Mediterranean Journal of Mathematics **13** (2016) 2073-2085

ON SIMILARITY TO NORMAL OPERATORS

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Abstract. This paper gives a characterization of the asymptotic limit A_T associated to a contraction T that is similar to a normal operator (Theorem 2). Extensions from contractions to power bounded operators intertwined to a contraction with a $\mathcal{C}_{0\cdot}$ completely nonunitary part (not necessarily a normaloid contraction) are considered as well (Theorem 1). It is also given a characterization of the asymptotic limit A_T for a hyponormal contraction T, and it is shown that if a hyponormal contraction has no nontrivial invariant subspace, then one of the defect operators is not finite-rank (Corollary 1).

1. Notation and Terminology

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ are complex (nontrivial) Hilbert spaces and $\mathcal{B}[\mathcal{H},\mathcal{K}]$ denotes the Banach space of all bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$. Norms in $\mathcal{H}$ and $\mathcal{K}$ and the induced uniform norm in $\mathcal{B}[\mathcal{H},\mathcal{K}]$ will all be denoted by de same symbol $\|\cdot\|$. Inner products in both in $\mathcal{H}$ and $\mathcal{K}$ will also be denoted by the same symbol $\langle\cdot\,;\cdot\rangle$. By an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. Set $\mathcal{B}[\mathcal{H}]=\mathcal{B}[\mathcal{H},\mathcal{H}]$, which is the C^* algebra of all operators on $\mathcal{H}$. Let $\mathcal{N}(T)\subseteq\mathcal{H}$ denote the kernel of $T\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ (i.e., $\mathcal{N}(T)=T^{-1}(\{0\})=\{x\in\mathcal{H}\colon Tx=0\}$), which is a subspace (i.e., a closed linear manifold) of $\mathcal{H}$, and let $\mathcal{R}(T)\subseteq\mathcal{K}$ denote the range of T (i.e., $\mathcal{R}(T)=T(\mathcal{H})$), which a linear manifold of $\mathcal{H}$. A transformation $T\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ is said to be finite-rank (or finite-dimensional) if its range $\mathcal{R}(T)$ is a finite-dimensional (thus closed) subspace of $\mathcal{K}$.

A contraction is a transformation $T\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $\|T\|\leq 1$ (i.e., such that $\|Tx\|\leq\|x\|$ for every x in $\mathcal{H}$). A power bounded is an operator $T\in\mathcal{B}[\mathcal{H}]$ such that $\sup_{n\geq 0}\|T^n\|<\infty$ (trivially, every contraction in $\mathcal{B}[\mathcal{H}]$ is power bounded). Let $T^*\in\mathcal{B}[\mathcal{K},\mathcal{H}]$ denote the adjoint of $T\in\mathcal{B}[\mathcal{H},\mathcal{K}]$. An operator $T\in\mathcal{B}[\mathcal{H}]$ is self-adjoint if $T^*=T$. A self-adjoint operator $T\in\mathcal{B}[\mathcal{H}]$ is nonnegative if $0\leq\langle Tx;x\rangle$ for every x in $\mathcal{H}$ (notation: $O\leq T$). Let $I\in\mathcal{B}[\mathcal{H}]$ denote the identity operator in $\mathcal{B}[\mathcal{H}]$. The defect operator of $T\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ is the self-adjoint $D_T=I-T^*T\in\mathcal{B}[\mathcal{H}]$, so that $T\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ is a contraction if and only if D_T is nonnegative (i.e., if and only if $T^*T\leq I$ — the defect operator of a contraction $T\in\mathcal{B}[\mathcal{H}]$ is sometimes defined as $(I-T^*T)^{\frac{1}{2}}\in\mathcal{B}[\mathcal{H}]$, however the closure of the ranges of both forms coincide). Let $\delta_T=\dim\mathcal{R}(D_T)$ and $\delta_{T^*}=\dim\mathcal{R}(D_{T^*})$ stand for the rank of the defect operators. An isometry is a transformation $V\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ such that $V^*V=I$ (i.e., such that $\|Vx\|=\|x\|$ for every x in $\mathcal{H}$, thus isometries are contractions whose defect operators are null), and a coisometry is a transformation whose adjoint is an isometry. A transformation $U\in\mathcal{B}[\mathcal{H},\mathcal{K}]$ is unitary if it is both an isometry and a coisometry (equivalently, if it is a surjective isometry, or still an invertible isometry).

Date: April 23, 2014; revised: July 7, 2015.
2010 *Mathematics Subject Classification.* Primary 47A15, 47A45; Secondary 47B15, 47B20.
Keywords. Asymptotic limits, similarity, normal operators, invariant subspaces.

