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Bishop's property (β), a commutativity theorem and the dynamics of class $\mathcal{A}(s,t)$ operators

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Abstract

Given a Hilbert space operator $A \in B(\mathcal{H})$ with polar decomposition $A = U|A|$, the class $\mathcal{A}(s,t)$, $0 < s,t \leq 1$, consists of operators $A \in B(\mathcal{H})$ such that $|A^*|^{2t} \leq (|A^*|^t|A|^{2s}|A^*|^t)^{\frac{t}{t+s}}$. Every class $\mathcal{A}(s,t)$ operator is paranormal; prominent amongst the subclasses of $\mathcal{A}(s,t)$ operators are the class $\mathcal{A}(\frac{1}{2},\frac{1}{2})$ consisting of w-hyponormal operators and the class $\mathcal{A}(1,1)$ consisting of (semi quasihyponormal [16, p. 93], or) class $\mathcal{A}$ operators. Our aim here is threefold. We prove that $\mathcal{A}(s,t)$ operators satisfy: (i) Bishop's property (β), thereby providing a proof of [6, Theorem 3.1], and (ii) a Putnam–Fuglede commutativity theorem, thereby answering a question posed in [17, Conjecture 2.4]; we prove also an extension of [3, Theorem 3.4] to prove that (iii) if an $\mathcal{A}(s,t)$ operator is weakly supercyclic then it is a scalar multiple of a unitary operator.

1. Introduction

Let $\mathcal{H}$ denote a complex infinite dimensional Hilbert space and let $B(\mathcal{H})$ denote the algebra of operators (equivalently, bounded linear transformations) on $\mathcal{H}$ into itself. An operator $A \in B(\mathcal{H})$, with polar decomposition $A = U|A|$, is *hyponormal* if $|A^*|^2 \leq |A|^2$, *p-hyponormal* for some $0 < p \leq 1$ if $|A^*|^{2p} \leq |A|^{2p}$, *w-hyponormal* (or, weakly hyponormal) if $(|A|^{\frac{1}{2}}U|A|U^*|A|^{\frac{1}{2}})^{\frac{1}{2}} \leq |A| \leq (|A|^{\frac{1}{2}}U^*|A|U|A|^{\frac{1}{2}})^{\frac{1}{2}}$, *class* $\mathcal{A}$ (or, *semi-quasihyponormal*) if $|A|^2 \leq |A^2|$ and *paranormal* if $||Ax||^2 \leq ||A^2x||$ for all unit vectors $x \in \mathcal{H}$. The following inclusion is well known:

$$\text{Hyponormal} \subset p\text{-hyponormal} \subset \text{w-hyponormal} \subset \text{class } \mathcal{A} \subset \text{paranormal}.$$

Mihai Putinar [20] proved that hyponormal operators satisfy (Bishop's) property (β), and this result has since been extended to w-hyponormal operators (see, for example, [4]). Paranormal operators do not satisfy property (β) [11], and this naturally leads to the question: Do class $\mathcal{A}$ operators satisfy property (β)? [6, Theorem 3.1] claims property (β) for class $\mathcal{A}$ operators, but the authors have recently withdrawn their claim. In this paper we prove that class $\mathcal{A}$ operators satisfy property (β) (even property $(\beta)_\epsilon$ under certain conditions).

If $A, B \in B(\mathcal{H})$ are normal operators, then the classical Putnam–Fuglede theorem says that $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ [16], where $\delta_{A,B} \in B(B(\mathcal{H}))$ is the generalized derivation $\delta_{A,B}(X) = AX - XB$. An asymmetric version of this classical result holds for classes of Hilbert space operators more general than the class of normal operators. Thus, if

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$A, B^* \in B(\mathcal{H})$ are hyponormal operators, then $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$ [23, 7]. Lee and Jeon [17] ask the question: Does $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$ hold for operators $A, B^* \in \mathcal{A}$? We answer this question in the affirmative to show that if A, B^* are class $\mathcal{A}(s,t)$, $0 < s,t \leq 1$, operators such that 0 is a normal eigenevalue of both A and B^*, then $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$. ($\mathcal{A}(s,t)$ operators are defined in the following section; we note that class $\mathcal{A}$ coincides with $\mathcal{A}(1,1)$.)

