of this paper is to establish conditions to distinguish weak supercyclicity from supercyclicity for operators acting on normed spaces, and also to provide conditions on weakly supercyclic operators to ensure weak stability. Sections 2 and 3 deal with notation and terminology, and also offer a broad view on supercyclicity in the weak and norm topologies. Auxiliary results are considered in Section 4. Thus Sections 2, 3 and 4 present a brief survey on supercyclicity emphasizing the role played by weak supercyclicity. The new results appear in Sections 5 and 6. Theorem 5.1 characterizes weakly l-sequentially supercyclic vectors that are not supercyclic for a power bounded operator, and Theorems 6.1 and 6.2 exhibit a criterion to extend the results on supercyclicity and strong stability of [3] to weak l-sequential supercyclicity and weak stability. The main result is Theorem 6.2, and special classes of operators on Hilbert space are considered Corollaries 5.2 and 6.1.

2. NOTATION AND TERMINOLOGY

Let \mathbb{F} stand either for the complex field \mathbb{C} or for the real field \mathbb{R} , and let \mathcal{X} be an infinite-dimensional normed space over \mathbb{F} . Let A^- denote the closure (in the norm topology of \mathcal{X}) of a set $A \subseteq \mathcal{X}$. A subspace of \mathcal{X} is a *closed* linear manifold of \mathcal{X} . If \mathcal{M} is a linear manifold of \mathcal{X} , then its closure \mathcal{M}^- is a subspace. By an operator on \mathcal{X} we mean a linear bounded (i.e., continuous) transformation of \mathcal{X} into itself. Let $\mathcal{B}[\mathcal{X}]$ be the normed algebra of all operators on \mathcal{X} . A subspace \mathcal{M} of \mathcal{X} is invariant for an operator $T \in \mathcal{B}[\mathcal{X}]$ (or T -invariant) if $T(\mathcal{M}) \subseteq \mathcal{M}$, and it is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{X}$. Let $\|T\|$ stand for the induced uniform norm of T in $\mathcal{B}[\mathcal{X}]$.

An operator $T \in \mathcal{B}[\mathcal{X}]$ is power bounded if $\sup_{n \geq 0} \|T^n\| < \infty$, it is a contraction if $\|T\| \leq 1$ (i.e., if $\|T^n x\| \leq \|x\|$ for every $x \in \mathcal{X}$ and every integer $n \geq 0$), and it is an isometry if $\|T^n x\| = \|x\|$ for every $x \in \mathcal{X}$ and every integer $n \geq 0$. (On an inner product space, a unitary operator is precisely an invertible isometry). An operator $T \in \mathcal{B}[\mathcal{X}]$ is weakly or strongly stable (notation: $T^n \xrightarrow{w} O$ or $T^n \xrightarrow{s} O$) if the \mathcal{X} -valued sequence $\{T^n x\}_{n \geq 0}$ converges weakly or strongly (i.e., or in the norm topology of \mathcal{X}) to zero for every $x \in \mathcal{X}$. In other words, if $T^n x \xrightarrow{w} 0$, which means $f(T^n x) \rightarrow 0$ for every f in the dual \mathcal{X}^* of \mathcal{X} , for every x in \mathcal{X} ; or $T^n x \rightarrow 0$, which means $\|T^n x\| \rightarrow 0$, for every x in \mathcal{X} . An operator $T \in \mathcal{B}[\mathcal{X}]$ is uniformly stable (notation: $T^n \xrightarrow{u} O$) if the $\mathcal{B}[\mathcal{X}]$ -valued sequence $\{T^n\}_{n \geq 0}$ converges to the null operator O in the induced uniform norm of $\mathcal{B}[\mathcal{X}]$, which means $\|T^n\| \rightarrow 0$.

The orbit $\mathcal{O}_T(y)$ of a vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the set

$$\mathcal{O}_T(y) = \bigcup_{n \geq 0} T^n y = \{T^n y \in \mathcal{X} : n \in \mathbb{N}_0\},$$

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where \mathbb{N}_0 denotes the set of nonnegative integers — we write $\bigcup_{n \geq 0} T^n y$ for the set $\bigcup_{n \geq 0} T^n(\{y\}) = \bigcup_{n \geq 0} \{T^n y\}$. The orbit $\mathcal{O}_T(A)$ of a set $A \subseteq \mathcal{X}$ under an operator T is the set $\mathcal{O}_T(A) = \bigcup_{n \geq 0} T^n(A) = \bigcup_{y \in A} \mathcal{O}_T(y)$. For any set $A \subseteq \mathcal{X}$ let $\text{span } A$ be the (linear) span of A (the linear manifold spanned by A). The projective orbit of a vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the orbit of the one-dimensional space spanned by the singleton $\{y\}$; that is, it is the orbit of span of $\{y\}$:

$$\mathcal{O}_T(\text{span}\{y\}) = \bigcup_{n \geq 0} T^n(\text{span}\{y\}) = \{\alpha T^n y \in \mathcal{X} : \alpha \in \mathbb{F}, n \in \mathbb{N}_0\}.$$

A vector y in \mathcal{X} is a *cyclic vector* for an operator T in $\mathcal{B}[\mathcal{X}]$ if \mathcal{X} is the smallest invariant subspace for T containing y . Equivalently, $y \in \mathcal{X}$ is a cyclic vector for T if its orbit spans \mathcal{X} :

$$(\text{span } \mathcal{O}_T(y))^- = \mathcal{X}.$$

Still equivalently, y is a cyclic vector for T if $\{p(T)y : p \text{ is a polynomial}\}^- = \mathcal{X}$, which means $\{Sy : S \in \mathcal{P}(T)\}^- = \mathcal{X}$, where $\mathcal{P}(T)$ is the algebra of all polynomials in T with scalar coefficients. An operator $T \in \mathcal{B}[\mathcal{X}]$ is a *cyclic operator* if it has a cyclic vector. Stronger forms of cyclicity are defined as follows. A vector y in \mathcal{X} is a *supercyclic vector* for an operator T in $\mathcal{B}[\mathcal{X}]$ if its projective orbit is dense in \mathcal{X} in the norm topology; that is, if

$$\mathcal{O}_T(\text{span}\{y\})^- = \mathcal{X}.$$

An operator T in $\mathcal{B}[\mathcal{X}]$ is a *supercyclic operator* if it has a supercyclic vector. Moreover, a vector y in \mathcal{X} is a *hypercyclic vector* for an operator T in $\mathcal{B}[\mathcal{X}]$ if the orbit of y is dense in \mathcal{X} in the norm topology; that is, if

$$\mathcal{O}_T(y)^- = \mathcal{X}.$$

An operator T in $\mathcal{B}[\mathcal{X}]$ is a *hypercyclic operator* if it has a hypercyclic vector.