A transformation $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ intertwines an operator $T \in \mathcal{B}[\mathcal{H}]$ to an operator $L \in \mathcal{B}[\mathcal{K}]$ if $XT = LX$ and, in this case, T is said to be intertwined to L via X; if there exists an X with dense range, then T is densely intertwined with L; if there exists a quasiinvertible X (i.e., if X is injective with dense range), then T is a quasiaffine transform of L. If T is a quasiaffine transform of L (via a quasiinvertible $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$) and L is a quasiaffine transform of T (via a quasiinvertible $Y \in \mathcal{B}[\mathcal{K},\mathcal{H}]$), then T and L are quasisimilar. If there is an invertible (or unitary) $X \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ — that is, if X is injective and surjective (or if X is a unitary transformation) — such that $XT = LX$, then T and L are similar (or unitarily equivalent).

If a sequence $\{T_n\}_{n\ge0}$ of transformations $T_n \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ converges to $T \in \mathcal{B}[\mathcal{H},\mathcal{K}]$ uniformly, strongly, or weakly, then this will be denoted by $T_n \xrightarrow{u} T$, $T_n \xrightarrow{s} T$, or $T_n \xrightarrow{w} T$ (or by $T = u\text{-}\lim T_n$, $T = s\text{-}\lim T_n$, or $T = w\text{-}\lim T_n$), respectively. An operator $T \in \mathcal{B}[\mathcal{H}]$ is uniformly, strongly, or weakly stable if the power sequence $\{T^n\}_{n\ge0}$ converges uniformly, strongly, or weakly to the null operator O; that is, $T^n \xrightarrow{u} O$, $T^n \xrightarrow{s} O$, or $T^n \xrightarrow{w} O$. If T is strongly stable, then T is said to be of class $\mathcal{C}_{0\cdot}$ — if T^* is strongly stable, then T is said to be of class $\mathcal{C}_{\cdot0}$, and if both T and T^* are strongly stable, then they are of class $\mathcal{C}_{00}$. Uniform stability implies strong stability, which implies weak stability, which in turn implies power boundedness. Uniform stability is equivalent to $r(T) < 1$, and power boundedness implies $r(T) \le 1$ (where $r(T)$ stands for the spectral radius of T).

If $T \in \mathcal{B}[\mathcal{H}]$ is a contraction, then $\{T^{*n}T^n\}_{n\ge0}$ is a bounded monotone sequence of self-adjoint operators (a nonincreasing sequence of nonnegative contractions) so that $T^{*n}T^n \xrightarrow{s} A_T$; that is, $\{T^{*n}T^n\}_{n\ge0}$ converges strongly to an operator $A_T \in \mathcal{B}[\mathcal{H}]$, which is a nonnegative contraction (i.e., $O \le A_T \le I$). Recall that T is of class $\mathcal{C}_{0\cdot}$ if and only if $A_T = O$. For a wide-ranging review on the properties of A_T see, e.g., [21, Chapter 3] and [23]. A characterization of nonnegative contractions that may be the strong limit of $\{T^{*n}T^n\}_{n\ge0}$ was recently achieved in [9, Theorems 6 and 7]. Sometimes A_T is referred to as the asymptotic limit of T, which can be extended from contractions to power bounded operators by means of Banach limits [15], [16], and beyond, for operators whose power sequence satisfies some regularity condition weaker than power boundedness [17], [18]. For a discussion on asymptotic limits of power bounded operators see [10], [11].

2. Introduction

We take as a starting point the 1947 classical Sz.-Nagy's result, which says that *an operator is similar to a unitary operator if and only if it is an invertible power bounded with a power bounded inverse* [30] (also see [32, Section IX.1]). Since a unitary is precisely an isometric (thus contractive) normal operator, this naturally motivates investigating generalizations (perhaps with restrictions) to (i) plain contractions, (ii) isometries, and (iii) normal operators. Extensions to plain contractions — *which operators are similar to contractions*? — has a long and successful story. This question was posed by Sz.-Nagy himself in [31], and has passed along some seminal papers such as [8], [14], [28], [4] (also see [13], [32, Section II.8], [21, Section 8.1], and the references therein). Further discussions on similarity to contractions can be found in [19], [20], [26]. As for similarity to isometry see [30] again, and for quasisimilarity to isometries, see [33], [34], [35]. Quasisimilarity to normal operators was investigated in [1] (also see [12]).