Finally, we consider *weakly supercyclic* operators in $B(\mathcal{H})$. Answering a question of Sanders [22, Question 4.6], Bayart and Matheron proved in [3, Theorem 3.4] that a weakly supercyclic hyponormal operator is necessarily a scalar multiple of a unitary operator. The argument used by Bayart and Matheron depends in an essential way on the Berger-Shaw theorem, a result (at least currently) not available for classes of operators such as w-hyponormal operators. Using an alternative argument, we prove that a weakly supercyclic class $\mathcal{A}(s,t)$ operator is a scalar multiple of a unitary operator.

2. Results

Given an $A \in B(\mathcal{H})$, we say that the operator A is a class $\mathcal{A}(s,t)$ $(0 < s,t)$ operator if

$$|A^*|^{2t} \leq (|A^*|^t|A|^{2s}|A^*|^t)^{\frac{t}{t+s}}.$$

Let A have the polar decomposition $A = U|A|$, and let $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ denote the (first) Aluthge transform of A. Then $A \in \mathcal{A}(\frac{1}{2},\frac{1}{2})$ if and only if $|(\tilde{A})^*| \leq |A| \leq |\tilde{A}|$ (i.e., $A \in \mathcal{A}(\frac{1}{2},\frac{1}{2})$ *if and only if A is w-hyponormal* [14]) and $A \in \mathcal{A}(1,1)$ if and only if $|A|^2 \leq |A^2|$ (i.e., $A \in \mathcal{A}(1,1)$ *if and only if A is semi-quasihyponormal* [16] *or class $\mathcal{A}$* [14]). It is known, [14, Theorem 4], that $A \in \mathcal{A}(s,t)$ implies $A \in \mathcal{A}(\alpha,\beta)$ for every $0 < s \leq \alpha$ and $0 < t \leq \beta$, and if $0 < s,t \leq 1$ then $A \in \mathcal{A}(s,t)$ implies $A^n \in \mathcal{A}(\frac{s}{n},\frac{t}{n})$ for every $n \in \mathbb{N}$. Consequently, for an operator $A \in \mathcal{A}(s,t)$, if $0 < s,t \leq \frac{1}{2}$ then A is w-hyponormal, and if $\frac{1}{2} < s,t \leq 1$ then A is a class $\mathcal{A}$ operator and A^2 is a w-hyponormal operator. Conclusion: If $A \in \mathcal{A}(s,t)$, $0 < s,t \leq 1$, then (either A or) A^2 is w-hyponormal.

2.1 Properties (β) and $(\beta)_\epsilon$ for $\mathcal{A}(s,t)$ operators

For a Banach space $\mathcal{X}$ and open subset $\mathcal{U}$ of $\mathbb{C}$, let $\mathcal{E}(\mathcal{U},\mathcal{X})$ (resp., $\mathcal{O}(\mathcal{U},\mathcal{X})$) denote the Fréchet space of all infinitely differentiable $\mathcal{X}$-valued functions on $\mathcal{U}$ endowed with the topology of uniform convergence of all derivatives on compact subsets of $\mathcal{U}$ (resp., of all analytic $\mathcal{X}$-valued functions on $\mathcal{U}$ endowed with the topology of uniform convergence on compact subsets of $\mathcal{U}$). We say that $A \in B(\mathcal{X})$ satisfies: (Bishop's) property (β) at $\lambda \in \mathbb{C}$ if there exists a neighbiourhood $\mathcal{N}$ of λ such that, for each open subset $\mathcal{U}$ of $\mathcal{N}$ and sequence $\{f_n\}$ of $\mathcal{X}$-valued functions in $\mathcal{O}(\mathcal{U},\mathcal{X})$,

$$(T-z)f_n(z) \longrightarrow 0 \ \text{ in } \ \mathcal{O}(\mathcal{U},\mathcal{X}) \Longrightarrow f_n(z) \longrightarrow 0 \ \text{ in } \ \mathcal{O}(\mathcal{U},\mathcal{X});$$

