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POWERS OF POSINORMAL OPERATORS

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ABSTRACT. Square of a posinormal operator is not necessarily posinormal. But (i) powers of quasiposinormal operators are quasiposinormal and, under closed ranges assumption, powers of (ii) posinormal operators are posinormal, (iii) of operators that are both posinormal and coposinormal are posinormal and coposinormal, and (iv) of semi-Fredholm posinormal operators are posinormal.

1. INTRODUCTION

Throughout this paper the term *operator* means a bounded linear transformation of a Hilbert space into itself. Posinormal operators where introduced in [10] as the class of operators T such that $TT^* = T^*QT$ for some nonnegative operator Q, which turns out to be equivalent to saying that $TT^* \leq \alpha^2 T^*T$ for some nonnegative real number α. It was noticed then that this was a very large class, including the dominant (and so the hyponormal) operators, as well as the invertible operators.

It is well known that the square of a hyponormal operator is not necessarily hyponormal. Since hyponormal operators are posinormal, it is sensible to ask whether the square of a posinormal operator is posinormal. Although open for a while, this question had been tackled before. For instance, an operator T is p-posinormal for some positive real number $p > 0$ if $(TT^*)^p \leq \alpha^2 (T^*T)^p$ for some positive real number $\alpha > 0$ (cf. [3], [9]), so that a 1-posinormal operator is posinormal. It was shown in [9, Corollary 4] that, for each integer $n \geq 1$, if T is p-posinormal, then T^n is $\frac{p}{n}$-posinormal. However, the original simple question remained unanswered, namely *is the square of a posinormal operator posinormal?* We show that this fails in general, and investigate conditions to ensure that natural powers of a posinormal operator are posinormal. In particular, we show that each natural power of a posinormal operator with finite descent is posinormal, natural powers of operators that are both posinormal and coposinormal are posinormal and coposinormal, natural powers of semi-Fredholm posinormal operators are posinormal, and every natural power of a quasiposinormal operator i quasiposinormal.

2. POSINORMAL OPERATORS

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{B}[\mathcal{H}]$ denote the Banach algebra of all operators on $\mathcal{H}$. If $\mathcal{M}$ is a linear manifold of $\mathcal{H}$ then $\mathcal{M}^-$ and $\mathcal{M}^\perp$ stand for closure and orthogonal complement of $\mathcal{M}$, respectively. For any $T \in \mathcal{B}[\mathcal{H}]$, set $\mathcal{N}(T) = \ker T = T^{-1}\{0\}$ (the kernel or null space of T, which is a subspace — that is, a closed linear manifold — of $\mathcal{H}$) and $\mathcal{R}(T) = \operatorname{ran} T = T(\mathcal{H})$ (the range of T, which is a linear manifold of $\mathcal{H}$). Let $T^* \in \mathcal{B}[\mathcal{H}]$ denote the adjoint of $T \in \mathcal{B}[\mathcal{H}]$. A nonnegative operator $Q \in \mathcal{B}[\mathcal{H}]$ is a self-adjoint (i.e., $Q^* = Q$) such that $0 \leq \langle Qx\,;x\rangle$ for every $x \in \mathcal{H}$, which is denoted by $O \leq Q$ (or $Q \geq O$), where $\langle\ ;\ \rangle$ stands for the inner product in $\mathcal{H}$, and O stands for the null operator. If A and B are operators

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on $\mathcal{H}$ such that $O \le A - B$, then we write $B \le A$. Recall that T^*T (and so TT^*) is always nonnegative. An operator T s normal if it commutes with its adjoint (i.e., $TT^* = T^*T$), hyponormal if $TT^* \le T^*T$, and cohyponormal if T^* is hyponormal. There are several equivalent definitions of posinormality as it will be listed in Definition 1, whose properties that will be required in the sequel will be presented in Proposition 1. For proofs concerning the equivalences in Definition 1 and the properties in Proposition 1, the reader is referred to [10, Theorems 2.1, 3.1, Corollary 2.3, Proposition 3.5], [8, Proposition 1, Remarks 1,2], and [4, Theorem 1, Proposition 3]). The main ingredient for proving the equivalences in Definition 1 is a classical result due to Douglas [2, Theorems 1,2], which reads as follows.

Lemma 1 [2]**.** *For arbitrary operators A and B in $\mathcal{B}[\mathcal{H}]$, the following assertions are pairwise equivalent.*

(a) $AA^* \le \alpha^2 BB^*$ *for some* $\alpha \ge 0$.
(b) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.
(c) *There exists* $C \in \mathcal{B}[\mathcal{H}]$ *such that* $A = BC$.

Definition 1 [10, 8, 4]**.** Take an arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$.

(a) T is *posinormal* if any of the following equivalent assertions are fulfilled.

(a_1) $TT^* = T^*QT$ for some $Q \ge O$.
(a_2) $TT^* \le T^*QT$ for some $Q \ge O$.
(a_3) $T = T^*L$ for some $L \in \mathcal{B}[\mathcal{H}]$.
(a_4) $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$.
(a_5) $TT^* \le \alpha^2 T^*T$ for some $\alpha \ge 0$.
(a_6) $\|T^*x\| \le \alpha\|Tx\|$ for some $\alpha \ge 0$ and every $x \in \mathcal{H}$.

(b) T is *coposinormal* if T^* is posinormal.

(c) T is *dominant* if any of the following equivalent assertions are fulfilled.

(c_1) $\lambda I - T$ is posinormal for every $\lambda \in \mathbb{C}$.
(c_2) $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$ for every $\lambda \in \mathbb{C}$.
(c_3) For each $\lambda \in \mathbb{C}$ there is a real number $\alpha_\lambda > 0$ such that $\|(\overline{\lambda} I - T^*)x\| \le \alpha_\lambda \|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$.

(d) T is *codominant* if T^* is dominant.

A further characterization for posinormality was worked out in [3, Theorem 2]. Basic properties of posinormal operators that will be required in the sequel are summarized in Proposition 1 below. Note from Definition 1 that

- *T is posinormal and coposinormal if and only if* $\mathcal{R}(T) = \mathcal{R}(T^*)$,
- *T is dominant and codominant if and only if* $\mathcal{R}(\lambda I - T) = \mathcal{R}(\overline{\lambda} I - T^*)$ *for all* λ.

Proposition 1 [10, 8, 4]. *Take an arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$.*

(a) *If T is posinormal, then*
(a_1) $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$,
(a_2) $\mathcal{N}(T^2) = \mathcal{N}(T)$.