Our focus in this paper will be on similarity to normal operators. A distinguished reference on this topic is [2]. Its main result, [2, Theorem 1.6.6], is the one that will be of interest to the present paper. This will be stated in Theorem 0 below. Before stating it we need some further standard notions. The resolvent set $\rho(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ is the set of all complex numbers λ such that $\lambda I - T$ is invertible (i.e., has a bounded inverse), and its spectrum $\sigma(T)$ is the complement $\sigma(T) = \mathbb{C}\backslash\rho(T)$. The resolvent function of $T \in \mathcal{B}[\mathcal{H}]$ is the function $R_T\colon \rho(T) \to \mathcal{B}[\mathcal{H}]$ defined by $R_T(\lambda) = (\lambda I - T)^{-1}$ for every λ in $\rho(T)$, and the spectral radius of $T \in \mathcal{B}[\mathcal{H}]$ is the nonnegative number $r(T) = \max_{\lambda\in\sigma(T)} |\lambda|$, which is bounded by the norm: $0 \leq r(T) \leq \|T\|$. The spectrum of a power bounded operator is a subset of the closed unit disk $\mathbb{D}^-$ (i.e., if $\sup_{n\geq 0} \|T^n\| \leq 1$, then $\sigma(T) \subseteq \mathbb{D}^-$). Let $d(\lambda, \sigma(T))$ denote the distance of a point $\lambda \in \mathbb{C}$ to the set $\sigma(T) \subset \mathbb{C}$. An operator $T \in \mathcal{B}[\mathcal{H}]$ is said to satisfy the linear resolvent growth if $\sup_{\lambda\in\rho(T)} \|R_T(\lambda)\|\, d(\lambda, \sigma(T)) < \infty$; that is, if

$$\sup_{\lambda\in\rho(T)} \|(\lambda I - T)^{-1}\|\, d(\lambda, \sigma(T)) < \infty.$$

An operator $T \in \mathcal{B}[\mathcal{H}]$ is normal if it commutes with its adjoint ($O = T^*T - TT^*$), quasinormal if $O = (T^*T - TT^*)T$, hyponormal if $O \leq T^*T - TT^*$ (equivalently, if $\|T^*x\| \leq \|Tx\|$ for every x in $\mathcal{H}$), paranormal if $\|Tx\|^2 \leq \|Tx\|^2\|x\|$ for every x in $\mathcal{H}$, and normaloid if $r(T) = \|T\|$. Every normal is quasinormal, every quasinormal is hyponormal, every hyponormal is paranormal, every paranormal is normaloid, and a unitary operator is precisely a normal isometry. Coquasinormal, cohyponormal, or coparanormal mean that the adjoint is quasinormal, hyponormal or paranormal.

Theorem 0. [2] *Let $T \in \mathcal{B}[\mathcal{H}]$ be a contraction with finite-rank defect operators for T and T^** (i.e., $\delta_T < \infty$, $\delta_{T^*} < \infty$). *The following assertions are equivalent.*

(i) *T is similar to a normal operator.*

(ii) $\sigma(T) \neq \mathbb{D}^-$ (so that $\delta_T = \delta_{T^*}$) *and T satisfies the linear resolvent growth condition.*

In this paper we will characterize the nonnegative contraction A_T associated to a contraction T which is similar to a normal operator. This follows as a particular case of a characterization of A_T for a power bounded operator T similar to a contraction with a $\mathcal{C}_{0\cdot}$ completely nonunitary part (also see [11]).

3. Preliminary Results

The auxiliary results in Propositions 1 and 2 below will be applied to prove the main result in Theorems 1 and 2 (next section).

Proposition 1. *If a normaloid operator is similar to a power bounded operator, then it is a contraction.*

Proof. Let $T \in \mathcal{B}[\mathcal{H}]$ be a power bounded operator. Suppose there exists an invertible transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ and a normaloid operator $N \in \mathcal{B}[\mathcal{K}]$ such that

$$XT = NX.$$

Since similarity preserves the spectrum, and since power boundedness implies spectral radius not grater than 1, we get

$$\|N\| = r(N) = r(X^{-1}NX) = r(T) \leq 1. \qquad \square$$

Proposition 2. *Let $N \in \mathcal{B}[\mathcal{K}]$ be a contraction and consider its Nagy–Foiaş–Langer decomposition,*

$$N = U \oplus G,$$

where U is unitary (the unitary part of N) and G is a completely nonunitary contraction (the completely nonunitary part of N). Suppose N is intertwined to an operator $T \in \mathcal{B}[\mathcal{H}]$,

$$YN = TY$$

for some nonzero transformation $Y \in \mathcal{B}[\mathcal{K}, \mathcal{H}]$. For each integer $n \geq 0$ set

$$A_n = Y^*T^{*n}T^nY$$

in $\mathcal{B}[\mathcal{K}]$. Then

$$A_n = B_n + C_n + D_n + E_n,$$

where

$$B_n = (O \oplus G^{*n})Y^*Y(O \oplus G^n),$$

$$C_n = (O \oplus G^{*n})Y^*Y(U^n \oplus O),$$

$$D_n = (U^{*n} \oplus O)Y^*Y(O \oplus G^n),$$

$$E_n = (U^{*n} \oplus O)Y^*Y(U^n \oplus O),$$

in $\mathcal{B}[\mathcal{K}]$. If

$$G^n \xrightarrow{s} O,$$

then

$$B_n \xrightarrow{s} O, \quad D_n \xrightarrow{s} O, \quad \text{and} \quad C_n \xrightarrow{w} O.$$