Versions in the weak topology of the above notions read as follows. Let A^{-w} denote the weak closure of a set $A \subseteq \mathcal{X}$ (i.e., the closure of A in the weak topology of \mathcal{X}). A vector y in \mathcal{X} is a *weakly cyclic vector* for an operator T in $\mathcal{B}[\mathcal{X}]$ if

$$(\text{span } \mathcal{O}_T(y))^{-w} = \mathcal{X},$$

and T in $\mathcal{B}[\mathcal{X}]$ is a *weakly cyclic operator* if it has a weakly cyclic vector. (Weak cyclicity, however, collapses to plain cyclicity according to Remark 3.1(f) below). A vector y in \mathcal{X} is a *weakly supercyclic vector* for an operator T in $\mathcal{B}[\mathcal{X}]$ if

$$\mathcal{O}_T(\text{span}\{y\})^{-w} = \mathcal{X},$$

and T in $\mathcal{B}[\mathcal{X}]$ is a *weakly supercyclic operator* if it has a weakly supercyclic vector. A vector y in \mathcal{X} is a *weakly hypercyclic vector* for an operator T in $\mathcal{B}[\mathcal{X}]$ if

$$\mathcal{O}_T(y)^{-w} = \mathcal{X},$$

and T in $\mathcal{B}[\mathcal{X}]$ is a *weakly hypercyclic operator* if it has a weakly hypercyclic vector. (For a treatise on hypercyclicity see [14].)

Remark 2.1. Although we will not deal with n -supercyclicity in this paper, we just pose definitions for sake of completeness: an operator $T \in \mathcal{B}[\mathcal{X}]$ is *n -supercyclic* (*weakly n -supercyclic*) if there is an n -dimensional subspace of \mathcal{X} whose orbit under T is dense (weakly dense) in \mathcal{X} . So a one-supercyclic (weakly one-supercyclic) is precisely a supercyclic (weakly supercyclic) operator. For each $n \geq 1$ there are examples of n -supercyclic operators that are not $(n-1)$ -supercyclic (see, e.g., [4, p.2]).

3. PRELIMINARIES

If T has a cyclic vector, then \mathcal{X} is separable (because \mathcal{X} is spanned by the countable set $\mathcal{O}_T(y)$ — see, e.g., [19, Proposition 4.9]), and so cyclic operators exist only on separable normed space; in particular, supercyclic and hypercyclic operators (as well as weakly cyclic, weakly supercyclic, and weakly hypercyclic) operators only exist on separable normed spaces (thus separability is not an assumption, but a consequence of cyclicity).

Remark 3.1. For each vector $y \in \mathcal{X}$ consider its punctured projective orbit; that is, its projective orbit under an operator $T \in \mathcal{B}[\mathcal{X}]$ excluding the origin,

$$\mathcal{O}_T(\text{span}\{y\}) \setminus \{0\} = \{\alpha T^n y \in \mathcal{X} : \alpha \in \mathbb{F} \setminus \{0\}, n \in \mathbb{N}_0\} \setminus \{0\}.$$

For each $z \in \mathcal{O}_T(\text{span}\{y\}) \setminus \{0\}$ the set $\mathcal{O}_T(\text{span}\{y\}) \setminus \mathcal{O}_T(\text{span}\{z\})$ is a finite union of one-dimensional subspaces of \mathcal{X} . So if $y \in \mathcal{X}$ is supercyclic (weakly supercyclic) for T , then every $z \in \mathcal{O}_T(\text{span}\{y\}) \setminus \{0\}$ is supercyclic (weakly supercyclic) for T :

- (a) if a vector y is supercyclic or weakly supercyclic for T , then so is $\alpha T^m y$ for every $0 \neq \alpha \in \mathbb{F}$ and every integer $m \geq 0$.

In particular, item (b) below on supercyclic and weakly supercyclic vectors are immediately verified, and item (c) is straightforward since supercyclicity is defined in terms of denseness in the norm topology (which is metrizable).

- (b) Every nonzero multiple of a supercyclic (weakly supercyclic) vector for an operator is again a supercyclic (weakly supercyclic) vector for it. Hence an operator is supercyclic (weakly supercyclic) if and only if any nonzero multiple of it is supercyclic (weakly supercyclic).
- (c) Since the norm topology is metrizable, a nonzero vector y in \mathcal{X} is a supercyclic vector for an operator T in $\mathcal{B}[\mathcal{X}]$ if and only if for every $x \in \mathcal{X}$ there exists an \mathbb{F} -valued sequence $\{\alpha_k\}_{k \geq 0}$ (which depends on x and y and consists of nonzero numbers) such that for some subsequence $\{T^{n_k}\}_{k \geq 0}$ of $\{T^n\}_{n \geq 0}$ the \mathcal{X} -valued sequence $\{\alpha_k T^{n_k} y\}_{k \geq 0}$ converges to x (in the norm topology):

$$\alpha_k T^{n_k} y \longrightarrow x.$$

(i.e., $\|\alpha_k T^{n_k} y - x\| \rightarrow 0$). If $\mathbb{F} = \mathbb{C}$ and $\{\alpha_k\}_{k \geq 0}$ is constrained to be \mathbb{R} -valued, then the notion of supercyclicity is referred to as \mathbb{R} -supercyclicity [6].

- (d) If a vector y is supercyclic (or hypercyclic) for an operator T , then y is supercyclic (or hypercyclic) for every positive power T^n of T [2, Theorems 1 and 2]. Hence if an operator is supercyclic (or hypercyclic) then so is every positive power T^n of it.
- (e) If an operator T is supercyclic (weakly supercyclic), then the set of all supercyclic (weakly supercyclic) vectors for it is dense (weakly dense).

Indeed, if y is a supercyclic (weakly supercyclic) vector for T , then the punctured projective orbit $\mathcal{O}_T(\text{span}\{y\}) \setminus \{0\}$ is dense (weakly dense) in \mathcal{X} . But according to item (a) $\mathcal{O}_T(\text{span}\{y\}) \setminus \{0\}$ is included in the set of all supercyclic (weakly supercyclic) vectors for T , and so the set of all supercyclic (weakly supercyclic) vectors is dense (weakly dense) in \mathcal{X} as well. In fact, weakly dense can be extended to dense (in the norm topology) [25, Proposition 2.1].

- (f) Denseness (in the norm topology) implies weak denseness (and the converse fails). However, if a set A is convex, then $A^- = A^{-w}$ (e.g., see [23, Theorem 2.5.16]), and so cyclicity coincides with weak cyclicity, since span is convex.
- (g) An operator has a nontrivial invariant subspace if and only if it has a nonzero noncyclic vector. Since cyclicity coincides with weak cyclicity, it follows that if every nonzero vector in \mathcal{X} is cyclic for T in any form of cyclicity discussed here (see Diagram 1 below), then T has no nontrivial invariant subspace.

Definitions of Section 2 and Remark 3.1(f) ensure the following relations.

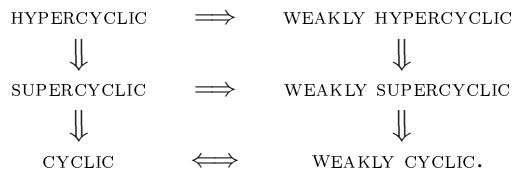


Diagram 1.