(Eschmeier–Putinar–Bishop's) property $(\beta)_\epsilon$ at $\lambda \in \mathbb{C}$ if there exists an $r > 0$ such that, for every open subset $\mathcal{U}$ of the open disc $\mathbf{D}(\lambda; r)$ of radius r centered at λ and sequence $\{f_n\}$ of $\mathcal{X}$-valued functions in $\mathcal{E}(\mathcal{U},\mathcal{X})$,

$$(T-z)f_n(z) \longrightarrow 0 \ \text{ in } \ \mathcal{E}(\mathcal{U},\mathcal{X}) \Longrightarrow f_n(z) \longrightarrow 0 \ \text{ in } \ \mathcal{E}(\mathcal{U},\mathcal{X}).$$

Recall that an operator $A \in B(\mathcal{X})$ has the single-valued extension property at a point $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc $\mathcal{D}$ centered at λ_0 the only analytic function $f : \mathcal{D} \longrightarrow \mathcal{X}$ satisfying $(A-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$; A has SVEP if it has SVEP at every $\lambda \in \mathtt{C}$. Evidently, property $(\beta)_\epsilon$ implies property (β), and operators satisfying property $(\beta)_\epsilon$ are subscalar [12]. It is well known that property (β) implies (Dunford's condition (C) , which in turn implies) SVEP [18, pp. 22–23].

Theorem 2.1 *If $A \in \mathcal{A}(s,t)$, $0 < s,t \leq 1$, then:*

(i) *A satisfies property (β).*

(ii) *A satisfies property $(\beta)_\epsilon$ whenever $0 < s,t \leq 1/2$.*

(iii) *If $\frac{1}{2} < s,t \leq 1$ and $\sigma(A)$ is contained in an angle $L < \pi$, then (again) A satisfies property $(\beta)_\epsilon$*

Proof. (i) As observed above, $A \in \mathcal{A}(s,t)$, $0 < s,t \leq 1$, implies A^2 is w-hyponormal. Recall from [4, Proposition 4.1] that w-hyponormal operators satisfy property (β); hence A^2 satisfies property (β). Recall now from [18, Theorem 3.3.9] that if T is a Banach space operator such that $f(T)$ satisfies property (β) for a function f analytic on, and non-constant on all connected components of, a neighbourhood of $\sigma(T)$, then T satisfies property (β): Hence A satisfies property (β).

(ii) and (iii) If $0 < s,t \leq 1/2$, then A is w-hypopnormal and so satisfies property $(\beta)_\epsilon$ [4, Proposition 4.1]. If, instead, $\frac{1}{2} < s,t \leq 1$, then A^2 satisfies property $(\beta)_\epsilon$. Recall from [8, Theorem 2.9] that if T is a Banach space operator and f is bi-holomorphic on a neighbourhood of $\sigma(T)$, then T satisfies property $(\beta)_\epsilon$ if and only if $f(T)$ satisfies property $(\beta)_\epsilon$: Hence A satisfies property $(\beta)_\epsilon$ (in this case also). $\square$

Remark 2.2 (i) We thank a referee for pointing out reference [15], where it is proved, using a totally different argument, that class $\mathcal{A}$ operators are subscalar of order 12.