Proof. Since $YN = TY$, it follows that $YN^n = T^nY$, and so $Y^*T^{*n} = N^{*n}Y^*$, and hence, for each n,

$$\begin{aligned} A_n = Y^*T^{*n}T^nY = N^{*n}Y^*YN^n &= (U^{*n} \oplus G^{*n})Y^*Y(U^n \oplus G^n) \\ &= B_n + C_n + D_n + E_n, \end{aligned}$$

where

$$B_n = (O \oplus G^{*n})Y^*Y(O \oplus G^n),$$

$$C_n = (O \oplus G^{*n})Y^*Y(U^n \oplus O),$$

$$D_n = (U^{*n} \oplus O)Y^*Y(O \oplus G^n),$$

$$E_n = (U^{*n} \oplus O)Y^*Y(U^n \oplus O).$$

Suppose $G^n \xrightarrow{s} O$. Since U^* is power bounded (and also is G and so is G^* as well), it follows that $B_n \xrightarrow{s} O$ and $D_n \xrightarrow{s} O$. Moreover, $C_n \xrightarrow{w} O$ because $C_n = D_n^*$. □

Recall that G always is weakly stable, $G^n \xrightarrow{w} O$, since it is a completely nonunitary contraction [7, p.55] or [21, p.106] — a contraction is completely nonunitary if its unitary part is missing in its Nagy–Foiaş–Langer decomposition.

Remark 1. Besides the results of Proposition 2 it will be necessary to ensure that $C_n \xrightarrow{s} O$. Consider the setup of Proposition 2: $N \in \mathcal{B}[\mathcal{K}]$ is a contraction, intertwined via $Y \in \mathcal{B}[\mathcal{K},\mathcal{H}]$ to an operator $T \in \mathcal{B}[\mathcal{H}]$ (i.e., $YN = TY$), whose completely nonunitary part G is strongly stable (i.e., $G^n \xrightarrow{s} O$). Suppose T is such that

$$T^{*n}T^n \xrightarrow{s} A_T$$

for some $O \leq A_T \in \mathcal{B}[\mathcal{H}]$. If, in addition,

$$\|E_n y\| \leq \|Y^*A_T\,Y y\|$$

for every $y \in \mathcal{K}$, then

$$E_n \xrightarrow{s} Y^*A_T\,Y,$$

which is equivalent to

$$C_n \xrightarrow{s} O.$$

Indeed, if $T^{*n}T^n \xrightarrow{s} A_T$, then $A_n \xrightarrow{s} A = Y^*A_T\,Y$. Since $B_n \xrightarrow{s} O$ and $D_n \xrightarrow{s} O$ (Proposition 2), it follows that $C_n + E_n \xrightarrow{s} A$. Since $C_n \xrightarrow{w} O$ (Proposition 2), it follows that $E_n \xrightarrow{w} A$. Now recall that *weak convergence from below implies strong convergence* (i.e., if $F_n \xrightarrow{w} F$ and $\|F_n x\| \leq \|Fx\|$ for every x, then $F_n \xrightarrow{s} F$, see e.g., [22, Problem 5.23(b)]). Therefore, if $\|E_n y\| \leq \|Ay\|$ for every $y \in \mathcal{K}$, then $E_n \xrightarrow{s} A$, which means that $C_n \xrightarrow{s} O$.

Remark 2. We did not assume that N is normal, and the assumptions "N is normaloid" and "X is invertible" have been used only in Proposition 1 to ensure that N is a contraction. By assuming that N is a contraction we may dismiss both the normaloidness assumption on N and the invertibility assumption on X.

(i) As for the normaloidness issue, there are some classical examples of normaloid contractions N for which the completely nonunitary part G is strongly stable. For instance, recall that paranormal (and so coparanormal) operators are normaloid, and also recall that if N is a coparanormal (in particular, a cohyponormal or, more particularly, a normal) contraction, then G is strongly stable; that is,

$$G^n \xrightarrow{s} O.$$

Indeed, if a contraction N is coparanormal, then its completely nonunitary part G is of class $\mathcal{C}_{0\cdot}$ [27]. This was first investigated for cohyponormal contractions in [29]. Thus if a contraction N is normal, then G is of class $\mathcal{C}_{00}$. Actually, the class of contractions for which the completely nonunitary part is of class $\mathcal{C}_{\cdot 0}$ goes beyond the paranormal (including k-paranormal, k-quasihyponormal and dominant contractions — e.g., [24] and the references therein).