Thus cyclicity (i.e., cyclicity in the norm topology), which coincides with weak cyclicity, is the weakest form (in the sense that it is implied by the other forms) of cyclicity among those notions of cyclicity considered here.

Proposition 3.1. *Diagram 1 is complete.*

Proof. (a) Classical examples. Consider the (complex separable) Hilbert space ℓ_+^2 (of all complex-valued square-summable sequences). Let S be the (canonical) unilateral shift (of multiplicity one) on ℓ_+^2 and take its adjoint S^* ; a backward unilateral shift on ℓ_+^2 . Both S and S^* are cyclic operators, and while S^{*2} is cyclic, S^2 is not cyclic [15, Problem 160], but S is not supercyclic [17, p.564] (actually, every isometry is not supercyclic — [3, Proof of Theorem 2.1]), while S^* is supercyclic (in fact, the adjoint of every injective unilateral weighted shift is supercyclic) [17, Theorem 3] but they are not hypercyclic (since S and so S^* , S^2 , and S^{*2} are contractions); although $2S^*$ is hypercyclic [15, Solution 168] (and so S^* must indeed be supercyclic). This shows that there is no upward arrow on the left-hand column.

(b) There are weakly supercyclic operators that are not supercyclic [25, Corollary to Theorem 2.2], and there are weakly hypercyclic operator that are not hypercyclic [9, Corollary 3.3]. (Examples were all built in [25] and [9] in terms of bilateral weighted shifts on ℓ^p for $2 \leq p < \infty$.) This shows that there is no leftward arrow between the two columns (except the lower row).

(c₁) There are weakly supercyclic operators that are not weakly hypercyclic. Indeed, it was shown in [25, Theorem 4.5] that hyponormal operators (on Hilbert space) are not weakly hypercyclic (neither supercyclic [8, Theorem 3.1]) but they can be weakly supercyclic: there are weakly supercyclic unitary operators [4, Example 3.6].

(c₂) Finally, to exhibit weakly cyclic operators that are not weakly supercyclic proceed, for instance, as follows. Every weakly supercyclic hyponormal operator is a multiple of a unitary [4, Theorem 3.4]. Since the canonical unilateral shift S , which is hyponormal, is cyclic (or, equivalently, weakly cyclic), and since it is a completely nonunitary isometry (and therefore not a multiple of a unitary), it is not weakly supercyclic. (Another proof is exhibited in the forthcoming Proposition 4.1(b).)

(c) From (c₁) and (c₂), there is no upward arrow on the right-hand column. \square

Remark 3.2. This refers to items in the proof of Proposition 3.1. All examples in item (a) were based on the unilateral shift. However, a basic example of a cyclic operator that is not supercyclic is given by normal operators: no normal operator is supercyclic [17, p.564] (actually, no hyponormal operator is supercyclic [8, Theorem 3.1]) and there exist cyclic normal operators on separable Hilbert spaces (by the Spectral Theorem — see, e.g., [20, Proof of Theorem 3.11]). Although all examples in item (b) were built in [9] and [25] in terms of bilateral weighted shifts on ℓ^p for $2 \leq p < \infty$, it was shown in [24, Theorem 6.3] that a bilateral weighted shifts on ℓ^p for $1 \leq p < 2$ is weakly supercyclic if and only if it is supercyclic. The arguments used in items (a) and (c) were based on hyponormal and cohyponormal operators (an operator $T \in \mathcal{B}[\mathcal{X}]$ on a Hilbert space \mathcal{X} is hyponormal if $\|T^*x\| \leq \|Tx\|$ for every $x \in \mathcal{X}$, where $T^* \in \mathcal{B}[\mathcal{X}]$ is the adjoint of T , and cohyponormal if its adjoint is hyponormal). Such a circle of ideas has been extended from hyponormal to paranormal operators and beyond [10, Corollary 3.1], [11, Theorem 2.7] but we refrain from going further than hyponormal operators here to keep up with the focus on weak supercyclicity (and plain supercyclicity) only.

4. AUXILIARY RESULTS

Items (a) and (b) of next lemma first appeared embedded in a proof of another result in [3, Proof of Theorem 2.1]. The proof's argument is to show that if an isometry has a supercyclic vector, then every vector is supercyclic, which leads to a contradiction if \mathcal{X} is a Banach space. We isolate this result in Lemma 4.1(a,b).

Lemma 4.1. *Let \mathcal{X} be an arbitrary (nonzero) normed space.*

- (a) *A supercyclic isometry on \mathcal{X} has no nontrivial invariant subspace.*
- (b) *An isometry on a complex Banach space is never supercyclic.*
- (c) *There exist isometries on a complex Hilbert space that are weakly supercyclic.*

Proof. (a) Let $V \in \mathcal{B}[\mathcal{X}]$ be an isometry on a normed space \mathcal{X} , which means $\|V^n z\| = \|z\|$ for every $z \in \mathcal{X}$ and every integer $n \geq 0$. Suppose V is supercyclic. Let $0 \neq y \in \mathcal{X}$ be a supercyclic vector for V (with no loss of generality set $\|y\| = 1$) and take an arbitrary nonzero $z \in \mathcal{X}$. Then there is a scalar-valued sequence of nonzero numbers $\{\alpha_k\}_{k \geq 0}$ such that $\alpha_k V^{n_k} y \rightarrow z$ for some subsequence $\{V^{n_k}\}_{k \geq 0}$ of $\{V^n\}_{n \geq 0}$. Take an arbitrary $\varepsilon > 0$ so that

$$\|\alpha_k V^{n_k} y - z\| < \varepsilon$$

for k large enough. Observe that $\{\alpha_k\}_{k \geq 0}$ is bounded (reason: since V is an isometry, $|\alpha_k| = \|\alpha_k V^{n_k} y\|$ and so boundedness of the convergent sequence $\{\alpha_k V^{n_k} y\}_{k \geq 0}$ implies boundedness of $\{\alpha_k\}_{k \geq 0}$). Thus set $\alpha = \sup_n |\alpha_n| > 0$. Take an arbitrary $\delta > 0$. Since the above displayed convergence holds for every $0 \neq z \in \mathcal{X}$, take an arbitrary nonzero $x \in \mathcal{X}$ so that for every $\delta > 0$ there exists a nonzero number β and a positive integer m for which

$$\|\beta V^m y - x\| < \delta.$$

Note that $\|x\| - \delta < |\beta|$ (in fact, $\|x\| - |\beta| = \|x\| - \|\beta V^m y\| \leq \|\beta V^m y - x\| < \delta$ since V is an isometry). Moreover, by the above inequality, for every $n_k \geq m$

$$\|\beta V^{n_k} y - V^{n_k - m} x\| = \|V^{n_k - m}(\beta V^m y - x)\| = \|\beta V^m y - x\| < \delta.$$