(b) *Every invertible* (*in fact, every injective with closed range*) *is posinormal.*

(c) *The class of hyponormal operators is properly included in the class of dominant operators, which is properly included in the class of posinormal operators.*

Remark 1. (a) Proposition 1(a_1) is an immediate consequence of Definition 1(a_6), and Proposition 1(a_2) has been verified in [4, Proposition 3] and [8, Remark 2].

(b) *An operator is surjective if and only if its adjoint is injective with closed range.* (Indeed, for any $A\in\mathcal{B}[\mathcal{H}]$, $\mathcal{R}(A)=\mathcal{H}$ if and only if $\mathcal{R}(A)$ is closed and dense, and $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A^*)$ is closed, and $\mathcal{R}(A)^-=\mathcal{H} \iff \mathcal{R}(A)^\perp=\{0\} \iff \mathcal{N}(A^*)=\{0\}$). Now observe that, if T^* is surjective, then T is trivially posinormal (cf. Definition 1(a_4)); equivalently. *if T is injective with closed range, then T is posinormal*, and this leads to Proposition 1(b).

(c) That the inclusions in Proposition 1(c) are all proper has been shown, for instance, in [8, p.5]. Since a normal operator is precisely an operator that is both hyponormal and cohyponormal, it is worth noticing in light of the proper inclusion in Proposition 1(c) that even the combined inclusion of dominant with codominant, and posinormal with coposinormal, remain proper. In other words,

$$\begin{aligned}\text{normal} &= \text{hyponormal}\cap\text{cohyponormal}\\ &\subset \text{dominant}\cap\text{codominant}\not\subseteq\text{hyponormal}\\ &\subset \text{posinormal}\cap\text{coposinormal}\not\subseteq\text{dominant}.\end{aligned}$$

In fact, a bilateral weighted shift on ℓ^2 with weights $\{|k|^{-1}\}_{-\infty}^{\infty}$ is quasinilpotent, posinormal, and coposinormal, and so it is dominant and codominant, but it is not hyponormal, thus showing that there exist nonhyponormal operators such that $\mathcal{R}(\lambda I-T)=\mathcal{R}(\overline{\lambda}I-T^*)$ for all $\lambda\in\mathbb{C}$. Moreover, for an example of a posinormal and coposinormal which is not dominant take an invertible nondominant operator; e.g., $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ where $\mathcal{R}(I-T)\not\subseteq\mathcal{R}(I-T^*)$; this can be generalized by taking the sum of $2I$ with a backward unilateral shift, also yielding an invertible nondominant.

3. An Auxiliary Result

Recall the notion of ascent of an operator. If $A\in\mathcal{B}[\mathcal{H}]$, then

(i) $\mathcal{N}(A^n)\subseteq\mathcal{N}(A^{n+1})$ for every integer $n\geq 0$, and

(ii) if $\mathcal{N}(A^{n_0})=\mathcal{N}(A^{n_0+1})$ for some integer $n_0\geq 0$, then $\mathcal{N}(A^n)=\mathcal{N}(A^{n+1})$ for every integer $n\geq n_0$,

where (i) is clear, and (ii) is well-known (see, e.g., [7, Lemma 5.29]). If there exists an integer $n_0\geq 0$ such that $\mathcal{N}(A^{n_0})=\mathcal{N}(A^{n_0+1})$, then the least integer for which the identity holds is the (finite) *ascent* of A — notation: $\operatorname{asc}(A)$ — so that $\mathcal{N}(A^n)=\mathcal{N}(A^{\operatorname{asc}(A)})$ for every $n\geq\operatorname{asc}(A)$; if there is no such an integer, then we write $\operatorname{asc}(A)=\infty$. Summing up: $\operatorname{asc}(A)=\min\{n\colon \mathcal{N}(A^{n+1})=\mathcal{N}(A^n)\}$.

Dually, recall the notion of descent of an operator. If $A\in\mathcal{B}[\mathcal{H}]$, then

(i′) $\mathcal{R}(A^{n+1})\subseteq\mathcal{R}(A^n)$ for every integer $n\geq 0$, and

(ii″) if $\mathcal{R}(A^{n_0+1})=\mathcal{R}(A^{n_0})$ for some integer $n_0\geq 0$, then $\mathcal{R}(A^{n+1})=\mathcal{R}(A^n)$ for every integer $n\geq n_0$.

where again (i′) is clear, and (ii″) is well-known (see, e.g., [7, Lemma 5.29]). If there exists an integer $n_0\geq 0$ such that $\mathcal{R}(A^{n_0+1})=\mathcal{R}(A^{n_0})$, then the least integer for which the identity holds is the (finite) *descent* of A — notation: $\operatorname{dsc}(A)$ — so that

$\mathcal{R}(A^n) = \mathcal{R}(A^{\mathrm{dsc}(A)})$ for every $n \geq \mathrm{dsc}(A)$; if there is no such an integer, then we write $\mathrm{dsc}(A) = \infty$. Summing up: $\mathrm{dsc}(A) = \min\big\{n\colon \mathcal{R}(A^{n+1}) = \mathcal{R}(A^n)\big\}$.

Remark 2. Thus what Proposition 1(a_2) says is

(a) A is posinormal $\Longrightarrow \mathrm{asc}(T) \leq 1$.

The following basic properties of ascent and descent are readily verified.

(b) $\mathrm{asc}(A) = 0 \iff A$ is injective and $\mathrm{dsc}(A) = 0 \iff A$ is surjective.

For arbitrary integers $j, k \geq 1$,

(c) $\mathrm{asc}(A^k) \leq j \iff \mathrm{asc}(A) \leq jk$ and $\mathrm{dsc}(A^k) \leq j \iff \mathrm{dsc}(A) \leq jk$.

(Indeed, $\mathcal{N}(A^{kn_0}) = \mathcal{N}(A^{k(n_0+1)}) \iff \mathcal{N}(A^{kn_0}) \subseteq \mathcal{N}(A^{kn_0+1}) \subseteq \cdots \subseteq \mathcal{N}(A^{kn_0+k}) = \mathcal{N}(A^{kn_0}) \iff \mathcal{N}(A^{kn_0}) = \mathcal{N}(A^{kn_0+1}) = \cdots = \mathcal{N}(A^{(kn_0+1)}) \Longrightarrow \{\mathrm{asc}(A^k) \leq n_0 \iff \mathrm{asc}(A) \leq kn_0\}$. By a similar argument: $\mathrm{dsc}(A^k) \leq n_0 \iff \mathrm{dsc}(A) \leq kn_0$.)