(ii) As for the power boundedness (or even contractiveness) of T, the assumption that $T^{*n}T^n \xrightarrow{s} A_T$ is always satisfied if T is a contraction and, in this case, the nonnegative A_T is a contraction. In fact, it is enough to assume simply that a contraction N is intertwined to an operator T such that the completely nonunitary part G of N is strongly stable (as we did in Proposition 2), and then impose that $T^{*n}T^n \xrightarrow{s} A_T$ (as in Remark 1), which implies that T must be power bounded.

Summing up, we may dismiss the assumptions of Proposition 1 for a while, and we may replace the assumptions of Remark 1 with the assumptions of the forthcoming Theorem 1 (where it is proved that $C_n \xrightarrow{s} O$ under weaker hypotheses), yielding the most general setup for our purpose so far.

4. Asymptotic Limits for T Intertwined to a Contraction N

Next we characterize the limit A_T for power bounded operators T such that $\{T^{*n}T^n\}_{n\geq 0}$ converges strongly (in particular, for contractions T), whenever a contraction N with a $\mathcal{C}_{0\cdot}$ completely nonunitary part is intertwined to T. Such a characterization is also considered in Theorem 2 for the special case when intertwinement is strengthened to similarity and the contraction N is coparanormal (in particular, if N is normal). The overall setup of Theorem 1 reads as follows. *Let a contraction N, whose completely nonunitary part G is strongly stable* (i.e., *G is of class $\mathcal{C}_{0\cdot}$*)*, be intertwined to an operator T such that $T^{*n}T^n \xrightarrow{s} A_T$ for some A_T.*

Theorem 1. *Let $N \in \mathcal{B}[\mathcal{K}]$, $T \in \mathcal{B}[\mathcal{H}]$, and $A_T \in \mathcal{B}[\mathcal{H}]$ be operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and consider the following properties.*

(a) $\|N\| \leq 1$,

(b) $N = U \oplus G$ *with* $G^n \xrightarrow{s} O$,

(c) $YN = TY$ *for some nonzero transformation* $Y \in \mathcal{B}[\mathcal{K},\mathcal{H}]$,

(d) $T^{*n}T^n \xrightarrow{s} A_T$.

If (d) *holds, then*

(1) *A_T is nonnegative, and T is power bounded.*

From now on suppose (a), (b), (c)*, and* (d) *hold. Then*

(2) $C_n \xrightarrow{s} O$,

where $C_n \in \mathcal{B}[\mathcal{K}]$ was defined in Proposition 2*, and*

(3)
$$Y^*A_TY = s\text{-}\lim U^{*n}Q\,U^n \oplus O,$$

where $O \leq Q$ denotes the upper left block of Y^*Y with respect to the Nagy–Foiaş–Langer decomposition for the contraction N in (b) (i.e., $Y^*Y = \left(\begin{smallmatrix} Q & S^* \\ S & R \end{smallmatrix}\right)$).

(4) *If, in addition to* (c)*, T and N are similar, then there is an invertible transformation $Y \in \mathcal{B}[\mathcal{K},\mathcal{H}]$ satisfying* (c)*, with inverse $X = Y^{-1} \in \mathcal{B}[\mathcal{H},\mathcal{K}]$, such that*

$$A_T = X^*(s\text{-}\lim U^{*n}Q\,U^n \oplus O)X,$$

so that, if N is completely nonunitary, then $A_T = O$ (i.e., *T is strongly stable*).

(4.i) *With respect to the same decomposition in* (b)*, Y is lower* (*upper*) *triangular if and only if X is* (i.e., $Y = \left(\begin{smallmatrix} Y_{11} & O \\ Y_{21} & Y_{22} \end{smallmatrix}\right)$ *if and only if* $X = \left(\begin{smallmatrix} X_{11} & O \\ X_{21} & X_{22} \end{smallmatrix}\right)$ *and* $Y = \left(\begin{smallmatrix} Y_{11} & Y_{12} \\ O & Y_{22} \end{smallmatrix}\right)$ *if and only if* $X = \left(\begin{smallmatrix} X_{11} & Y_{12} \\ O & X_{22} \end{smallmatrix}\right)$)*. If they are lower triangular, then*

$$A_T = s\text{-}\lim X_{11}^*U^{*n}Q\,U^nX_{11} \oplus O.$$

(4.ii) *If they are upper triangular, then*

$$Q = Y_{11}^*Y_{11} = (X_{11}X_{11}^*)^{-1},$$

and $Q = I$ if and only if X_{11} (*equivalently, Y_{11}*) *is unitary. In this case*

$$A_T = X^*(I \oplus O)X = X^*A_NX,$$

where $N^{*n}N^n \xrightarrow{s} A_N = I \oplus O$ (i.e., $A_N = s\text{-}\lim N^{*n}N^n$).