In particular, take any δ such that $0 < \delta < \frac{\varepsilon\|x\|}{\alpha+\varepsilon}$. Thus $\delta\alpha < \varepsilon(\|x\| - \delta) < \varepsilon|\beta|$. Multiply both sides of the above inequality by $\frac{|\alpha_k|}{|\beta|}$ to get

$$\left\| \alpha_k V^{n_k} y - \frac{\alpha_k}{\beta} V^{n_k-m} x \right\| < \delta \frac{|\alpha_k|}{|\beta|} \leq \delta \frac{\alpha}{|\beta|} < \varepsilon$$

for every $n_k \geq m$. Therefore, since $\|\alpha_k V^{n_k} y - z\| < \varepsilon$ for k large enough,

$$\left\| \frac{\alpha_k}{\beta} V^{n_k-m} x - z \right\| \leq \left\| \frac{\alpha_k}{\beta} V^{n_k-m} x - \alpha_k V^{n_k} y \right\| + \|\alpha_k V^{n_k} y - z\| < 2\varepsilon$$

for k large enough, which means $\frac{\alpha_k}{\beta} V^{n_k-m} x \rightarrow z$, and so there exists a sequence $\{\alpha_j\}_{j \geq 0}$ of nonzero numbers such that $\alpha_j V^{n_j} x \rightarrow z$ for some subsequence $\{V^{n_j}\}_{j \geq 0}$ of $\{V^n\}_{n \geq 0}$. Since z and x are arbitrary nonzero vectors in \mathcal{X} , this ensures that every vector in \mathcal{X} is supercyclic for V , and hence V has no nontrivial invariant subspace (cf. Remark 3.1(g)).

(b) The result in item (a) leads to a contradiction if \mathcal{X} is a complex Banach space because in this case isometries do have nontrivial invariant subspaces. In fact, if an isometry V on a Banach space is not surjective, then $\mathcal{R}(V)$ is a nontrivial invariant (hyperinvariant, actually) subspace for V because on a Banach space isometries have closed range (see, e.g., [19, Problem 4.41(d)]). On the other hand, since isometries are always injective, if V is a surjective isometry then it is invertible (whose inverse also is an isometry) so that $\|V^n\| = \|V^{-n}\| = 1$ for every $n \geq 0$. Thus surjective isometries are power bounded with a power bounded inverse. But *a nonscalar invertible power bounded operator on a complex Banach space with a power bounded inverse has a nontrivial invariant (hyperinvariant, actually) subspace*. (See, e.g., [1, Theorem 10.79] — this is the Banach space counterpart of a well-known result due to Sz.-Nagy which says: *an invertible power bounded operator on a Hilbert space with a power bounded inverse is similar to a unitary operator*; see, e.g., [18, Corollary 1.16]). Thus an isometry on a complex Banach space has a nontrivial invariant subspace (see also [13, Theorem J]) and so it cannot be supercyclic according to (a).

(c) There are weakly supercyclic unitary operators on a complex Hilbert space [4, Example 3.6] (see also [26, Theorem 2] and [28, Theorem 1.2]). Thus there are (invertible) weakly supercyclic isometries on a Hilbert space. \square

If \mathcal{X} is a Hilbert space, then a completely nonunitary contraction is a contraction such that no restriction of it to a reducing subspace is unitary (i.e., such that every direct summand of it is not unitary), and a completely nonunitary isometry (i.e., a pure isometry) is precisely a unilateral shift of some multiplicity. These are consequences of Nagy–Foiş–Langer decomposition for contractions and von Neumann–Wold decomposition for isometries (see e.g., [29, pp.3,8] or [18, pp.76,81]).

Proposition 4.1. (a) *A weakly supercyclic isometry on a Hilbert space is unitary.*
 (b) *Every unilateral shift on a Hilbert space is not weakly supercyclic.*

Proof. (a) If a hyponormal operator is weakly supercyclic, then it is a multiple of a unitary [4, Theorem 3.4]. Since isometries on a Hilbert space are hyponormal with norm 1, then a weakly supercyclic isometry on a Hilbert space is unitary.

(b) A unilateral shift, of any multiplicity, on a Hilbert space is a completely nonunitary isometry, thus not weakly supercyclic by item (a). \square

5. WEAK AND STRONG SUPERCYCLICITY

The weak counterpart of the supercyclicity criterion described in Remark 3.1(c) was considered in [7] (also in [4] implicitly), and it was referred to as weak 1-sequential supercyclicity in [28]. Although there are reasons for such a terminology, we will change it here to weak l-sequential supercyclicity, replacing the numeral “1” with the letter “l” for “limit”. A nonzero vector y in \mathcal{X} is a *weakly l-sequentially supercyclic vector* for an operator T in $\mathcal{B}[\mathcal{X}]$ if for every $x \in \mathcal{X}$ there exists an \mathbb{F} -valued sequence $\{\alpha_k\}_{k \geq 0}$ (which depends on x and y and consists of nonzero numbers) such that for some subsequence $\{T^{n_k}\}_{k \geq 0}$ of $\{T^n\}_{n \geq 0}$ the \mathcal{X} -valued sequence $\{\alpha_k T^{n_k} y\}_{k \geq 0}$ converges weakly to x . That is,

$$\alpha_k T^{n_k} y \xrightarrow{w} x.$$

This means the projective orbit $\mathcal{O}_T(\text{span}\{y\})$ of the vector y under T is weakly l-sequentially dense in \mathcal{X} in the following sense. For any set $A \subseteq \mathcal{X}$ let A^{-wl} denote the set of all weak limits of weakly convergent A -valued sequences, and A is said to be weakly l-sequentially dense in \mathcal{X} if $A^{-wl} = \mathcal{X}$. Thus y is a weakly l-sequentially supercyclic vector for T if and only if

$$\mathcal{O}_T(\text{span}\{y\})^{-wl} = \mathcal{X}.$$

An operator T in $\mathcal{B}[\mathcal{X}]$ is a *weakly l-sequentially supercyclic operator* if it has a weakly l-sequentially supercyclic vector. Observe that

$$\text{SUPERCYCLIC} \implies \text{WEAKLY L-SEQUENTIALLY SUPERCYCLIC} \implies \text{WEAKLY SUPERCYCLIC},$$

and the converses fail (see, e.g., [28, pp.38,39], [5, pp.259,260]).

We will be dealing with normed spaces \mathcal{X} with the following property: an \mathcal{X} -valued sequence $\{x_k\}_{k \geq 0}$ converges strongly (i.e., in the norm topology) if and only if it converges weakly and the norm sequence $\{\|x_k\|\}_{k \geq 0}$ converges to the limit's norm; that is, $x_k \rightarrow x \iff \{x_k \xrightarrow{w} x \text{ and } \|x_k\| \rightarrow \|x\|\}$. We say that a normed space \mathcal{X} that has the above property is a normed space of type 1. (Also called a Radon–Riesz space, whose property is called the Radon–Riesz property — see, e.g., [23, Definition 2.5.26]). Hilbert spaces are Banach spaces of type 1 [15, Problem 20].