Lemma 2. *Take any operator $A \in \mathcal{B}[\mathcal{H}]$ and an arbitrary integer $k \geq 1$. If*

$$\mathrm{asc}(A) \leq k \;\; \textit{and} \;\; \mathrm{dsc}(A) < \infty \quad \textit{or} \quad \mathrm{asc}(A) < \infty \;\; \textit{and} \;\; \mathrm{dsc}(A) \leq k,$$

then

$$\mathrm{dsc}(A) = \mathrm{asc}(A) \leq k,$$

and so

$$\mathcal{R}(A^n) = \mathcal{R}(A^k) \quad \textit{and} \quad \mathcal{N}(A^n) = \mathcal{N}(A^k) \quad \textit{for each integer } n \geq k.$$

If, in addition, $\mathcal{R}(A^n)$ is closed for every n, then

$$\mathrm{dsc}(A^*) = \mathrm{asc}(A^*) \leq k,$$

and so

$$\mathcal{R}(A^{*n}) = \mathcal{R}(A^{*k}) \quad \textit{and} \quad \mathcal{N}(A^{*n}) = \mathcal{N}(A^{*k}) \quad \textit{for each integer } n \geq k.$$

Proof. Take an arbitrary $A \in \mathcal{B}[\mathcal{H}]$. Consider the following auxiliary results.

Claim (i). $\mathrm{asc}(A) < \infty$ and $\mathrm{dsc}(A) < \infty \Longrightarrow \mathrm{asc}(A) = \mathrm{dsc}(A)$.

Proof of Claim (i). See, e.g., [12, Theorem 6.2]. □

Claim (ii).

(a) $\mathrm{dsc}(A^*) < \infty \Longrightarrow \mathrm{asc}(A) < \infty$,

(b) $\mathrm{asc}(A) < \infty \Longrightarrow \mathrm{dsc}(A^*) < \infty$ if $\mathcal{R}(A^n)$ is closed for every integer $n \geq 1$,

(c) $\mathrm{asc}(A) < \infty \not\Longrightarrow \mathrm{dsc}(A^*) < \infty$ if $\mathcal{R}(A^n)$ is not closed for some integer $n \geq 1$.

Proof of Claim (ii). Take an arbitrary positive integer n.

(a) If $\mathrm{asc}(A) = \infty$, then $\mathcal{N}(A^n) \subset \mathcal{N}(A^{n+1})$ so that $\mathcal{N}(A^{n+1})^\perp \subset \mathcal{N}(A^n)^\perp$ (since $\mathcal{N}(\cdot)$ is closed — indeed, $\mathcal{M} \subset \mathcal{N} \Longrightarrow \mathcal{N}^\perp \subseteq \mathcal{M}^\perp$ and $\mathcal{N}^\perp = \mathcal{M}^\perp \Longrightarrow \mathcal{M}^- = \mathcal{N}^-$). Equivalently, $\mathcal{R}(A^{*(n+1)})^- \subset \mathcal{R}(A^{*n})^-$. As $\mathcal{R}(A^{*n+1}) \subseteq \mathcal{R}(A^{*n})$, the above proper inclusion ensures the proper inclusion $\mathcal{R}(A^{*(n+1)}) \subset \mathcal{R}(A^{*n})$. So $\mathrm{dsc}(A^*) = \infty$, and

$$\mathrm{asc}(A) = \infty \Longrightarrow \mathrm{dsc}(A^*) = \infty.$$

(b) If $\operatorname{dsc}(A)=\infty$, then $\mathcal{R}(A^{n+1})\subset\mathcal{R}(A^n)$. Suppose $\mathcal{R}(A^n)$ is closed so that $\mathcal{R}(A^{n+1})\subset\mathcal{R}(A^n)$ implies $\mathcal{R}(A^n)^\perp\subset\mathcal{R}(A^{n+1})^\perp$. That is, $\mathcal{N}(A^{*n})\subset\mathcal{N}(A^{*(n+1)})$. Hence $\operatorname{asc}(A^*)=\infty$. Therefore

$$\operatorname{dsc}(A)=\infty \implies \operatorname{asc}(A^*)=\infty \quad \text{if } \mathcal{R}(A^n) \text{ is closed for every integer } n\geq 1.$$

Dually (as $A^{**}=A$ and $\mathcal{R}(A^n)$ closed $\iff$ $\mathcal{R}(A^{*n})$ closed),

$$\operatorname{dsc}(A^*)=\infty \implies \operatorname{asc}(A)=\infty \quad \text{if } \mathcal{R}(A^n) \text{ is closed for every integer } n\geq 1,$$

(c) To verify (c) consider the following example. Take A such that $\mathcal{N}(A^*)=\{0\}$ and $\mathcal{R}(A^*)\neq\mathcal{R}(A^*)^-=\mathcal{H}$. Then $\mathcal{N}(A)=\mathcal{R}(A^*)^\perp=\{0\}$, and hence $\operatorname{asc}(A)=0$. We show that $\operatorname{dsc}(A^*)=\infty$.

Since $\mathcal{R}(A^*)\neq\mathcal{R}(A^*)^-=\mathcal{H}$, take $v\in\mathcal{H}\backslash\mathcal{R}(A^*)$. Suppose $\operatorname{dsc}(A^*)<\infty$, say, suppose $\operatorname{dsc}(A^*)=n$. Then $\mathcal{R}(A^{*n})=\mathcal{R}(A^{*n+1})$, and so there exists $w\in\mathcal{H}$ such that $A^{*n+1}w=A^{*n}v$. Thus $A^{*n}(A^{*n}w-v)=0$ so that $A^*w=v$ (since $\operatorname{asc}(A^*)=0\implies\mathcal{N}(A^{*n})=\{0\}$). Hence $v\in\mathcal{R}(A^*)$, which is a contradiction. Thus $\operatorname{dsc}(A^*)=\infty$. □

Claim (iii). $\quad \operatorname{dsc}(A)<\infty \implies \operatorname{asc}(A^*)\leq\operatorname{dsc}(A)$.