(ii) A Banach space operator A satisfies property $(\beta)_\epsilon$ if and only if it is subscalar [12]; hence operators $A \in \mathcal{A}(s,t)$ satisfying either of the conditions in Theorem 2.1 (ii) and (iii) are subscalar and so satisfy the condition that their quasinilpotent part $H_0(A-\lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} ||(A-\lambda)^n x||^{\frac{1}{n}} = 0\}$ is nilpotent for all complex λ (see [1, p. 175] and [18]). It is well known [9] that paranormal operators are simply polaroid (i.e., the isolated points of the spectrum of a paranormal operator are rank one poles of the resolvent of the operator); hence $\mathcal{A}(s,t)$, $0 < s,t \leq 1$, operators are simply polaroid. This, coupled with the fact that class $\mathcal{A}(s,t)$ $(0 < s,t \leq 1)$ operators have SVEP, implies that the operators in this class satisfy *Weyl's theorem* (indeed, *generalized Weyl's theorem*). (We shall say no more about Weyl's or generalized Weyl's theorem, and refer the reader to [9] for more information.)

(iii) The fact that $\mathcal{A}(s,t)$ operators, $0 < s,t \leq 1$, satisfy property (β) has a number of other interesting consequences. Thus, if $A, B \in \mathcal{A}(s,t)$ $(0 < s,t \leq 1)$ satisfy $AX = XB$ and $BY = YA$ for some operators $X, Y \in B(\mathcal{H})$ with dense range, then A and B have the same spectrum, the same (Fredholm) essential spectrum, and the same (Fredholm) index at every point of their Fredholm domain [18, Theorem 3.7.15]; if $N \in B(\mathcal{H})$ is an algebraic operator (i.e., there exists a non-trivial polynomial $p(z)$ such that $p(N) = 0$) which commutes with A, in particular if N is a nilpotent which commutes with A, then $A+N$ satisfies property (β) (resp., $(\beta)_\epsilon$ if A satisfies $(\beta)_\epsilon$) [8, Theorem 2.4]. Again, if $\sigma(A)$ is thick, then A has a non-trivial invariant subspace [18, Theorem 2.6.12]. Here a compact subset $F \subseteq \mathtt{C}$ is *thick* if there exists a non-empty, bounded and open set $E \subseteq \mathtt{C}$ in which F is dominating; F is dominating in E if $||f||_\infty = \sup\{\mathrm{f}(\lambda| : \lambda \in \mathrm{F} \cap \mathrm{E}\}$ for all $f \in P^\infty(E)$, where $P^\infty(U) \subset H^\infty(U)$ is the weak star closure of complex polynomials on U [18, p. 180].

2.2 A Commutativity theorem for $\mathcal{A}(s,t)$ operators

The classical Putnam–Fuglede commutativity theorem [16] says that if $A, B \in B(\mathcal{H})$ are normal operators, then $AX = XB$ implies $A^*X = XB^*$ for every $X \in B(\mathcal{H})$. An asymmetric version of this theorem holds for classes of Hilbert space operators more general than the class of normal operators (see, for example, [7]). In their consideration of an asymmetric Putnam–Fuglede commutativity theorem for quasi-class $\mathcal{A}$ operators (i.e., operator $A \in \mathcal{A}(1,1)$ such that $A^*|A|^2A \leq A^*|A^2|A$), Lee and Jeon [17] ask the question of whether $AX = XB$ implies $A^*X = XB^*$ for operator $A, B^* \in \mathcal{A}$. We answer this question here. Observe that if $\delta_{A,B}(X) = 0 \Longrightarrow \delta_{A^*,B^*}(X) = 0$ for all $X \in B(\mathcal{H})$, then (also) $\delta_{A,B}(AX) = 0 \Longrightarrow \delta_{A^*,B^*}(AX) = 0$; hence $(A^*A - AA^*)X = X(B^*B - BB^*) = AXX^* - XX^*A = X^*XB - BX^*X = 0$. We have:

Lemma 2.3 [24] *Let $A, B \in B(\mathcal{H})$. If $X \in \delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$, then $\overline{ran(X)}$ reduces A, $\ker(X)^\perp$ reduces B, and $A|_{\overline{ran(X)}}$ and $B|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.*

