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PF PROPERTY AND PROPERTY (β) FOR PARANORMAL OPERATORS

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Abstract

For a pair of Hilbert space operators $A, B \in B(\mathcal{H})$, let $\delta_{A,B}$ and $\Delta_{A,B}$ in $B(B(\mathcal{H}))$ be defined by $\delta_{A,B}(X) = AX - XB$ and $\Delta_{A,B}(X) = AXB - X$ for each $X \in B(\mathcal{H})$. An operator $A \in B(\mathcal{H})$ satisfies the Putnam–Fuglede properties δ and Δ (notation: $A \in \mathrm{PF}(\delta)$ and $A \in \mathrm{PF}(\Delta)$) if, for every isometry $V \in B(\mathcal{H})$ for which the equation $\delta_{A,V^*}(X) = 0$ or $\Delta_{A,V^*}(X) = 0$ has a nontrivial solution $X \in B(\mathcal{H})$, the solution X also satisfies $\delta_{A^*,V}(X) = 0$, respectively $\Delta_{A^*,V}(X) = 0$. It is proved that, if an operator $A \in B(\mathcal{H})$ is hereditarily normaloid, satisfies (Dunford's) condition (C), and the contractive parts of A have a $\mathcal{C}_{\cdot 0}$ completely nonunitary part, then A lies in $\mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$. This is applied to prove that a paranormal operator may not satisfy condition (C) (and hence paranormal operators do not satisfy (Bishop's) property (β)). Also, as an application to the dynamics of power bounded operators in $B(\mathcal{H})$ (nontrivially) satisfying property $\mathrm{PF}(\delta)$, we prove that a number of classes of Hilbert space operators, including classes consisting of dominant, paranormal and $*$-paranormal operators, are not n-supercyclic.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, and let $B(\mathcal{H})$ stand for the Banach algebra of all bounded linear operators on $\mathcal{H}$ into itself. By a subspace we mean a *closed* linear manifold of $\mathcal{H}$. A part of an operator $A \in B(\mathcal{H})$ is a restriction of it to an invariant subspace. Let $\mathcal{M}$ be a subspace of $\mathcal{H}$. Suppose $\mathcal{M}$ is A-invariant (i.e., $A(\mathcal{M}) \subseteq \mathcal{M}$). We say that a part $A|_{\mathcal{M}}$ of A is contractive if $A|_{\mathcal{M}}$ is a contraction in $B(\mathcal{M})$, and that $A|_{\mathcal{M}}$ is reducing if $\mathcal{M}$ reduces A (i.e., if $\mathcal{M}$ is a reducing subspace, which means that $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^{\perp}$ both are A-invariant). Recall [23, Problem 4.10]), if $A \in B(\mathcal{H})$ is a contraction, then every unitary part of A (i.e., every restriction of A to an A-invariant subspace $\mathcal{M}$ such that $A|_{\mathcal{M}}$ is unitary) is reducing. In other words, if the restriction of a Hilbert space contraction to an invariant subspace is unitary, then the subspace is reducing. This, however, fails for a general operator $A \in B(\mathcal{H})$.

In this paper we investigate Putnam–Fuglede commutativity properties (shortened to PF-properties) for operators $A \in B(\mathcal{H})$ such that A is hereditarily normaloid (i.e., every part of A is normaloid) and satisfies (Dunford's) condition (C). Combining commutativity properties and (Dunford's) condition (C) has been successfully explored before (e.g., [10, Lemma 3.1], [13, Lemma 2.1] and [32, Theorem 2]). Here we prove that if the contractive parts of A have $\mathcal{C}_{\cdot 0}$ completely nonunitary part, then $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$ (where $\mathrm{PF}(\delta)$ and $\mathrm{PF}(\Delta)$ are classes of operators having especial Putnam–Fuglede commutativity properties that will be defined in the next section). As an application, we

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prove that if A is p-hyponormal ($0 < p \leq 1$), or w-hyponormal, or totally paranormal, then $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$. (We prove also that M-hyponormal, more generally dominant, and $*$-paranormal operators satisfy property $\mathrm{PF}(\delta)$.) Furthermore, we prove that *a paranormal operator may not satisfy* (*Dunford's*) *condition* (C). *Hence a paranormal operator may not satisfy* (*Bishop's*) *property* (β) as well, pointing out thereby the need for correction in, for instance, [36], [9], and [16].