Proof of Claim (iii). Consider the argument in the proof of Claim (ii-a). So $\operatorname{dsc}(A)=n_0$ implies $\mathcal{R}(A^n)=\mathcal{R}(A^{n_0})$ for every $n\geq n_0$. Thus $\mathcal{R}(A^n)^-=\mathcal{R}(A^{n_0})^-$. Equivalently, $\mathcal{N}(A^{*n})=\mathcal{N}(A^{*n_0})$ (as $\mathcal{R}(\,\cdot\,)^\perp=\mathcal{N}(\,\cdot\,^*)$), which implies $\operatorname{asc}(A^n)\leq n_0$. □

If $\operatorname{asc}(A)\leq k$ and $\operatorname{dsc}(A)<\infty$ (or if $\operatorname{asc}(A)<\infty$ and $\operatorname{dsc}(A)\leq k$), then

$$\operatorname{dsc}(A)=\operatorname{asc}(A)\leq k$$

by Claim (i). Moreover, this implies that $\operatorname{asc}(A^*)\leq\operatorname{dsc}(A)\leq k$ by Claim (iii). Now suppose $\mathcal{R}(A^n)$ is closed for every n. Since $\operatorname{asc}(A)\leq k$, we get $\operatorname{dsc}(A^*)<\infty$ by Claim (ii-b). Then, since $\operatorname{asc}(A^*)\leq k$, Claim (i) ensures that

$$\operatorname{dsc}(A^*)=\operatorname{asc}(A^*)\leq k.$$

The range and kernel identities follow from the definition of ascent and descent. □

Lemma 2 will be needed in the next section.

4. Powers of a Posinormal Operator

We begin with an example of a posinormal T whose square is not posinormal.

Notation: since $A^{*n}=A^{n*}$ for every $A\in\mathcal{B}[\mathcal{H}]$ and every $n\geq 1$, we will denote the adjoint of A^n by A^{*n} for every positive integer n.

Example 1. Set $P=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $P_k=\frac{1}{k}\begin{pmatrix}k-1&\sqrt{k-1}\\\sqrt{k-1}&1\end{pmatrix}$ so that $(P+P_k)^2=\frac{1}{k}\begin{pmatrix}4k-3&2\sqrt{k-1}\\2\sqrt{k-1}&1\end{pmatrix}$ in $\mathcal{B}[\mathbb{C}^2]$ for each positive integer k, where P and each P_k are orthogonal projections. Set $A=\bigoplus_k P$ and $B=A+\bigoplus_k P_k=\bigoplus_k(P+P_k)$ in $B[\ell^2_+(\mathbb{C}^2)]$ so that

$$O\leq A\leq B.$$

Since $O\leq A^{1/2}A^{1/2}\leq B^{1/2}B^{1/2}$, Lemma 1 ensures that

$$\mathcal{R}(A^{1/2})\subseteq\mathcal{R}(B^{1/2}).$$

If $O\leq(P+P_k)^2-\beta P$ for some integer $k\geq 1$, then $\beta\leq\frac{1}{k}$. This implies that there is no constant $\alpha>0$ for which $0\neq\bigoplus_k P\leq\alpha^2\bigoplus_k(P+P_k)^2$. Thus (since $A=A^2$)

there is no $\alpha \geq 0$ such that $AA = A \leq \alpha^2 B^2 = \alpha^2 BB$, which means that

$$\mathcal{R}(A) \nsubseteq \mathcal{R}(B)$$

by Lemma 1. Now consider the operator $T \in \mathcal{B}[\ell^2_+(\ell^2_+(\mathbb{C}^2))]$ defined by

$$T = \begin{pmatrix} O & & & & & \\ A^{1/2} & O & & & & \\ & A^{1/2} & O & & & \\ & & B^{1/2} & O & & \\ & & & B^{1/2} & \ddots & \\ & & & & \ddots & \end{pmatrix},$$

where every entry not directly below the main block diagonal is null. Thus

$$T^2 = \begin{pmatrix} O & & & & & \\ O & O & & & & \\ A & O & O & & & \\ & B^{1/2}A^{1/2} & O & O & & \\ & & B & O & \ddots & \\ & & & B & \ddots & \\ & & & & \ddots & \end{pmatrix}.$$

Observe that

$$\begin{aligned} \mathcal{R}(T) &= \{0\} \oplus \mathcal{R}(A^{1/2}) \oplus \mathcal{R}(A^{1/2}) \oplus \mathcal{R}(B^{1/2}) \oplus \textstyle\bigoplus_{k=5}^{\infty} \mathcal{R}(B^{1/2}), \\ \mathcal{R}(T^*) &= \mathcal{R}(A^{1/2}) \oplus \mathcal{R}(A^{1/2}) \oplus \mathcal{R}(B^{1/2}) \oplus \mathcal{R}(B^{1/2}) \oplus \textstyle\bigoplus_{k=5}^{\infty} \mathcal{R}(B^{1/2}), \\ \mathcal{R}(T^2) &= \{0\} \oplus \{0\} \oplus \mathcal{R}(A) \oplus \mathcal{R}(B^{1/2}A^{1/2}) \oplus \textstyle\bigoplus_{k=5}^{\infty} \mathcal{R}(B), \\ \mathcal{R}(T^{*2}) &= \mathcal{R}(A) \oplus \mathcal{R}(A^{1/2}B^{1/2}) \oplus \mathcal{R}(B) \oplus \mathcal{R}(B) \oplus \textstyle\bigoplus_{k=5}^{\infty} \mathcal{R}(B). \end{aligned}$$

Since $\mathcal{R}(A^{1/2}) \subseteq \mathcal{R}(B^{1/2})$, it follows that $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$, and so T is posinormal. Since $\mathcal{R}(A) \nsubseteq \mathcal{R}(B)$, it follows that $\mathcal{R}(T^2) \nsubseteq \mathcal{R}(T^{*2})$, and so T^2 is not posinormal.

When we refer to a power of an operator we mean a positive integer power. Now we investigate under which conditions powers of posinormal operators remain posinormal. Theorem 1 below ensures that every power of an operator is eventually posinormal if it has a posinormal power and a power with finite descent; and also that every power of an operator having a posinormal power and a coposinormal power is eventually both posinormal and coposinormal. These hold under the assumption that all ranges are closed. Let k, m, n stand for positive integers.