(4.iii) *If they are lower and upper* (i.e., *block diagonal with respect to the same decomposition in* (b)), *and if* Y_{11} (*equivalently,* X_{11}) *is unitary, then*

$$A_T = A_N = I \oplus O.$$

Proof. (1) If (d) holds, then A_T must be nonnegative because the cone of all nonnegative operators is weakly, thus strongly, closed. Moreover, since $O \le A_T$, it follows by (d) that $\|T^n x\| \to \|A_T{}^{\frac{1}{2}} x\|$ for every x in $\mathcal{H}$, and so $\sup_n \|T^n\| < \infty$ by the Banach–Steinhaus Theorem.

(2) Assumption (a), (b), (c), and (d) are enough to ensure that $A_n \xrightarrow{s} Y^* A_T Y$, $B_n \xrightarrow{s} O$, and $D_n \xrightarrow{s} O$ according Proposition 2, and hence

$$E_n + C_n = A_n - B_n - D_n \xrightarrow{s} Y^* A_T Y.$$

With respect to the same decomposition in (b) write $Y^*Y = \left(\begin{smallmatrix} Q & S^* \\ S & R \end{smallmatrix}\right)$, so that $C_n = \left(\begin{smallmatrix} O & O \\ G^{*n} S\, U^n & O \end{smallmatrix}\right)$ and $E_n = \left(\begin{smallmatrix} U^{*n} Q\, U^n & O \\ O & O \end{smallmatrix}\right)$. Thus

$$E_n + C_n = \begin{pmatrix} U^{*n} Q\, U^n & O \\ G^{*n} S\, U^n & O \end{pmatrix}.$$

Claim. There exists a bounded linear transformation Z such that

$$G^{*n} S\, U^n \xrightarrow{s} Z.$$

Proof. First observe the following string of equivalences: $\left(\begin{smallmatrix} W_n & O \\ Z_n & O \end{smallmatrix}\right) \xrightarrow{s} \left(\begin{smallmatrix} W & \Theta \\ Z & \Theta' \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} W_n - W & -\Theta \\ Z_n - Z & -\Theta' \end{smallmatrix}\right) x \to 0$ for every $x \iff \|(W_n - W)u - \Theta v, (Z_n - Z)u - \Theta' v\|^2 \to 0$ for every $x = (u, v) \iff \|(W_n - W)u - \Theta v\|^2 + \|(Z_n - Z)u - \Theta' v\|^2 \to 0$ for every (u, v). Set $u = 0$, and get $\Theta = O$ and $\Theta' = O$. So $\|(W_n - W)u\|^2 + \|(Z_n - Z)u\|^2 \to 0$ for every u, which means that $W_n \xrightarrow{s} W$ and $Z_n \xrightarrow{s} Z$. Since $E_n + C_n \xrightarrow{s} Y^* A_T Y$, it then follows that there exist W and Z such that

$$\begin{pmatrix} W & O \\ Z & O \end{pmatrix} = \begin{pmatrix} s\text{-}\lim U^{*n} Q\, U^n & O \\ s\text{-}\lim G^{*n} S\, U^n & O \end{pmatrix} = s\text{-}\lim \begin{pmatrix} U^{*n} Q\, U^n & O \\ G^{*n} S\, U^n & O \end{pmatrix} = Y^* A_T Y. \ \square$$

The above Claim ensures that there is a $C = \left(\begin{smallmatrix} O & O \\ Z & O \end{smallmatrix}\right)$ such that $C_n \xrightarrow{s} C$, and so $C_n \xrightarrow{w} C$. But $C_n \xrightarrow{w} O$ (Proposition 2), and therefore $C = O$. Outcome:

$$C_n \xrightarrow{s} O.$$

(3) Since $C_n + E_n \xrightarrow{s} Y^* A_T Y$, $C_n \xrightarrow{s} O$, and $E_n \xrightarrow{s} \left(\begin{smallmatrix} W & O \\ O & O \end{smallmatrix}\right) = W \oplus O$, we get

$$Y^* A_T Y = W \oplus O = s\text{-}\lim U^{*n} Q\, U^n \oplus O.$$

(4) The result in (4) follows at once from (3).