Theorem 5.1. *Suppose T is an operator on a type 1 normed space \mathcal{X} . If*

- (a) T is power bounded,
- (b) $y \in \mathcal{X}$ is a weakly l-sequentially supercyclic vector for T ,
- (c) $y \in \mathcal{X}$ is not a supercyclic vector for T ,

then

- (d) every nonzero $f \in \mathcal{X}^*$ is such that either
 - (d₁) $\liminf_n |f(T^n y)| = 0$, or
 - (d₂) $\limsup_k |f(T^{n_k} y)| < \|f\| \limsup_k \|T^{n_k} y\|$ for some subsequence $\{T^{n_k}\}_{k \geq 0}$ of $\{T^n\}_{n \geq 0}$.

Proof. First we need the following auxiliary result.

Claim 1. If $z_k, z \in \mathcal{X}$, where \mathcal{X} is a normed space of type 1, and if the sequence $\{z_k\}_{k \geq 0}$ is such that $z_k \xrightarrow{w} z$ and $z_k \not\rightarrow z$, then $\|z\| < \limsup_k \|z_k\|$.

Proof. If \mathcal{X} is an arbitrary normed space, then $z_k \xrightarrow{w} z$ implies $\|z\| \leq \liminf_k \|z_k\|$ (see, e.g., [16, Proposition 46.1]). Thus, if $z_k \not\rightarrow z$ and \mathcal{X} is a normed space of type 1, then $\|z_k\| \not\rightarrow \|z\|$ so that $\|z\| \leq \liminf_k \|z_k\| < \limsup_k \|z_k\|$. \square

Now consider assumptions (a), (b), and (c), and suppose (d) fails; that is, suppose the contradictory (\not{d}) of (d) holds:

(\not{d}) there exists a nonzero $f_0 \in \mathcal{X}^*$ such that

$$(\not{d}_1) \quad 0 < \liminf_n |f_0(T^n y)| \quad \text{and}$$

$$(\not{d}_2) \quad \limsup_k |f_0(T^{n_k} y)| = \|f_0\| \limsup_k \|T^{n_k} y\| \quad \text{for every subsequence } \{T^{n_k}\}_{k \geq 0} \text{ of } \{T^n\}_{n \geq 0}.$$

According to assumption (b) let $0 \neq y \in \mathcal{X}$ be a weakly 1-sequentially supercyclic vector for T . Thus for every $x \in \mathcal{X}$ there exists a scalar-valued sequence $\{\beta_\ell\}_{\ell \geq 0}$ (depending on x and y) such that

$$\beta_\ell T^{n_\ell} y \xrightarrow{w} x$$

for some subsequence $\{T^{n_\ell}\}_{\ell \geq 0}$ of $\{T^n\}_{n \geq 0}$. By assumption (c) suppose y is not a supercyclic vector for T . So there exists a nonzero vector $x_0 \in \mathcal{X}$ such that

$$\gamma_\ell T^{n_\ell} y \not\rightarrow x_0$$

for every sequence of numbers $\{\gamma_\ell\}_{\ell \geq 0}$ and every subsequence $\{T^{n_\ell}\}_{\ell \geq 0}$ of $\{T^n\}_{n \geq 0}$. Then there is a scalar-valued sequence $\{\alpha_j\}_{j \geq 0}$ (depending on x_0 and y) such that

$$\alpha_j T^{n_j} y \xrightarrow{w} x_0$$

for some subsequence $\{T^{n_j}\}_{j \geq 0}$ of $\{T^n\}_{n \geq 0}$, and

$$\alpha_i T^{n_i} y \not\rightarrow x_0$$

for every subsequence $\{T^{n_i}\}_{i \geq 0} = \{T^{n_{j_i}}\}_{i \geq 0}$ of $\{T^{n_j}\}_{j \geq 0}$ and every subsequence $\{\alpha_i\}_{i \geq 0} = \{\alpha_{j_i}\}_{i \geq 0}$ of $\{\alpha_j\}_{j \geq 0}$. Next consider assumption (\not{d}) which says: there is a nonzero $f_0 \in \mathcal{X}^*$ (which depends on y) satisfying (\not{d}_1) and (\not{d}_2) . Since $\alpha_j T^{n_j} y \xrightarrow{w} x_0$ we get $|f_0(x_0)| = \lim_j |f_0(\alpha_j T^{n_j} y)|$ for every $f \in \mathcal{X}^*$. In particular,

$$|f_0(x_0)| = \lim_j |f_0(\alpha_j T^{n_j} y)| = \lim_j |\alpha_j| |f_0(T^{n_j} y)|.$$

Hence $\limsup_j |\alpha_j| < \infty$ by (\not{d}_1) . (Indeed, $0 < \liminf_n |f_0(T^n y)| \in \mathbb{R}$ by (a) and (\not{d}_1) and so for every $\varepsilon \in (0, \liminf_n |f_0(T^n y)|)$ there exists a positive integer n_ε such that if $n \geq n_\varepsilon$ then $\varepsilon < |f_0(T^n y)|$, and hence $\limsup_j |\alpha_j| < \infty$ since $|f_0(x_0)| \in \mathbb{R}$.) Thus there is a subsequence $\{\alpha_k\}_{k \geq 0} = \{\alpha_{j_k}\}_{k \geq 0}$ of $\{\alpha_j\}_{j \geq 0}$ such that

$$\{|\alpha_k|\}_{k \geq 0} \text{ converges.}$$

Take this subsequence $\{\alpha_k\}_{k \geq 0}$ of $\{\alpha_j\}_{j \geq 0}$ and take the corresponding subsequence $\{T^{n_k}\}_{k \geq 0}$ of $\{T^{n_j}\}_{j \geq 0}$ so that $\alpha_k T^{n_k} y \xrightarrow{w} x_0$ (because $\alpha_j T^{n_j} y \xrightarrow{w} x_0$). Then

$$\alpha_k T^{n_k} y \xrightarrow{w} x_0 \quad \text{and} \quad \alpha_k T^{n_k} y \not\rightarrow x_0.$$

Therefore, according to Claim 1,

$$\|x_0\| < \limsup_k \|\alpha_k T^{n_k} y\|.$$

Again, since $\alpha_k T^{n_k} y \xrightarrow{w} x_0$, it follows that

$$|f_0(x_0)| = \lim_k |f_0(\alpha_k T^{n_k} y)|.$$

Note: if $\{\xi_k\}_{k \geq 0}$ and $\{\zeta_k\}_{k \geq 0}$ are bounded sequences of nonnegative real numbers such that $\{\xi_k\}_{k \geq 0}$ converges, then $\lim_k \xi_k \limsup_k \zeta_k = \limsup_k \xi_k \zeta_k$. (In fact, if

$\xi_k \rightarrow \xi$ then for every $\varepsilon > 0$ there exists an integer $k_\varepsilon \geq 1$ such that if $k \geq k_\varepsilon$ then $\xi \zeta_k - \xi_k \zeta_k < \varepsilon \zeta_k \leq \varepsilon \sup_k \zeta_k$, and hence $\lim_k \xi_k \limsup_k \zeta_k \leq \limsup_k \xi_k \zeta_k \leq \limsup_k \xi_k \limsup_k \zeta_k = \lim_k \xi_k \limsup_k \zeta_k$. Thus, since $\sup_k |f_0(T^{n_k}y)| < \infty$ by assumption (a) and since $\{\alpha_k\}_{k \geq 0}$ converges, we get

$$\begin{aligned} \limsup_k |\alpha_k| \limsup_k |f_0(T^{n_k}y)| &= \lim_k |\alpha_k| \limsup_k |f_0(T^{n_k}y)| \\ &= \limsup_k |\alpha_k| |f_0(T^{n_k}y)| \\ &= \limsup_k |f_0(\alpha_k T^{n_k}y)| \\ &= \lim_k |\alpha_k f_0(T^{n_k}y)|. \end{aligned}$$