Theorem 1. *Take $T \in \mathcal{B}[\mathcal{H}]$. Suppose $\mathcal{R}(T^n)$ is closed for every $n \geq 1$.*

(a) *If T^k is posinormal for some $k \geq 1$ and $\operatorname{dsc}(T^m) < \infty$ for some $m \geq 1$, then T^n is posinormal for every $n \geq k$.*

(b) *If T^k is posinormal for some $k \geq 1$ and T^{*m} is posinormal for some $m \geq k$, then T^n is posinormal for every $n \geq k$ and coposinormal for every $n \geq m$.*

Proof. (a) Let T^k be posinormal for some $k \geq 1$, so that $\operatorname{asc}(T^k) \leq 1$ (cf. Remark 2(a)), for which $\operatorname{dsc}(T^m) < \infty$ for some $m \geq 1$. Since $\operatorname{asc}(T^k) \leq 1$ if and only if $\operatorname{asc}(T) \leq k$ and $\operatorname{dsc}(T^m) < \infty$ if and only if $\operatorname{dsc}(T) < \infty$ (cf. Remark 2(c)),

$$\operatorname{asc}(T) \leq k \text{ and } \operatorname{dsc}(T) < \infty.$$

Suppose $\mathcal{R}(T^n)$ is closed for every $n \geq 1$. Then by Lemma 2

$$\operatorname{dsc}(T) \leq k \;\text{ and }\; \operatorname{dsc}(T^*) \leq k.$$

Therefore, since $\mathcal{R}(T^k) \subseteq \mathcal{R}(T^{*k})$ (i.e., since T^k is posinormal), we get

$$\mathcal{R}(T^n) = \mathcal{R}(T^k) \subseteq \mathcal{R}(T^{*k}) = \mathcal{R}(T^{*n}),$$

implying that T^n is posinormal, for every integer $n \geq k$.

(b) If T^k is posinormal for some $k \geq 1$ and T^{*m} is posinormal (i.e., T^m is coposinormal) for some $m \geq 1$, then $\operatorname{asc}(T^k) \leq 1$ and $\operatorname{asc}(T^{*m}) \leq 1$ by Remark 2(a) and so

$$\operatorname{asc}(T) \leq k \;\text{ and }\; \operatorname{asc}(T^*) \leq m$$

by Remark 2(c). Suppose $\mathcal{R}(T^n)$ is closed for every $n \geq 1$, and so is $\mathcal{R}(T^{*n})$ (since these ranges are closed together). Thus by Claim (ii-b) in the proof of Lemma 2,

$$\operatorname{dsc}(T^*) < \infty \;\text{ and }\; \operatorname{dsc}(T) < \infty.$$

Applying Claim (i) in the proof of Lemma 2,

$$\operatorname{dsc}(T) \leq k \;\text{ and }\; \operatorname{dsc}(T^*) \leq m.$$

Thus by Claim (iii) in the proof of Lemma 2,

$$\operatorname{asc}(T^*) \leq k \;\text{ and }\; \operatorname{asc}(T) \leq m.$$

So applying Claim (i) in the proof of Lemma 2 once again,

$$\operatorname{dsc}(T) \leq k \;\text{ and }\; \operatorname{dsc}(T^*) \leq k \qquad \text{and} \qquad \operatorname{dsc}(T^*) \leq m \;\text{ and }\; \operatorname{dsc}(T) \leq m.$$

Since $\operatorname{dsc}(T) \leq k$ and $\operatorname{dsc}(T^*) \leq k$ (so that $\mathcal{R}(T^n) = \mathcal{R}(T^k)$ and $\mathcal{R}(T^{*n}) = \mathcal{R}(T^{*k})$ for every $n \geq k$), and since T^k is posinormal (so that $\mathcal{R}(T^k) \subseteq \mathcal{R}(T^{*k})$),

$$\mathcal{R}(T^n) = \mathcal{R}(T^k) \subseteq \mathcal{R}(T^{*k}) = \mathcal{R}(T^{*n}),$$

and so T^n is posinormal for every $n \geq k$. Since $\operatorname{dsc}(T^*) \leq m$ and $\operatorname{dsc}(T) \leq m$ (so that $\mathcal{R}(T^{*n}) = \mathcal{R}(T^{*m})$ and $\mathcal{R}(T^n) = \mathcal{R}(T^{*m})$ for every $n \geq m$), and since T^{*m} is posinormal (so that $\mathcal{R}(T^{*m}) \subseteq \mathcal{R}(T^m)$),

$$\mathcal{R}(T^{*n}) = \mathcal{R}(T^{*m}) \subseteq \mathcal{R}(T^m) = \mathcal{R}(T^n),$$

and hence T^{*n} is posinormal for every $n \geq m$. □

An important particular case of Theorem 1 for $k = m = 1$ reads as follows.

Corollary 1. *Take $T \in \mathcal{B}[\mathcal{H}]$. Suppose $\mathcal{R}(T^n)$ is closed for every $n \geq 1$.*

(a) *If T is posinormal and* $\operatorname{dsc}(T) < \infty$*, then T^n is posinormal for every $n \geq 1$.*

(b) *If T is posinormal and coposinormal, then T^n is posinormal and coposinormal for every $n \geq 1$.*

Example 1 and Theorem 1(a) (or Corollary 1(a)) suggest the existence of posinormal operators T with $\operatorname{dsc}(T) = \infty$. Posinormal operators T with $\operatorname{dsc}(T) = \infty$, however, do not need to have a nonposinormal square. A typical example is the canonical unilateral shift T of multiplicity 1 acting on ℓ^2_+, which is hyponormal, and hence posinormal. Since T is an isometry, it is injective, and so $\operatorname{asc}(T) = 0$. Moreover, for each positive integer n. $\mathcal{R}(T^n) = \ell^2_+ \ominus \mathbb{C}^n$ (which is closed because T^n is an isometry) so that $\operatorname{dsc}(T) = \infty$. Furthermore, T^n is a unilateral shift of multiplicity n, thus hyponormal, and so posinormal. The next theorem shows that this argument can be extended along the same line to injective unilateral weighted shifts S, so that $\operatorname{dsc}(S) = \infty$, although $\mathcal{R}(S)$ is not necessarily closed, and $\operatorname{asc}(S) = 0$.