If $AX = XA^*$ implies $A^*X = XA$ for an operator $A \in B(\mathcal{H})$, then necessarily $A^{-1}(0) \subseteq {A^*}^{-1}(0)$. Hence the Putnam–Fuglede theorem fails for operators $A, B^* \in \mathcal{A}(s,t)$, $0 < s, t \leq 1$, such that 0 is not a normal eigenvalue (i.e., such that the eigenspace corresponding to the eigenvalue is not reducing). However, if $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and ${B^*}^{-1}(0) \subseteq B^{-1}(0)$, then the presence of 0 is in the point spectrum of either of A (resp., B^*) implies that $A = 0 \oplus A_1$ (resp., $B^* = 0 \oplus B_1^*$) for some injective $\mathcal{A}(1,1)$ operators A_1 (resp., B_1^*). Hence, if 0 is in the point spectrum of both A and B^*, and if $X \in \delta^{-1}_{A,B}(0)$ has the (corresponding) matrix representation $X = [Y_{ij}]^1_{i,j=0}$, then $\delta_{A_1,B_1}(Y_{11}) = 0$, and it is seen that $\delta_{A,B}(X) = 0$ implies $\delta_{A^*,B^*}(X) = 0$ if and only if $\delta_{A_1,B_1}(Y_{11}) = 0$ implies $\delta_{A_1^*,B_1^*}(Y_{11}) = 0$. Since a similar statement holds for when only one (or, none) of A and B^* has 0 in its point spectrum, we may as well assume that A, B^* are injective $\mathcal{A}(1,1)$ operators (and prove $\delta_{A,B}(X) = 0 \Longrightarrow \delta_{A^*,B^*}(X) = 0$).

Let, as above, $\tilde{A}$ denote the (first) Aluthge transform of $A \in B(\mathcal{H})$, and let $\tilde{\tilde{A}}$ denote the second Aluthge transform of A. (Thus, if $\tilde{A}$ has the polar decomposition $\tilde{A} = V|\tilde{A}|$, then $\tilde{\tilde{A}} = |\tilde{A}|^{\frac{1}{2}}V|\tilde{A}|^{\frac{1}{2}}$.) If an $A \in B(\mathcal{H})$ is w-hyponormal, then its second Aluthge transform $\tilde{\tilde{A}}$ is hyponormal; if $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ then $(A-\lambda)^{-1}(0) \subseteq (A^* - \overline{\lambda})^{-1}(0)$ for all complex λ in the point spectrum of A, and if $\tilde{\tilde{A}}$ is normal then $A = \tilde{A} = \tilde{\tilde{A}}$ [2]. The proof of the following (known) lemma is included for completeness.

Lemma 2.4 *If $A, B^* \in B(\mathcal{H})$ are w-hyponormal operators such that $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and ${B^*}^{-1}(0) \subseteq B^{-1}(0)$, then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$.*

Proof. As seen above, we may assume A and B^* to be injective and prove $X \in \delta^{-1}_{A,B}(0) \Longrightarrow X \in \delta^{-1}_{A^*,B^*}(0)$. Evidently, $\overline{\mathrm{ran(X)}}$ is invariant for A and $\mathrm{ker(X)}^\perp$ is invariant for B^*; let $A_1 = A|_{\overline{\mathrm{ran(X)}}}$, $B_1^* = B^*|_{\mathrm{ker(X)}^\perp}$, and define $X_1 : \mathrm{ker(X)}^\perp \to \overline{\mathrm{ran(X)}}$ by setting $X_1x = Xx$ for each $x \in \mathrm{ker(X)}^\perp$. Then A_1 and B_1^* are w-hyponormal, and X_1 is a quasiaffinty. Let $C = \tilde{\tilde{A_1}}$, $D^* = \tilde{\tilde{B_1^*}}$ and $Y = |\tilde{A_1}|^{\frac{1}{2}}|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}}|\tilde{B_1^*}|^{\frac{1}{2}}$; then $\delta_{A,B}(X) = 0$ implies $\delta_{C,D}(Y) = 0$, where C, D^* are hyponormal operators in $B(\mathcal{H})$. Applying the asymmetric Putnam–Fuglede theorem for hyponormal operators [7, 23] we have $\delta_{C^*,D^*}(Y) = 0$, and hence (by Lemma 2.3) $C = A_1$ and $D = B_1$ are (unitarily) equivalent normal operators. But then $CY - YD = 0$ implies $|\tilde{A_1}|^{\frac{1}{2}}|A_1|^{\frac{1}{2}}(A_1^*X_1 - X_1B_1^*)|B_1^*|^{\frac{1}{2}}|\tilde{B_1^*}|^{\frac{1}{2}} = |A_1|(A_1^*X_1 - X_1B_1^*)|B_1^*| = 0$. Since A_1, B_1^* are injective, $A_1^*X_1 - X_1B_1^* = 0$. Conclusion: $AX - XB = 0$ implies $\overline{\mathrm{ran(X)}}$ reduces A,