Then, by the above three displayed expressions and (d_2) ,

$$\begin{aligned} \lim_k |f_0(\alpha_k T^{n_k}y)| &= |f_0(x_0)| \leq \|f_0\| \|x_0\| \\ &< \|f_0\| \limsup_k \|\alpha_k T^{n_k}y\| \\ &\leq \limsup_k |\alpha_k| \|f_0\| \limsup_k \|T^{n_k}y\| \\ &= \limsup_k |\alpha_k| \limsup_k |f_0(T^{n_k}y)| \\ &= \lim_k |f_0(\alpha_k T^{n_k}y)|, \end{aligned}$$

which is a contradiction. Therefore, (a), (b) and (c) imply (d). \square

Corollary 5.1. *If a power bounded operator T on a type 1 normed space \mathcal{X} is such that T is not supercyclic, then either*

- (i) T is not weakly l -sequentially supercyclic, or
- (ii) if $y \in \mathcal{X}$ is a weakly l -sequentially supercyclic vector for T , then every nonzero $f \in \mathcal{X}^*$ is such that either

$$\begin{aligned} \liminf_n |f(T^n y)| &= 0, \quad \text{or} \\ \limsup_k |f(T^{n_k} y)| &< \|f\| \limsup_k \|T^{n_k} y\| \text{ for some subsequence} \\ &\{T^{n_k}\}_{k \geq 0} \text{ of } \{T^n\}_{n \geq 0}. \end{aligned}$$

Proof. Immediate by Theorem 5.1. \square

Remark 5.1. This remark deals with operators on Hilbert spaces as it will be considered in the forthcoming Corollaries 5.2 and 6.1. Whenever we refer to a Hilbert space, the inner product in it will be denoted by $\langle \cdot ; \cdot \rangle$.

(a) Although hyponormal operators are never supercyclic [8, Theorem 3.1], neither weakly hypercyclic [25, Theorem 4.5], there exist weakly l -sequentially supercyclic hyponormal operators. In fact, every weakly supercyclic (in particular, every weakly l -sequentially supercyclic) hyponormal operator is a multiple of a unitary [4, Theorem 3.4], and there exist weakly supercyclic (in fact, weakly l -sequentially supercyclic) unitary operators [4, Example 3.6, pp.10,12] (see also [28, Question 1]). Thus a weakly l -sequentially supercyclic hyponormal contraction must be unitary. Corollary 5.2 below gives a condition for a hyponormal contraction (or a unitary operator) to be weakly l -sequentially supercyclic.

(b) The above discussion guarantees the existence of weakly l -sequentially supercyclic power bounded operators that are not supercyclic (see also [25, Corollary to Theorem 2.2]). Thus alternative (ii) in Corollary 5.1 cannot be dismissed. Actually, the existence of weakly l -sequentially supercyclic unitary operators (which are never supercyclic) shows alternative (ii) in Corollaries 5.2 and 5.3 below cannot

be dismissed as well; that is, it shows the existence of unitary operators satisfying condition (ii) in Corollaries 5.1, 5.2, and 5.3.

Corollary 5.2. *If a power bounded operator T on a Hilbert space \mathcal{X} is hyponormal, then it is a contraction and either*

- (i) T is not weakly l -sequentially supercyclic, or
- (ii) if $y \in \mathcal{X}$ is a weakly l -sequentially supercyclic vector for T , then T is unitary and every nonzero $z \in \mathcal{X}$ is such that either

$$\liminf_n |\langle T^n y; z \rangle| = 0, \quad \text{or}$$

$$\limsup_k |\langle T^{n_k} y; z \rangle| < \|z\| \|y\| \text{ for some subsequence } \{T^{n_k}\}_{k \geq 0} \text{ of } \{T^n\}_{n \geq 0}.$$

Proof. It is well known that if T is hyponormal, then it is normaloid (i.e., $\|T\|^n = \|T^n\|$ for every $n \geq 1$), and every power bounded normaloid operator is a contraction. A hyponormal operator on a Hilbert space is not supercyclic [8, Theorem 3.1]. Then apply Corollary 5.1 (replacing $f(x)$ with $\langle x; z \rangle$ according to the Riesz Representation Theorem in Hilbert space), and recall that a weakly supercyclic hyponormal contraction is unitary (cf. Remark 5.1), thus an isometry. \square

Corollary 5.3. *If T is an isometry on a type 1 Banach space \mathcal{X} , then either*

- (i) T is not weakly l -sequentially supercyclic, or
- (ii) if $y \in \mathcal{X}$ is a weakly l -sequentially supercyclic vector for T , then every nonzero $f \in \mathcal{X}^*$ is such that either

$$\liminf_n |f(T^n y)| = 0, \quad \text{or}$$

$$\limsup_k |f(T^{n_k} y)| < \|f\| \|y\| \text{ for some subsequence } \{T^{n_k}\}_{k \geq 0} \text{ of } \{T^n\}_{n \geq 0}.$$

Proof. If T is an isometry on a Banach space, then it is not supercyclic [3, Proof of Theorem 2.1] (Lemma 4.1). Thus apply Corollary 5.1. (In a Hilbert space setting this is a particular case of Corollary 5.2, where T is an invertible isometry). \square

6. WEAK SUPERCYCLICITY AND STABILITY

It was proved in [3, Theorem 2.1] that a power bounded operator T on a Banach space \mathcal{X} such that $\|T^n x\| \not\rightarrow 0$ for every $0 \neq x \in \mathcal{X}$ (i.e., a power bounded operator of class \mathcal{C}_1 .) has no supercyclic vector. The next result is a weak version of it.

Theorem 6.1. *If a power bounded operator T on a type 1 normed space \mathcal{X} is such that $T^n x \xrightarrow{w} 0$ for every $0 \neq x \in \mathcal{X}$, then either*

- (i) T has no weakly l -sequentially supercyclic vector, or
- (ii) if $y \in \mathcal{X}$ is a weakly l -sequentially supercyclic vector for T , then every nonzero $f \in \mathcal{X}^*$ for which $f(T^n y) \not\rightarrow 0$ is such that either

$$\liminf_n |f(T^n y)| = 0, \quad \text{or}$$

$$\limsup_k |f(T^{n_k} y)| < \|f\| \limsup_k \|T^{n_k} y\| \text{ for some subsequence } \{T^{n_k}\}_{k \geq 0} \text{ of } \{T^n\}_{n \geq 0}.$$

Proof. Consider the following result.

Claim 1. If a power bounded operator on any normed space is such that $T^n x \xrightarrow{w} 0$ for every $0 \neq x \in \mathcal{X}$, then it has no supercyclic vector.