Theorem 2. *If an injective unilateral weighted shift S is posinormal, then S^n is posinormal for every integer $n \geq 1$.*

Proof. Let

$$S = \operatorname{shift}(\{\omega_k\}_{k=1}^{\infty}) = \begin{pmatrix} 0 & & & & \\ \omega_1 & 0 & & & \\ & \omega_2 & 0 & & \\ & & \omega_3 & 0 & \\ & & & \omega_4 & \ddots \\ & & & & \ddots \end{pmatrix}$$

be a unilateral weighted shift on ℓ^2_+, which is injective if and only if the weight sequence $\{\omega_k\}$ has no zero term (i.e., $\omega_k \neq 0$ for every $k \geq 1$). Suppose S is an injective unilateral weighted shift. It is know that

$$S \text{ is posinormal if and only if } \sup_{k\geq 1} \tfrac{|\omega_k|}{|\omega_{k+1}|} < \infty$$

[8, p.4]. This can be extended to every integer power of injective unilateral weighted shifts as follows. Take an arbitrary integer $n \geq 1$. Observe that

$$S^n S^{*n} = \operatorname{diag}\big(0, \ldots, 0, \textstyle\prod_{k=1}^{n} |\omega_k|^2, \prod_{k=2}^{n+1} |\omega_k|^2, \ldots\big),$$

a diagonal operator on ℓ^2_+ with zeros at the first n entries, and

$$S^{*n} S^n = \operatorname{diag}\big(\textstyle\prod_{k=1}^{n} |\omega_k|^2, \ldots, \prod_{k=n}^{2n-1} |\omega_k|^2, \prod_{k=n+1}^{2n} |\omega_k|^2, \prod_{k=n+2}^{2n+1} |\omega_k|^2, \ldots\big),$$

another diagonal operator on ℓ^2_+. According to Definition 1(a_5), for each n the operator S^n is posinormal if and only if there exists a nonnegative number α_n (constant with respect to the variable k) such that $S^n S^{*n} \leq \alpha_n^2 S^{*n} S^n$. This means that $\prod_{k=j+1}^{n+j} |\omega_k|^2 \leq \alpha_n^2 \prod_{k=n+j+1}^{2n+j} |\omega_k|^2$ for every $j \geq 0$. Equivalently,

$$\frac{\prod_{k=j+1}^{n+j} |\omega_k|}{\prod_{k=n+j+1}^{2n+j} |\omega_k|} \leq \alpha_n \quad \text{for every} \quad j \geq 0.$$

Therefore,

$$S^n \text{ is posinormal if and only if } \sup_{j\geq 0} \frac{\prod_{k=j+1}^{n+j} |\omega_k|}{\prod_{k=n+j+1}^{2n+j} |\omega_k|} < \infty.$$

Since

$$\sup_{j\geq 0} \frac{\prod_{k=j+1}^{n+j} |\omega_k|}{\prod_{k=n+j+1}^{2n+j} |\omega_k|} \leq \bigg(\sup_{k\geq 1} \frac{|\omega_k|}{|\omega_{k+n}|}\bigg)^n \leq \bigg(\sup_{k\geq 1} \frac{|\omega_k|}{|\omega_{k+1}|}\bigg)^{n^2},$$

it follows the claimed result: if S is posinormal, then S^n is posinormal. □

If T is invertible, then T^n is posinormal for every $n \geq 1$. (Indeed, if T is invertible, then T^n is invertible for every $n \geq 1$, and hence posinormal every $n \geq 1$). Recall that T, T^*, T^*T and TT^* are invertible (or not) together.

Another special class of posinormal operators for which the square is again posinormal will be given in Theorem 3(c) below. Consider the class of all posinormal operators such that TT^* commutes with T^*T. Trivial examples: normal operators, or multiples of isometries (whose powers are clearly normal, or multiple of an isometry, respectively, thus posinormal). In fact, every posinormal operator T such that $T^*T = p(TT^*)$ (or $TT^* = p(T^*T)$) for some polynomial p lies in this class. Note that TT^* *commutes with* T^*T *if and only if* TT^*T^*T *is self-adjoint* (which happens *if and only if* TT^*T^*T *is nonnegative*, because the product of commuting nonnegative operators is again nonnegative). Thus, in particular, if the nonnegative operators T^*T and TT^* are both diagonal (diagonalized with respect to the same orthonormal basis for $\mathcal{H}$), then they must commute.

Theorem 3. *Take an arbitrary operator* $T \in \mathcal{B}[\mathcal{H}]$ *so that*

$\mathcal{R}(TT^*) \subseteq \mathcal{R}(T^*T)$ *if and only if* $(TT^*)^2 \leq \beta^2 (T^*T)^2$ *for some constant* $\beta > 0$.

Now suppose T *is posinormal.*

(a) *If* $\mathcal{R}(T^*T) = \mathcal{R}(T^*)$*, then* T^2 *is posinormal and* $\mathcal{R}(TT^*) \subseteq \mathcal{R}(T^*T)$.

(b) *If* $\mathcal{R}(TT^*) \subseteq \mathcal{R}(T^*T)$*, then* T^2 *is posinormal.*

(c) *If* T^*T *and* TT^* *commute, then* T^2 *and* T^3 *are posinormal.*

Proof. Take A and B in $\mathcal{B}[\mathcal{H}]$. If A and B are self-adjoint (as it is the case for TT^* and T^*T), then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ is equivalent to $A^2 \leq \beta^2 B^2$ for some $\beta > 0$ according to Lemma 1. Recall that $\mathcal{R}(A^*A) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(A^*A)^- = \mathcal{R}(A^*)^-$ for every operator A in $\mathcal{B}[\mathcal{H}]$.

Suppose T is posinormal, which means that $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$; equivalently, there exists a constant $\alpha > 0$ such that $\|T^*y\| \leq \alpha \|Ty\|$ for every $y \in \mathcal{H}$; still equivalently, there exists a constant $\alpha > 0$ such that $TT^* \leq \alpha^2 T^*T$ (cf. Definition 1).

(a) Since $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$, it follows that if $\mathcal{R}(T^*T) = \mathcal{R}(T^*)$, then $\mathcal{R}(T^2) \subseteq \mathcal{R}(T) \subseteq \mathcal{R}(T^*) = \mathcal{R}(T^*T) = T^*(\mathcal{R}(T)) \subseteq T^*(\mathcal{R}(T^*)) = \mathcal{R}(T^{*2})$, and T^2 is posinormal. Moreover, $\mathcal{R}(TT^*) \subseteq \mathcal{R}(T) \subseteq \mathcal{R}(T^*) = \mathcal{R}(T^*T)$, completing the proof of (a).