$\ker(\mathrm{X})^{\perp}$ reduces B, and $A|_{\overline{\mathrm{ran}(\mathrm{X})}}$ and $B|_{\ker(\mathrm{X})^{\perp}}$ are unitarily equivalent normal operators (consequently, A and B^* can not be pure or completely non-normal). Now decompose A, B^* into their normal and pure parts by $A = A_n \oplus A_p$ (with respect to some decomposition $\mathcal{H} = \mathcal{H}_n \oplus \mathcal{H}_p$) and $B^* = B_n^* \oplus B_p^*$ (with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$); Let $X : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_n \oplus \mathcal{H}_p$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then $A_i X_{ij} = X_{ij} B_j$ implies $A_i^* X_{ij} = X_{ij} B_j^*$, where $A_i = A_n$ if $i = 1$, $A_i = A_p$ if $i = 2$, $B_j = B_n$ if $j = 1$, $B_j = B_p$ if $j = 2$ and $1 \leq i, j \leq 2$. Since A_p and B_p are pure, it follows from our conclusion above that $X_{ij} = 0$ for all $1 \leq i, j \leq 2$ except for $i = j = 1$. But then $A^* X = X B^*$. □

Our promised theorem follows.

Theorem 2.5 *If $A \in \mathcal{A}(s_1, t_1)$ and $B^* \in \mathcal{A}(s_2, t_2)$, $0 < s_1, s_2, t_1, t_2 \leq 1$, are such that $A^{-1}(0) \subseteq A^{*-1}(0)$ and $B^{*-1}(0) \subseteq B^{-1}(0)$, then $\delta_{A,B}^{-1}(0) \subseteq d_{A^*,B^*}^{-1}(0)$.*

Proof. Without loss of generality, we may assume that both A, B^* are $\mathcal{A}(1,1)$ and $A^2, {B^*}^2$ are w-hyponormal. If $\delta_{A,B}(X) = 0$ for some $X \in B(\mathcal{H})$ and injective $A, B^* \in \mathcal{A}(1,1)$, then $\delta_{A^2,B^2}(X) = 0$, where A^2 and ${B^*}^2$ are injective w-hyponormal operators. Apply Lemma 2.4 to conclude that $\delta_{{A^*}^2,{B^*}^2}(X) = 0$. Hence (see Lemma 2.3) $\overline{\mathrm{ran}(X)}$ reduces A^2, $\ker(X)^{\perp}$ reduces B^2, and $A^2|_{\overline{\mathrm{ran}(X)}}$ and $B^2|_{\ker(X)^{\perp}}$ are unitarily equivalent normal operators Let $A_1 = A|_{\overline{\mathrm{ran}(X)}}$ and $B_1^* = B^*|_{\ker(X)^{\perp}}$. Observe that if M is an invariant closed subspace of $A \in \mathcal{A}(1,1)$ and P is the orthogonal projection onto M, then

$$|A|_M|^2 = P|A|^2 P \leq P|A^2|P = P({A^*}^2 A^2)^{\frac{1}{2}} \leq (P{A^*}^2 A^2 P)^{\frac{1}{2}} = (P{A^*}^2 P P A^2 P)^{\frac{1}{2}} = |A|_M^2|.$$