(4.i,ii,iii) To verify items (i,ii,iii) of (4), first observe that Y and $X = Y^{-1}$ are lower (upper) triangular together. Indeed, a simple algebraic manipulation shows that if $\left(\begin{smallmatrix} X_{11} & O \\ X_{21} & X_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right) = \left(\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} X_{11} & O \\ X_{21} & X_{22} \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & O \\ O & I \end{smallmatrix}\right)$, then $Y_{11} = X_{11}^{-1}$, $Y_{12} = O$, $Y_{21} = -X_{22}^{-1} X_{21} X_{22}^{-1}$, and $Y_{22} = X_{22}^{-1}$. Thus Y and X are lower triangular together. Dually, Y and X are upper triangular together. Now set $M_n = U^{*n} Q\, U^n$ and $M = s\text{-}\lim M_n$. If Y and X are lower triangular, then $X^*(M \oplus O)X = X_{11}^* M X_{11} \oplus O$, implying (4.i). (Indeed, $\|X_{11}^*(M_n - M)X_{11}u\| \le \|X_{11}\|\,\|(M_n - M)X_{11}u\| \to 0$.) If Y and X are upper triangular, then $Y^*Y = \left(\begin{smallmatrix} Q & S^* \\ S & R \end{smallmatrix}\right)$ with $Q = Y_{11}^* Y_{11}$, $S = Y_{12}^* Y_{11}$, and $R = Y_{12}^*Y_{12} + Y_{22}^*Y_{22}$. Also, if $Q = I$, then Y_{11} is an invertible isometry, which means that it is unitary (the converse is trivial). This implies, by (4), the expression for A_T in (4.ii). The result in (4.iii) follows from (4.i) and (4.ii). $\square$

Observe that in Theorem 1 the operator T was not supposed to be a contraction (there are power bounded noncontractions that satisfy (d); trivial example: nilpotent noncontractions), and N was not supposed to be even normaloid; and items (4.ii,iii) do not require that X is unitary.

Theorem 2. *Let $N \in \mathcal{B}[\mathcal{K}]$, $T \in \mathcal{B}[\mathcal{H}]$, and $A_T \in \mathcal{B}[\mathcal{H}]$ be operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.*

(i) *If T satisfies property* (d) *in the statement of Theorem* 1*, and if T is similar to a coparanormal operator N, then T is power bounded, N is a contraction whose completely nonunitary part is strongly stable, and A_T is given as in part* (4) *of the statement of Theorem* 1*.*

(ii) *If a contraction T is similar to a normal operator N, then N is a contraction whose completely nonunitary part is strongly stable, and A_T is given as in part* (4) *of the statement of Theorem* 1*.*

Proof. Consider properties (a), (b), (c) under similarity, and (d) as in the statement of Theorem 1.

(i) Suppose T is such that property (d) holds so that T is power bounded (according to the proof of part (1) in Theorem 1), and suppose N is coparanormal. Paranormal operators are normaloid, and so are coparanormal operators. If a normaloid N is similar to a power bounded T, then N is a contraction by Proposition 1, so that properties (d) and (c) under similarity imply property (a). If a contraction N is coparanormal, then property (b) holds [27] by uniqueness of the Nagy–Foiaş–Langer decomposition for contractions (i.e., the completely nonunitary part of a coparanormal contraction is of class $\mathcal{C}_{0\cdot}$ — see Remark 2(i)). Thus if (d) holds and T is similar to a coparanormal N, then properties (a), (b), (c) under similarity, and (d) hold, and so Theorem 1 ensures that A_T is given as in part (4).

(ii) Since contractions do satisfy property (d), and since normal operators are coparanormal (and also paranormal), item (ii) is a particular case of item (i). $\square$

Remark 3. According to Theorem 2, if T is a contraction similar to a normal operator N, then T and N satisfy the assumptions of Theorem 1. What else can be said if we assume that N is normal (and perhaps that T is a contraction and Y is invertible)? Here is a result that can be helpful. As we saw in Remark 2(i) a normal contraction N has a completely nonunitary part G of class $\mathcal{C}_{0\cdot}$ (i.e., $G^n \xrightarrow{s} O$), where G actually is a $\mathcal{C}_{00}$-contraction. In fact more is true as stated below, because if a normal operator is strongly stable, then it is a proper contraction of class $\mathcal{C}_{00}$. Indeed, consider the following result from [6, Lemma 2].

Let L be any operator and consider the following assertions.

(i) *L is strongly stable.*

(ii) *L is a proper contraction* (i.e., $\|Lx\| < \|x\|$ *whenever* $x \neq 0$)*.*

(iii) *L is a $\mathcal{C}_{00}$-contraction.*

If L is paranormal, then (i) *implies* (ii) *and* (iii). *If L is quasinormal, then assertions* (i), (ii), *and* (iii) *are pairwise equivalent.*

The above statement allows us to straighten Theorem 2 for the case of N normal. (Recall: the completely nonunitary part of a normal operator is normal.)