A generalization of p-hyponormal, w-hyponormal, dominant and $*$-paranormal operators is obtained by considering their k-quasi variants [14]. (Thus, $A \in B(\mathcal{H})$ is k-quasi-p-hyponormal for some integer $k \geq 1$ if $A^{*k}(|A|^{2p} - |A^*|^{2p})A^k \geq 0$.) It is seen that these generalizations also satisfy property $\mathrm{PF}(\delta)$. As a final application of our results, we consider the dynamics of power bounded operators $A \in B(\mathcal{H})$ satisfying property $\mathrm{PF}(\delta)$. For such operators it is seen that if the set $\big\{V \in B(\mathcal{H})\colon V \text{ isometric}, \delta^{-1}_{A,V^*}(0) \neq \{0\}\big\} \neq \varnothing$, then neither of A and A^* is finitely supercyclic.

2. PF Property

Let $\mathcal{H}$ and $\mathcal{K}$ stand for Hilbert spaces, $B(\mathcal{H},\mathcal{K})$ for the Banach space of all bounded linear transformation from $\mathcal{H}$ into $\mathcal{K}$, and let $B(\mathcal{H})$ and $B(\mathcal{K})$ denote the Banach algebras $B(\mathcal{H},\mathcal{H})$ and $B(\mathcal{K},\mathcal{K})$, respectively. Let the upper star * stand for the adjoint of a Hilbert space operator. An operator $A \in B(\mathcal{H})$ has Putnam–Fuglede property δ (property $\mathrm{PF}(\delta)$, for short) if, whenever the equation

$$XA^* = VX$$

holds for some isometry $V \in B(\mathcal{K})$ and some $X \in B(\mathcal{H},\mathcal{K})$, then

$$XA = V^*X.$$

That is, an operator A on $\mathcal{H} \neq \{0\}$ has property $\mathrm{PF}(\delta)$ if either A^* is not intertwined to any isometry on any $\mathcal{K} \neq \{0\}$ or, if $X \neq 0$ intertwines A^* to an isometry V, then the same X also intertwines A to the coisometry V^*. From now on suppose $\mathcal{H} = \mathcal{K}$. For every pair of operators $A, B \in B(\mathcal{H})$, consider the transformer $\delta_{A,B}\colon B(\mathcal{H}) \to B(\mathcal{H})$, defined by

$$\delta_{A,B}(X) = AX - XB$$

for every $X \in B(\mathcal{H})$, which is called the generalized derivation. By taking the adjoint in both sides of the definition of property $\mathrm{PF}(\delta)$, it follows that, for the particular case of $\mathcal{H} = \mathcal{K}$, an operator A has property $\mathrm{PF}(\delta)$, or $A \in \mathrm{PF}(\delta)$, if and only if, for every isometry $V \in B(\mathcal{H})$,

$$\delta_{A,V^*}(X) = 0 \implies \delta_{A^*,V}(X) = 0 \quad (\text{equivalently, } \ker(\delta_{A,V^*}) \subseteq \ker(\delta_{A^*,V})),$$

which can be written as,

$$\delta^{-1}_{A,V^*}(0) \subseteq \delta^{-1}_{A^*,V}(0).$$

For each $A, B \in B(\mathcal{H})$ consider the transformer $\Delta_{A,B}\colon B(\mathcal{H}) \to B(\mathcal{H})$, defined by

$$\Delta_{A,B}(X) = AXB - X$$

for every $X \in B(\mathcal{H})$, which is called the elementary operator. Motivated by the preceding definition one says that an operator $A \in B(\mathcal{H})$ satisfies the Putnam–Fuglede property Δ (property $\mathrm{PF}(\Delta)$, for short), or $A \in \mathrm{PF}(\Delta)$, if

$$\Delta_{A,V^*}(X) = 0 \implies \Delta_{A^*,V}(X) = 0 \quad (\text{equivalently, } \ker(\Delta_{A,V^*}) \subseteq \ker(\Delta_{A^*,V})),$$

for every isometry $V \in B(\mathcal{H})$, which again can be written as,

$$\Delta^{-1}_{A,V^*}(0) \subseteq \Delta^{-1}_{A^*,V}(0).$$

The classical Putnam–Fuglede commutativity theorem motivates the above notions of Properties PF(δ) and PF(Δ). Indeed, recall from the classical Putnam–Fuglede commutativity theorem [23, p.84] that if A, B are normal, then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$. A similar result holds for $\Delta_{A,B}$: if A, B are normal, then $\Delta^{-1}_{A,B}(0) \subseteq \Delta^{-1}_{A^*,B^*}(0)$. This symmetric version of the Putnam–Fuglede commutativity theorem has an asymmetric version valid for classes of Hilbert space operators more general than the class of normal operators. Thus, [11], if A is dominant and B^* is hyponormal (i.e., if B is cohyponormal), then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$.