Hence A_1 and B_1^* are $\mathcal{A}(1,1)$ operators. The normality of A_1^2 implies (by [21, Theorem 1]) the existence of normal operators E, F and a positive injective operator G which commutes with F such that

$$A_1 = E \oplus \begin{pmatrix} F & G \\ 0 & -F \end{pmatrix} = E \oplus H.$$

We prove that the upper triangular block matrix operator H in this expression is the trivial operator. Suppose thus that H is non-trivial. The operator H being an $\mathcal{A}(1,1)$ operator,

$$|H|^2 = \begin{pmatrix} |F|^2 & F^*G \\ GF & G^2 + |F|^2 \end{pmatrix} \leq \begin{pmatrix} |F|^2 & 0 \\ 0 & |F|^2 \end{pmatrix} = |H^2|$$

$$\Longrightarrow 0 \leq \begin{pmatrix} 0 & -F^*G \\ -FG & -G^2 \end{pmatrix} \Longleftrightarrow G = 0.$$

Since G is injective, this contradiction implies that H acts on the trivial space. Hence A_1 is normal. Similarly, B_1 is normal. Applying the classical Putnam–Fuglede theorem for normal operators it follows that $\delta_{A_1^*,B_1^*}(X_1) = 0$. Since the non-zero eigenvalues of an $\mathcal{A}(1,1)$ operator are normal (i.e., the corresponding eigenspaces are reducing) [6, Lemma 2.3], this implies that if $EZ - FZ = 0$ implies $E^*Z - ZF^* = 0$ for some $E, F \in \mathcal{A}(1,1)$, then neither of E and F^* can be pure. Decomposing $A, B^* \in \mathcal{A}(1,1)$ into their normal and pure parts and arguing as in the proof of Lemma 2.4 it is now seen that $\delta_{A^*,B^*}(X) = 0$. □

It is well known (indeed, easily proved) that hyponormal, p-hyponormal ($0 < p \leq 1$) and quasihyponormal operators are $\mathcal{A}(s,t)$ operators; hence either of the hypotheses that A and B^* are $\mathcal{A}(s,t)$, $0 < s, t \leq 1$, operators in Theorem 2.5 may be replaced by the hypothesis that the operator is hyponormal or p-hyponormal or quasihyponormal. An interesting case of Theorem 2.5 is obtained in the case in which $B^* = V$ is an isometry. Recall from [11] that an operator $A \in B(\mathcal{H})$ satisfies property PF(δ) (resp., property PF($\triangle$)) if for every isometry $V \in B(\mathcal{H})$ for which the equation $\delta_{A,V^*}(X) = AX - XV^* = 0$ (resp., the equation $\triangle_{A,V^*}(X) = AXV^* - X = 0$) has a non-trivial solution $X \in B(\mathcal{H})$, the solution X also satisfies $\delta_{A^*,V}(X) = 0$ (resp., $\triangle_{A^*,V}(X) = 0$). It is known that $A \in B(\mathcal{H})$ satisfies property PF(δ) if and only if it satisfies property PF($\triangle$) [10]. Evidently, $\mathcal{A}(s,t)$ $(0 < s,t \leq 1)$ operators satisfy property PF(δ). Recall, however, that paranormal operators fail to satisfy property PF(δ) [19, 11].