Proof. If an operator T on a normed space \mathcal{X} is such that $T^n x \xrightarrow{w} 0$ for some (for every) $0 \neq x \in \mathcal{X}$, then it is clear that $T^n x \not\xrightarrow{w} 0$ for some (for every) $0 \neq x \in \mathcal{X}$ (strong convergence implies weak convergence to the same limit). It was proved in [3, Theorem 2.1] that a if power bounded operator T on a Banach space \mathcal{X} is such that $\|T^n x\| \not\xrightarrow{w} 0$ for every $0 \neq x \in \mathcal{X}$, then it has no supercyclic vector, whose proof survives in any normed space. \square

Thus under the theorem hypothesis, Claim 1 ensures T has no supercyclic vector. If, in addition, \mathcal{X} is a type 1 normed space and T does not satisfy condition (i) — that is, if T has a weakly l-sequentially supercyclic vector y — then condition (ii) holds by Theorem 5.1 (or Corollary 5.1). \square

It was proved in [3, Theorem 2.2] by using [3, Theorem 2.1] that a *Banach-space supercyclic power bounded operator is strongly stable*, whose proof in fact does not require completeness. Theorem 6.2 below is a weak version of it based on Theorem 6.1. Weakly l-sequentially supercyclic contractions on Hilbert space are characterized in Corollary 6.1 as a consequence of Theorem 6.2.

Theorem 6.2. *If a power bounded operator T on a type 1 normed space \mathcal{X} is weakly l-sequentially supercyclic, then either*

- (i) T is weakly stable, or
- (ii) if $y \in \mathcal{X}$ is a weakly l-sequentially supercyclic vector for T such that $T^n y \xrightarrow{w} 0$, then for every nonzero $f \in \mathcal{X}^*$ such that $f(T^n y) \not\xrightarrow{w} 0$ either
 - $\liminf_n |f(T^n y)| = 0$, or
 - $\limsup_k |f(T^{n_k} y)| < \|f\| \limsup_k \|T^{n_k} y\|$ for some subsequence $\{T^{n_k}\}_{k \geq 0}$ of $\{T^n\}_{n \geq 0}$.

Proof. First we show: if (i) fails, then there is a weakly l-sequentially supercyclic vector y such that $T^n y \xrightarrow{w} 0$. That is, if $T^n x \xrightarrow{w} 0$ for some $x \in \mathcal{X}$, then the set $\{y \in \mathcal{X} : y \text{ is a weakly l-sequentially supercyclic vector for } T \text{ such that } T^n y \xrightarrow{w} 0\}$ is nonempty.

Claim 1. Suppose T is a power bounded weakly l-sequentially supercyclic operator on a normed space \mathcal{X} . If there exists a vector $x \in \mathcal{X}$ such that $T^n x \xrightarrow{w} 0$, then there exists a weakly l-sequentially supercyclic vector $y \in \mathcal{X}$ for T such that $T^n y \xrightarrow{w} 0$.

Proof. Let $Y \subseteq \mathcal{X}$ denote the set of all weakly l-sequentially supercyclic vectors for an operator T , and so T is weakly l-sequentially supercyclic if and only if $Y \neq \emptyset$. The same argument of Remark 3.1(a) ensures $\mathcal{O}_T(\text{span}\{y\}) \setminus \{0\} \subseteq Y$ for every $y \in Y$ (see [22, Lemma 5.1]). Since $(\mathcal{O}_T(\text{span}\{y\}) \setminus \{0\})^{-wl} = \mathcal{X}$ for every $y \in Y$ (definition of weak l-sequential supercyclicity), then $Y \neq \emptyset \implies Y^{-wl} = \mathcal{X}$. However, more is true. Denseness is attained in the norm topology (cf. [22, Theorem 5.1]):

$$Y \neq \emptyset \implies Y^- = \mathcal{X}.$$

(A weak version of the above implication was considered in [25, Proposition 2.1], where Y is replaced by the set of all weakly supercyclic vectors — see Remark 3.1(e)). Take an arbitrary $x \in \mathcal{X}$. If $Y^- = \mathcal{X}$, then there is a Y -valued sequence $\{y_k\}$ such that $\|y_k - x\| \rightarrow 0$. If $T_n y \xrightarrow{w} 0$ for every $y \in Y$, for some sequence $\{T_n\}_{n \geq 0}$ of operators, which means $f(T_n y) \rightarrow 0$ for every f in the dual \mathcal{X}^* of \mathcal{X} and every y in Y , then $|f(T_n y_k)| \rightarrow 0$ for every f in \mathcal{X}^* and every integer k . Therefore since

$$|f(T_n x)| \leq |f(T_n(y_k - x))| + |f(T_n y_k)| \leq \|f\| \sup_n \|T_n\| \|y_k - x\| + |f(T_n y_k)|$$

for every $f \in \mathcal{X}^*$ and every $x \in \mathcal{X}$, we get $T_n x \xrightarrow{w} 0$ for every $x \in \mathcal{X}$. So if there is an $x \in \mathcal{X}$ such that $T_n x \xrightarrow{w} 0$, then there is a $y \in Y$ such that $T_n y \xrightarrow{w} 0$. \square

Now let T be an operator on a type 1 normed space \mathcal{X} and consider the following assumptions.

- (a) T is power bounded, and set $\beta = \sup_n \|T^n\| > 0$.
- (b) T is weakly l-sequentially supercyclic.
- (c) If $y \in \mathcal{X}$ is a weakly l-sequentially supercyclic vector for T such that $T^n y \xrightarrow{w} 0$, then for some $f_0 \in \mathcal{X}^*$ with $\|f_0\| = 1$ such that $f_0(T^n y) \not\rightarrow 0$,
 - (c₁) $0 < \liminf_n |f_0(T^n y)|$ and
 - (c₂) $\limsup_k |f_0(T^{n_k} y)| = \|f_0\| \limsup_k \|T^{n_k} y\|$ for every subsequence $\{T^{n_k}\}_{k \geq 0}$ of $\{T^n\}_{k \geq 0}$.

Claim 2. If $y \in \mathcal{X}$ is weakly l-sequentially supercyclic for T , then $T^n y \xrightarrow{w} 0$.