A operator $A \in B(\mathcal{H})$ is a contraction if $\|A\| \leq 1$. Every contraction can be decomposed into a direct sum $A = U \oplus C$, where U is the unitary part of A and C is the completely non-unitary (shortened, henceforth, to cnu) part of A (i.e., C has no unitary part itself). A contraction C is of class $C_{\cdot 0}$ if its adjoint is strongly stable (i.e., if $\|C^{*n}x\| \to 0$ for every $x \in \mathcal{H}$). Property PF(δ) gives rise to a characterization of contractions A with $C_{\cdot 0}$ completely non-unitary part [12, Lemma 1]: *A contraction $A \in B(\mathcal{H})$ has $C_{\cdot 0}$ cnu part if and only if* $A \in$ PF(δ) (see also [26, Theorem 1]). For a survey along this line see [25]. Since $A \in$ PF(δ) if and only if $A \in$ PF(Δ), whenever A is power bounded [17, Theorem 2.3], it then follows that: *A contraction $A \in B(\mathcal{H})$ has $C_{\cdot 0}$ cnu part if and only if* $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$.

3. Property (β)

For every point ν in the complex plane $\mathbb{C}$ let $U_\nu \subseteq \mathbb{C}$ denote an arbitrary open neighborhood of ν. Let f denote an arbitrary $\mathcal{H}$-valued analytic function on an open subset of $\mathbb{C}$. An operator $A \in B(\mathcal{H})$ is said to have the single valued extension property (SVEP, for short) at a point $\mu \in \mathbb{C}$ if for every $U_\mu \subseteq \mathbb{C}$ the only solution $f\colon U_\mu \to \mathcal{H}$ to the equation

$$(A - \lambda)f(\lambda) = 0 \quad \text{for every} \quad \lambda \in U_\mu$$

is the null function ($f = 0$). We say that A has SVEP if A has SVEP at every $\mu \in \mathbb{C}$. Since every operator A has SVEP at every point of the resolvent set $\rho(T) = \mathbb{C}\backslash\sigma(A)$, it follows that A has SVEP if it has SVEP at every point of the spectrum $\sigma(A)$. The local resolvent set of an operator $A \in B(\mathcal{H})$ at a vector $x \in \mathcal{H}$ is the set

$$\rho_A(x) = \big\{\mu \in \mathbb{C}\colon\ (A - \lambda)f(\lambda) = x \ \text{ for every } \ \lambda \in U_\mu \ \text{ for some } \ f\colon U_\mu \to \mathcal{H}\big\};$$

that is, $\rho_A(x)$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f\colon U \to \mathcal{H}$ which satisfies $(A - \lambda)f(\lambda) = x$ for all $\lambda \in U$; the local spectrum $\sigma_A(x)$ of A at x is then the complement of local resolvent set: $\sigma_A(x) = \mathbb{C}\backslash\rho_A(x)$ (see, e.g., [8]). Take an arbitrary closed subset F of $\mathbb{C}$. Consider the (not necessarily closed) linear manifold of $\mathcal{H}$,

$$X_A(F) = \big\{x \in \mathcal{H}\colon\ \sigma_A(x) \subseteq F\big\} = \big\{x \in \mathcal{H}\colon\ U_F = \mathbb{C}\backslash F \subseteq \rho_A(x)\big\},$$

called the local (analytic) spectral subspace of A associated to F, and set

$$\mathcal{X}_A(F) = \big\{x \in \mathcal{H}\colon\ (A - \lambda)f(\lambda) = x \ \text{ for every } \ \lambda \in U_F = \mathbb{C}\backslash F \ \text{ for some } \ f\colon U_F \to \mathcal{H}\big\},$$