(b) Since $\mathcal{R}(TT^*) \subseteq \mathcal{R}(T^*T)$ is equivalent to saying that there is a $\beta > 0$ such that $(TT^*)^2 \leq \beta^2 (T^*T)^2$, which in turn is equivalent to $\|TT^*x\|^2 \leq \beta^2 \|T^*Tx\|^2$ for every $x \in \mathcal{H}$, it follows that if $\mathcal{R}(TT^*) \subseteq \mathcal{R}(T^*T)$ and T is posinormal, then

$$\|T^*T^*x\| \leq \alpha \|TT^*x\| \leq \alpha\beta \|T^*Tx\| \leq \alpha^2 \beta \|TTx\|$$

for every $x \in \mathcal{H}$, and so T^2 is posinormal, which proves (b).

Recall: if Q and R are operators in $\mathcal{B}[\mathcal{H}]$ such that $O \leq Q \leq R$, and if $QR = RQ$, then $O \leq QR$ and $O \leq Q^2 \leq R^2$ (see, e.g., [6, Problems 5.59 and 5.60]).

(c) Since T is posinormal, it follows that $O \leq TT^* \leq \alpha^2 T^*T$. If the nonnegative operators TT^* and T^*T commute, then $(TT^*)^2 \leq \alpha^4 (T^*T)^2$, which means that $\|TT^*x\|^2 \leq \alpha^4 \|T^*Tx\|^2$ for every $x \in \mathcal{H}$; equivalently, $\mathcal{R}(TT^*) \subseteq \mathcal{R}(T^*T)$. Thus T^2 is posinormal by (b) with

$$\|T^*T^*x\| \leq \alpha \|TT^*x\| \leq \alpha^3 \|T^*Tx\| \leq \alpha^4 \|TTx\|$$

for every $x \in \mathcal{H}$. Take an arbitrary $x \in \mathcal{H}$. The above inequalities imply that

$$\|T^*T^*T^*x\| \leq \alpha^4 \|TTT^*x\| \quad \text{and} \quad \|T^*T^*Tx\| \leq \alpha^4 \|TTTx\|.$$

However, since TT^* and T^*T commute, and since T is posinormal,

$$\begin{aligned} \|TTT^*x\|^2 &= \langle TTT^*x\,; TTT^*x \rangle = \langle T^*TTT^*x\,; TT^*x \rangle = \langle TT^*T^*Tx\,; TT^*x \rangle \\ &= \langle T^*T^*Tx\,; T^*TT^*x \rangle \leq \|T^*T^*Tx\| \|T^*TT^*x\| \leq \alpha \|T^*T^*Tx\| \|TTT^*x\|. \end{aligned}$$

Therefore, $\|TTT^*x\| \leq \alpha \|T^*T^*Tx\|$, so that

$$\|T^*T^*T^*x\| \leq \alpha^4 \|TTT^*x\| \leq \alpha^5 \|T^*T^*Tx\| \leq \alpha^9 \|TTTx\|,$$

and hence T^3 is posinormal. □

When is the product of two commuting posinormal operators posinormal?

Remark 3. (a) The collection of all posinormal operators is a cone in $\mathcal{B}[\mathcal{H}]$ (i.e., γT is posinormal for any $\gamma \geq 0$ whenever T is posinormal).

(b) Sum of two posinormal operators may not be posinormal. Clear: if T is not posinormal and λ is in the resolvent set of T, then λI and $T - \lambda I$ are both invertible, thus posinormal.

(c) Orthogonal direct sums of posinormal operators are trivially posinormal, and tensor products of posinormal operators are posinormal as well [5, Theorem 4].

(d) Product of two posinormal operators is not necessarily posinormal. For commuting operators, see Example 1. For operators that do not commute, consider, for instance, a unilateral weighted shift, which is the product of two noncommuting posinormal operators, namely, a diagonal (normal) and the canonical unilateral shift (hyponormal); but examples of (injective) unilateral weighted shifts that are not posinormal were exhibited in [8, p.4]. Therefore, this shows that even the product of a positive operator and a quasinormal (in particular, and a hyponormal) operator may not be posinormal.

(e) It is worth noticing that, if S and T commute, and if ST is posinormal, then

$$\mathcal{R}(ST) \subseteq \mathcal{R}(S) \cap \mathcal{R}(T) \cap \mathcal{R}(S^*) \cap \mathcal{R}(T^*).$$

(If S and T commute, then $\mathcal{R}(ST) \subseteq \mathcal{R}(S) \cap \mathcal{R}(T)$ and $\mathcal{R}(T^*S^*) \subseteq \mathcal{R}(T^*) \cap \mathcal{R}(S^*)$, so that, if ST is posinormal, then $\mathcal{R}(ST) \subseteq \mathcal{R}((ST)^*) = \mathcal{R}(T^*S^*)$.)

Theorem 4. *Suppose T is posinormal.*

(a) *If S is posinormal and S^* and T commute, then ST is posinormal.*

(b) *If S is normal and S and T commute, then ST is posinormal.*

Proof. (a) If T and S are posinormal in $\mathcal{B}[\mathcal{H}]$, and if $TS^* = S^*T$, then there exist positive constants α_T and α_S such that $\|(ST)^*x\| = \|T^*S^*x\| \leq \alpha_T\|TS^*x\| = \alpha_T\|S^*Tx\| \leq \alpha_T\alpha_S\|STx\|$ for every $x \in \mathcal{H}$, and so ST is posinormal.

(b) If T is posinormal, S is normal, and $ST = TS$, then the Fuglede Theorem ensures that $S^*T = TS^*$ (see, e.g., [7, Corollary 3.19]), so that (b) follows from (a) since S is posinormal. □

5. Powers of a Quasiposinormal Operator

Definition 2. Take a arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$.

(a) T is *quasiposinormal* if any of the following equivalent assertions are fulfilled.

- (a_1) $\mathcal{R}(T)^- \subseteq \mathcal{R}(T^*)^-$.
- (a_2) $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$.

(b) T is *coquasiposinormal* is T^* is quasiposinormal.

The above equivalence is readily verified. In fact, take an arbitrary operator A on $\mathcal{H}$, an arbitrary pair of linear manifolds $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$, and recall that $A^{**} = A$, $\mathcal{R}(A)^- = \mathcal{N}(A^*)^\perp$, $\mathcal{N}(A) = \mathcal{N}(A)^-$, and $\mathcal{M}^\perp \subseteq \mathcal{N}^\perp$ if and only if $\mathcal{N}^- \subseteq \mathcal{M}^-$. Thus $\mathcal{R}(A)^- \subseteq \mathcal{R}(A^*)^-$ if and only if $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$.