Proof. Under assumption (b) there exists a weakly l-sequentially supercyclic unit vector $y \in \mathcal{X}$ ($\|y\| = 1$) for T . Suppose

$$T^n y \xrightarrow{w} 0.$$

Under assumptions (a) and (c) Theorem 6.1 says there exists a unit vector v (i.e., $\|v\| = 1$) in \mathcal{X} such that $T^n v \xrightarrow{w} 0$. Since y is weakly l-sequentially supercyclic, there is a sequence $\{\alpha_k\}_{k \geq 0}$ of nonzero numbers such that $\alpha_k T^{n_k} y \xrightarrow{w} v$ for some subsequence $\{T^{n_k}\}_{k \geq 0}$. So, $f(\alpha_k T^{n_k} y) \rightarrow f(v)$ for every $f \in \mathcal{X}^*$. Take a unit vector $f \in \mathcal{X}^*$ ($\|f\| = 1$) for which $\frac{1}{2} \leq |f(v)| \leq 1$ (recall: $1 = \|v\| = \sup_{\|f\|=1} |f(v)|$). Thus there exists a positive integer k_f such that if $k \geq k_f$ then $||f(\alpha_k T^{n_k} y)| - |f(v)|| \leq |f(\alpha_k T^{n_k} y) - f(v)| < \frac{|f(v)|}{2}$, and hence $|f(v)| - |f(\alpha_k T^{n_k} y)| < \frac{|f(v)|}{2}$, which implies $\frac{1}{4} \leq \frac{|f(v)|}{2} = |f(v)| - \frac{|f(v)|}{2} < |f(\alpha_k T^{n_k} y)| \leq \|f\| |\alpha_k| \sup_n \|T^n\| \|y\| = \beta |\alpha_k|$ according to (a). Thus, for k large enough,

$$\frac{1}{4\beta} < |\alpha_k|.$$

Now take any unit vector $f_0 \in \mathcal{X}^*$ ($\|f_0\| = 1$) satisfying assumption (c) so that, according to (c₁), there exists a positive number δ such that

$$\delta < \liminf_n |f_0(T^n y)|.$$

Next take any positive γ such that $\gamma < \frac{\delta}{4\beta^2}$. Take an arbitrary integer $m \geq 0$. Note that $|f_0(\alpha_k T^{n_k+m} y - T^m v)| = |f_0(T^m(\alpha_k T^{n_k} y - v))| = |(T^{m*} f_0)(\alpha_k T^{n_k} y - v)| = |f_m(\alpha_k T^{n_k} y - T v)|$ for $f_m = T^{m*} f_0 \in \mathcal{X}^*$, where $T^{m*} \in \mathcal{B}[\mathcal{X}^*]$ is the normed-space adjoint of $T^m \in \mathcal{B}[\mathcal{X}]$ (see, e.g., [27, Section 3.2]). Since $\alpha_k T^{n_k} y \xrightarrow{w} v$, there is a positive integer k_m such that if $k \geq k_m$ then $|f_m(\alpha_k T^{n_k} y - v)| < \beta \frac{\gamma}{2}$. Thus, for any $m \geq 0$ and k large enough,

$$|f_0(\alpha_k T^{n_k+m} y - T^m v)| < \beta \frac{\gamma}{2}.$$

Finally, since $T^n v \xrightarrow{w} 0$, take m sufficiently large such that

$$|f_0(T^m v)| < \beta \frac{\gamma}{2}.$$

Then, by the above four displayed inequalities, for k and m large enough,

$$\begin{aligned}
\frac{\delta}{4\beta} &< \frac{\liminf_k |f_0(T^{n_k+m}y)|}{4\beta} \\
&< \liminf_k |f_0(T^{n_k+m}y)| \inf_k |\alpha_k| \leq \liminf_k |f_0(\alpha_k T^{n_k+m}y)| \\
&\leq \liminf_k |f_0(\alpha_k T^{n_k+m}y - T^m v)| + \liminf_k |f_0(T^m v)| \\
&< \beta \frac{\gamma}{2} + \beta \frac{\gamma}{2} = \beta \gamma < \frac{\delta}{4\beta},
\end{aligned}$$

which is a contradiction. Therefore if $y \in \mathcal{X}$ is weakly l-sequentially supercyclic for T , then $T^n y \xrightarrow{w} 0$. \square

By Claim 1 (which depends on assumptions (a) and (b)) if $T^n y \xrightarrow{w} 0$ for every weakly l-sequentially supercyclic vector y in \mathcal{X} for T , then $T^n x \xrightarrow{w} 0$ for every x in \mathcal{X} . So the result in Claim 2 (which depends on assumptions (a), (b) and (c)) ensures T is weakly stable. Thus, under assumptions (a) and (b), T is weakly stable if assumption (c) holds; that is if assumptions (a) and (b) hold and if T is not weakly stable, then assumption (c) fails; equivalently, assumption (ii) holds. \square

For a power bounded operator supercyclicity implies strong stability [3, Theorem 2.2]. Theorem 6.2 prompts the question. Consider a power bounded operator T . *Does weak l-sequential supercyclicity implies weak stability?*

$$\begin{array}{ccc}
T \text{ IS SUPERCYCLIC} & \implies & T^n \xrightarrow{s} O \\
\Downarrow & & \Downarrow \\
T \text{ IS WEAKLY L-SEQUENTIALLY SUPERCYCLIC} & \stackrel{?}{\implies} & T^n \xrightarrow{w} O.
\end{array}$$

In particular, *can alternative (ii) be dismissed from Theorems 6.1 and 6.2?*

Corollary 6.1. *If a contraction T on a Hilbert space is weakly l-sequentially supercyclic, then either*

- (i) T is weakly stable, or
- (ii) if y is a weakly l-sequentially supercyclic vector for the unitary part U of T such that $U^n y \xrightarrow{w} 0$, then for every nonzero z such that $\langle U^n y; z \rangle \neq 0$ either $\liminf_n |\langle U^n y; z \rangle| = 0$, or $\limsup_k |\langle U^{n_k} y; z \rangle| < \|z\| \|y\|$ for some subsequence $\{U^{n_k}\}_{k \geq 0}$ of $\{U^n\}_{n \geq 0}$.

Proof. Let T be a contraction on a Hilbert space \mathcal{X} . By the Nagy–Folias–Langer decomposition for Hilbert-space contractions (see, e.g., [29, p.8] or [18, p.76]), \mathcal{X} admits an orthogonal decomposition $\mathcal{X} = \mathcal{U}^\perp \oplus \mathcal{U}$, where T is uniquely a direct sum of a completely nonunitary contraction $C = T|_{\mathcal{U}^\perp} \in \mathcal{B}[\mathcal{U}^\perp]$ and a unitary operator $U = T|_{\mathcal{U}} \in \mathcal{B}[\mathcal{U}]$ (where any of these parcels may be missing):

$$T = C \oplus U,$$

where C is the completely nonunitary part of T and U is the unitary part of T . Every completely nonunitary contraction is weakly stable (see, e.g., [12, p.55] or [18, p.106]). Thus T is weakly stable if and only if U is weakly stable; that is,

$$C^n \xrightarrow{w} O \quad \text{and} \quad T^n \xrightarrow{w} O \quad \text{if and only if} \quad U^n \xrightarrow{w} O.$$

Suppose U acts on a nonzero space (otherwise the result is trivially verified). If $T = C \oplus U$ is weakly l-sequentially supercyclic (or supercyclic), then both C and U are weakly l-sequentially supercyclic. Thus the result follows by Theorem 6.2 and by the Riesz Representation Theorem in Hilbert space, since U is an isometry. \square

Weak l -sequential supercyclicity and weak stability for unitary operators are discussed in [21, Theorem 5.1] in terms of a condition similar to the so-called angle criterion for supercyclicity — see, e.g., [5, Theorem 9.1].

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