called the *glocal* (analytic) spectral subspace of A associated to F (*glocal* because the analytic function in its definition is globally defined but dependant on each $x \in \mathcal{H}$), which is a hyperinvariant linear manifold of A such that $\mathcal{X}_A(F) \subseteq X_A(F)$. The operator A has SVEP precisely when $\mathcal{X}_A(F) = X_A(F)$ for every closed set $F \subseteq \mathbb{C}$ [28, Proposition 1] — also see [30, p.220]. An operator $A \in B(\mathcal{H})$ is said to satisfy (Dunford's) condition (C) if $X_A(F)$ is closed for every closed set $F \subseteq \mathbb{C}$ (cf. [18, p.2135, XVI.1]). If $\mathcal{H} = \mathcal{X}_A(\overline{U_1}) + \mathcal{X}_A(\overline{U_2})$ for every open covering $\{U_1, U_2\}$ of $\mathbb{C}$, then A is said to satisfy property (δ) (or has the decomposition property (δ)). The operator A is decomposable if A has property (δ) and $\mathcal{X}_A(F)$ is closed for every closed set $F \subseteq \mathbb{C}$. In fact, an operator is decomposable if and only if it has both property (δ) and condition (C) [8, Proposition 1.3.8]. Let $\mathcal{A}(U, \mathcal{H})$ denote the collection of all $\mathcal{H}$-valued analytic functions defined on an open subset U of $\mathbb{C}$. An operator $A \in B(\mathcal{H})$ has (Bishop's) property (β) if, for every open set $U \subseteq \mathbb{C}$, the map

$$\big(A - (\cdot)\big) f(\cdot) \colon \mathcal{A}(U, \mathcal{H}) \to \mathcal{H} \;\text{ is injective and has closed range.}$$

Equivalently, an operator $A \in B(\mathcal{H})$ satisfies property (β) if, for every open set $U \subseteq \mathbb{C}$ and every sequence $\{f_n\}$ of functions in $\mathcal{A}(U, \mathcal{H})$, convergence of the sequence $\big\{\big(A - (\cdot)\big) f_n(\cdot)\big\}$ to 0 in $\mathcal{A}(U, \mathcal{H})$, implies convergence to zero in $\mathcal{A}(U, \mathcal{H})$ for the sequence $\{f_n\}$. In other words, $A \in B(\mathcal{H})$ satisfies property (β) if, for every open set $U \in \mathbb{C}$ and every sequence of analytic functions $f_n : U \to \mathcal{H}$ such that

$$\big(A - (\cdot)\big) f_n(\cdot) \to 0 \quad \text{as} \quad n \to \infty$$

locally uniformly on U (i.e., uniformly on compact subsets of U), then $f_n(\cdot) \to 0$ as $n \to \infty$ locally uniformly on U [6, Definition 8] — actually, the original definition in [6] was slightly different but equivalent). Properties (β) and (δ) are dual of each other [2, Theorems 19 and 21]: A satisfies property (β) if and only if A^* satisfies property (δ). Property (β) implies condition (C), which in turn implies SVEP [28, Proposition 1.2]:

$$\text{property } (\beta) \;\subseteq\; \text{condition } (C) \;\subseteq\; \text{SVEP}.$$

However, condition (C) does not imply property (β) [31]. Let $\sigma_{AP}(A) = \{\lambda \in \mathbb{C} \colon A - \lambda$ is not bounded below$\} \subseteq \sigma(A)$ be the approximate point spectrum of A. If A satisfies property (δ), then $\sigma_{AP}(A) = \sigma(A)$ [29, Corollary 1.7]. According to the above mentioned duality, if A^* satisfies (Bishop's) property (β), then $\sigma_{AP}(A) = \sigma(A)$.