2.3 Weak supercyclicity

A Banach space operator $A \in B(\mathcal{X})$ is *n-supercyclic* for some $n \in \mathbb{N}$ if $\mathcal{X}$ has an n-dimensional subspace M with dense orbit $\mathrm{Orb_M(A)} = \bigcup_{\mathrm{m}\in\mathbb{N}} \mathrm{A^mM}$; a 1-supercyclic operator is supercyclic, and we say that A is *weakly supercyclic* if there exists a vector $x \in \mathcal{X}$, with M the corresponding one dimensional subspace generated by x, such that $\mathrm{Orb_M(A)} = \mathbb{C}.\mathrm{Orb_x(A)}$ is weakly dense (equivalently, if the scaled orbit of x is dense in the weak topology) in $\mathcal{X}$. Recall from [5] that paranormal, hence also $\mathcal{A}(s,t)$ $(0 < s,t \leq 1)$, operators are not supercyclic. Recall also from Bayart and Matheron [3, Theorem 3.4] that a weakly supercyclic hyponormal operator is necessarily a scalar multiple of a unitary operator. The argument used by Bayart and Matheron to prove this result depends in essential way on the Berger–Shaw theorem (relating the trace of the commutator of a hyponormal operator to the Lebesgue area of its spectrum). Using an alternative argument, involving property (β), we prove in the following that a similar result holds for weakly supercyclic $\mathcal{A}(s,t)$ operators. We start by recalling some complementary results and terminology.

An operator A is normaloid if $r(A) = ||A||$, where $r(A) = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}}$ is the spectral radius of A. It is well known that paranormal, in particular $A(s,t)$ $(0 < s,t \leq 1)$ operators, operators are normaloid [13, Proposition 54.6]. Furthermore, the inverse (whenever it exists) of a paranormal operator is paranormal (hence, normaloid).

Weak supercyclicity implies the existence of a weakly cyclic vector (indeed, a vector $x \in \mathcal{X}$ is a 1-weakly supercyclic vector for $A \in B(\mathcal{X})$ if and only if x is a cyclic vector for A), and since cyclicity in the weak topology is equivalent to cyclicity in the strong topology, every weakly supercyclic operator has a strongly cyclic vector. Furthermore, see [22, Proposition 2.1], the set of weakly supercyclic vectors of an operator is (not only dense in the weak operator topology, but also) dense in the strong operator topology. (Thus, weakly supercyclic operator have a dense range.) In view of this, and the fact that $\mathcal{A}(s,t)$ $(0 < s,t \leq 1)$ operators satisfy property (β), it follows from the argument of the proof of [18, Proposition 3.3.18] that:

Lemma 2.6 *If an $A \in \mathcal{A}(s,t)$, $0 < s,t \leq 1$, is weakly supercyclic, then $|\lambda| = 1$ for every $\lambda \in \sigma(A)$.*

The following theorem generalizes [3, Theorem 3.4]. Let $\mathcal{D}_r$ denote the disc of radius r centered at the origin in the complex plane, and let $\partial\mathcal{D}_r$ denote its boundary.

Theorem 2.7 *If an $A \in \mathcal{A}(s,t)$, $0 < s,t \leq 1$, is weakly supercyclic, then it is a scalar multiple of a unitary operator.*

Proof. Since operators $A \in \mathcal{A}(s,t)$ are paranormal and paranormal operators are normaloid, we may assume that $A \in \mathcal{A}(s,t)$ is a weakly supercyclic contraction which satisfies property

(β). Hence, by Lemma 2.6, $\sigma(A) \subseteq \partial\mathcal{D}_1$. Consequently, A^{-1} is well defined as a bounded paranormal operator such that $\sigma(A^{-1}) \subseteq \partial\mathcal{D}_1$. But then $||A|| = ||A^{-1}|| = 1$; hence

$$||x|| = ||A^{-1}Ax|| \leq ||A^{-1}||||Ax|| = ||Ax|| \leq ||x||$$

for all $x \in \mathcal{H}$, which implies that A is an invertible isometry, hence a unitary. □

Recall from [11, Corollary 5.2] that paranormal operators do not satisfy property (β); hence the argument above does not extend to paranormal operators. The question however still remains: Does the weak supercyclicity of a paranormal operator force it to be a scalar multiple of a unitary?

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