As every surjective operator is trivially coposinormal, every injective operator is trivially quasiposinormal. In particular, *every injective unilateral weighted shift is quasiposinormal*. Along this line it is also worth remarking that *if T is not quasiposinormal* (or not coquasiposinormal so that either T or T^* is not injective), *then T has a nontrivial invariant subspace* (cf. [8, Section 5]).

Clearly, every posinormal is quasiposinormal (either by Definitions 1(a_4) and 2(a_1), or by Proposition 1(a_1) and Definition 2(a_2)). The converse holds for operators with closed range: if $\mathcal{R}(T)$ is closed (equivalently, if $\mathcal{R}(T^*)$ is closed) and if T is quasiposinormal, then T is posinormal. If T is posinormal and $\operatorname{dsc}(T) < \infty$, then T^n posinormal by Corollary 1(a), so that T^n is quasiposinormal. More is true.

Theorem 5. *If T is quasiposinormal, then T^n is quasiposinormal for every $n \geq 1$.*

Proof. The result in Proposition 1(a_2), namely, $\mathcal{N}(T^2) = \mathcal{N}(T)$ *whenever T is posinormal*, can be extended to quasiposinormal operators. Indeed, the very same proof in [8, Remark 2] survives: *if T is quasiposinormal, then $\mathcal{N}(T^2) = \mathcal{N}(T)$*. This means (as in Remark 2(a)) that

$$\textit{if } T \textit{ is quasiposinormal, then } \operatorname{asc}(T) \leq 1,$$

which implies that $\mathcal{N}(T^n) = \mathcal{N}(T)$ (by the definition of ascent). Summing up:

$$\mathcal{N}(T) \subseteq \mathcal{N}(T^*) \quad \Longrightarrow \quad \mathcal{N}(T^2) = \mathcal{N}(T) \quad \Longrightarrow \quad \mathcal{N}(T^n) = \mathcal{N}(T)$$

for every $n \geq 1$. Therefore, if $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$ (i.e., if T is quasiposinormal), then

$$\mathcal{N}(T^n) = \mathcal{N}(T) \subseteq \mathcal{N}(T^*) \subseteq \mathcal{N}(T^{*n}),$$

so that T^n is quasiposinormal, for every $n \geq 1$. □

Since posinormality implies quasiposinormality, and since quasiposinormality and closed range imply posinormality, we get the following immediate consequences of Theorem 5. (Recall again: $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^*)$ is closed).

Corollary 2. *If T is posinormal, then T^n is quasiposinormal for every $n \geq 1$.*

Corollary 3. *If T is posinormal and $\mathcal{R}(T^n)$ is closed for every integer $n \geq 1$, then T^n is posinormal.*

By Corollary 3, assumption $\operatorname{dsc}(T) < \infty$ in Corollary 1(a) can be dismissed. The next result is reminiscent of Fredholm Theory, a special case of Corollary 3.

Theorem 6. *If a semi-Fredholm operator T is posinormal, then T^n is posinormal* (and semi-Fredholm) *for every integer $n \geq 1$.*

Proof. Note that the above statement is equivalent to the following one: *if T is posinormal, if $\mathcal{R}(T)$ is closed, and if* $\dim \mathcal{N}(T) < \infty$ *or* $\dim(T^*) < \infty$*, then T^n is posinormal for every integer $n \geq 1$.*

Indeed, suppose T is posinormal. Corollary 2 says that T^n is quasiposinormal for every $n \geq 1$. In addition, suppose $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ or $\mathcal{N}(T^*)$ is finite-dimensional, which means that T is semi-Fredholm (see, e.g., [7, Corollary 5.2]). Since T is semi-Fredholm, it follows that T^n is semi-Fredholm, and this implies that $\mathcal{R}(T^n)$ is closed, for every $n \geq 1$ (see, e.g., [7, Corollaries 5.2 and 5.5] — also see [1, Corollary 2]). Being quasiposinormal with a closed range (so that the range of T^{*n} is also closed), T^n is posinormal for each $n \geq 1$. □

The notion of supraposinormal operators was recently introduced and investigated in [11]: an operator T is *supraposinormal* if there exist nonnegative operators P and Q, at least one of them with dense range, such that $TPT^* = T^*QT$ — a posinormal operator is a particular case of a supraposinormal with $P = I$, and a coposinormal operator is a particular case of a supraposinormal with $Q = I$. It is clear that if T is posinormal or coposinormal, then it is quasiposinormal or coquasiposinormal. However, it was shown in [11, Theorem 1] that *a supraposinormal operator is quasiposinormal or coquasiposinormal* (according to whether P or Q has dense range, respectively). This leads to another consequence of Theorem 5.

Corollary 4. *If T is supraposinormal, then T^n or T^{*n} is quasiposinormal for every integer $n \geq 1$.*

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References

1. R. Bouldin, *The product of operators with closed range*, Tôhoku Math. J. **25** (1973), 359–363.
2. R.G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415
3. M. Ito, *Characterization of posinormal operators*, Nihonkai Math. J. **11** (2000), 97–101.
4. I.H. Jeon, S.H. Kim, E. Ko and J.E. Park, *On positive-normal operators*, Bull. Korean Math. Soc. **39** (2002), 33–41.
5. C.S. Kubrusly, *Tensor product of proper contraction, stable and posinormal operators*, Publ. Math. Debrecen **71** (2007), 425–437.
6. C.S. Kubrusly, *The Elements of Operator Theory*, Birkhäuser/Springer, New York, 2011.
7. C.S. Kubrusly, *Spectral Theory of Operators on Hilbert Spaces*, Birkhäuser/Springer, New York, 2012.
8. C.S. Kubrusly and B.P. Duggal, *On posinormal operators*, Adv. Math. Sci. Appl. **17** (2007), 131–148.
9. M.Y. Lee and S.H. Lee, *On powers of p-posinormal operators*, Sci. Math. Jpn. **64** (2006), 97–101.
10. H.C. Rhaly, Jr., *Posinormal operators*, J. Math. Soc. Japan **46** (1994), 587–605.
11. H.C. Rhaly, Jr., *A superclass of the posinormal operators*, New York J. Math, **30** (2014), 497–506.
12. A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis*, Wiley, New York, 1980.

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