If a contraction $T \in \mathcal{B}[\mathcal{H}]$ is similar to a normal operator $N \in \mathcal{B}[\mathcal{K}]$, then N is a contraction whose completely nonunitary part is a proper contraction of class $\mathcal{C}_{00}$, and

$$A_T = X^*(s\text{-}\lim U^{*n} Q\, U^n \oplus O)X,$$
$$A_{T^*} = Y(s\text{-}\lim U^n Q^{-1}\, U^{*n} \oplus O)Y^*,$$

where X, Y, U, and Q are defined as in Theorem 1, and T is of class $\mathcal{C}_{00}$ (i.e., *$A_T = A_{T^*} = O$*) *if N is completely nonunitary.*

Corollary 1. *If $T \in \mathcal{B}[\mathcal{H}]$ is a hyponormal contraction for which T and T^* have finite-rank defect operators* (i.e., *$\delta_T < \infty$ and $\delta_{T^*} < \infty$*), *then*

(i) *T has a nontrivial invariant subspace,*

(ii) *the spectrum of T is a proper subset of the closed unit disk if and only if T is similar to normal contraction $N \in \mathcal{B}[\mathcal{K}]$ and, in this case,*

$$A_T = X^*(s\text{-}\lim U^{*n} Q\, U^n \oplus O)X$$

for the invertible transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ that intertwines T to N (i.e., *$XT = NX$*), *where Q denotes the upper left block of $(XX^*)^{-1} = \left(\begin{smallmatrix} Q & S^* \\ S & R \end{smallmatrix}\right)$, and U the is unitary part of the contraction N.*

Proof. (i) Since similarity preserves nontrivial invariant subspace, if an operator has no nontrivial invariant subspace, then it is not similar to a normal operator (for normal operators have nontrivial invariant subspace by the Spectral Theorem). Also, if a contraction T has no nontrivial invariant subspace, then $\partial\mathbb{D} \nsubseteq \sigma(T)$ by the Brown–Chevreau–Pearcy Theorem [3], and so $\sigma(T) \neq \mathbb{D}^-$. If an operator is hyponormal, then $\|(\lambda I - T)^{-1}\| = d(\lambda, \sigma(T))^{-1}$ for every $\lambda \in \rho(T)$ (see, e.g., [7, p.11] or [22, Problem 6.14(b)]), and therefore it satisfies the linear resolvent growth condition. Hence, by the Benamara–Nikolski Theorem (Theorem 0), the defect operator of T or the defect operator of T^* is not finite-rank. Conclusion:

If a hyponormal contraction T has no nontrivial invariant subspace, then $\delta_T = \infty$ or $\delta_{T^*} = \infty$ (one of the defect operators is not finite-rank).

(ii) Suppose T is a hyponormal contraction for which the defect operators of T and of T^* are finite-rank. Since T is hyponormal, it satisfies the resolvent growth condition (as we saw in item (i) above). Therefore, $\sigma(T) \subset \mathbb{D}^-$ (proper inclusion) if and only if T is similar to a normal operator N, by another application of Theorem 0. Moreover, since contractions satisfy assumption (d) of Theorem 1, and normal operators are coparanormal, the claimed result follows from Theorem 2. $\square$

A collection of properties that a possible hyponormal contraction without a nontrivial invariant subspace must possess has been discussed in [23]. The preceding corollary adds another one. In connection with it, it is worth remarking on the following points. First recall that if T is a $\mathcal{C}_{00}$-contraction, or if an operator T

is similar to a normal operator, then $\delta_T = \delta_{T^*}$ [32, Proof of Theorem 5.2] or [5, Corollary 3.6], and [2, Corollary 1.2.2]. Also recall the following well-known result.

If a contraction T of class $\mathcal{C}_{00}$ has no nontrivial invariant subspace, then $\delta_T = \infty$ or $\delta_{T^*} = \infty$ (one of the defect operators is not finite-rank),

equivalently, then $\delta_T = \infty$ [32, Theorems VI.5.2 and III.6.5] or [5, Proposition 3.15 and Corollary 4.9]. However, it is not known whether a hyponormal contraction without a nontrivial invariant subspace is of class $\mathcal{C}_{00}$ (it lies in $\mathcal{C}_{00} \cup \mathcal{C}_{10}$ [25, Theorem 1], but it remains as an open question whether it lies in $\mathcal{C}_{00}$).

Acknowledgment

I thank György Pál Gehér for his contribution, which began with the paper topic itself, passed through Proposition 1, and included many sensible remarks.

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