4. When Condition (C) $\subseteq$ $\mathbf{PF}(\delta) \cap \mathbf{PF}(\Delta)$

Take an arbitrary $A \in B(\mathcal{H})$. The operator A is normaloid if its norm coincides with its spectral radius, $\|A\| = r(A)$, and A is hereditarily normaloid if every part of A is normaloid. An operator A (not necessarily a contraction) is of class $\mathcal{C}_{0\cdot}$ if $\inf_{n \geq 0} \|A^n x\| = 0$ for every $x \in \mathcal{H}$, of class $\mathcal{C}_{1\cdot}$ if $\inf_{n \geq 0} \|A^n x\| > 0$ for every $0 \neq x \in \mathcal{H}$, of class $\mathcal{C}_{\cdot 0}$ if A^* is of class $\mathcal{C}_{0\cdot}$, and of class $\mathcal{C}_{\cdot 1}$ if A^* is of class $\mathcal{C}_{1\cdot}$. An operator A is said to be of class $\mathcal{C}_{\alpha\,\beta}$ if A is of class $\mathcal{C}_{\alpha\cdot} \cap \mathcal{C}_{\cdot\beta}$ for any combination $\alpha, \beta = 0, 1$. We say that a part $A|_{\mathcal{M}}$ of A is contractive if $A|_M$ is a contraction, and that a contractive part $A|_M$ of A has a $C_{\cdot 0}$ completely nonunitary part (i.e., a $C_{\cdot 0}$ cnu part), if the cnu part of the contraction $A|_M$ is of class $\mathcal{C}_{\cdot 0}$ (i.e., if the cnu part of $A|_M$ is a $\mathcal{C}_{\cdot 0}$-contraction). We shall need the following results.

Lemma 4.1 *Let $A \in B(\mathcal{H})$ be a contraction. The following assertions are equivalent.*

(a) *A has a $\mathcal{C}_{\cdot 0}$ cnu part.*

(b) *$A \in \mathrm{PF}(\delta)$.*

(c) *$A \in \mathrm{PF}(\Delta)$.*

Thus, a contraction $A \in B(\mathcal{H})$ has a $\mathcal{C}_{\cdot 0}$ cnu part if and only if $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$.

Proof. A contraction A has a $\mathcal{C}_{\cdot 0}$ cnu part if and only if $A \in \mathrm{PF}(\delta)$ [26, Theorem 1], and a power bounded operator A is in $\mathrm{PF}(\delta)$ if and only if it is in $\mathrm{PF}(\Delta)$ [17, Theorem 2.3]. $\square$

Contractions belonging to a subclass $\mathcal{S}$ of $B(\mathcal{H})$ may possess property $\mathrm{PF}(\delta)$ without operators (not necessarily contraction operators) in $\mathcal{S}$ possessing property $\mathrm{PF}(\delta)$. Thus paranormal contractions (i.e., contractions $A \in B(\mathcal{H})$ such that $\|Ax\|^2 \leq \|A^2x\|$ for all unit vectors $x \in \mathcal{H}$) have $\mathcal{C}_{\cdot 0}$ cnu part [33, 12, 15, 34], and so belong to $\mathrm{PF}(\delta)$ by the above lemma. For a general (not a contraction) paranormal operator $A \in B(\mathcal{H})$, if one assumes that 0 is a normal eigenvalue of A (i.e., if $\ker(\mathrm{A}) \subseteq \ker(\mathrm{A}^*)$), then a necessary and sufficient condition for $A \in \mathrm{PF}(\delta)$ is that the unitary subspaces of A reduce A [17]. Unlike the result in the preceding lemma, the following result does not require that A is power bounded (i.e., it does not require that $\sup_{n\geq 0}\|A^n\| < \infty$).

Lemma 4.2 *For every operator $A \in B(\mathcal{H})$, if $A \in \mathrm{PF}(\delta)$, then $A \in \mathrm{PF}(\Delta)$.*

Proof. See [17]. $\square$

Lemma 4.3 *If $X \in \delta^{-1}_{A,B}(0) \cap \delta^{-1}_{A^*,B^*}(0)$ for some $A, B \in B(\mathcal{H})$, then $\overline{\mathrm{ran\ X}}$ reduces A, $\ker X^{\perp}$ reduces B, and $A|_{\overline{\mathrm{ran\ X}}}$ and $B|_{\ker X^{\perp}}$ are unitarily equivalent normal operators.*

Proof. This is a well known consequence of the original Putnam–Fuglede Theorem, where the normality assumption is replaced with the condition $X \in \delta^{-1}_{A,B}(0) \cap \delta^{-1}_{A^*,B^*}(0)$, which always happens under normality assumption by the Putnam–Fuglede Theorem, and is indeed enough to complete the proof. See e.g., [24, Corollary 6.50]. $\square$

The following theorem is our main result.

Theorem 4.4 *If an operator $A \in B(\mathcal{H})$ is such that*

(i) *A is hereditarily normaloid,*

(ii) *the contractive parts of A have a $\mathcal{C}_{\cdot 0}$ cnu part,*

(iii) *A satisfies (Dunford's) condition (C),*

then $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$.

Proof. In view of Lemma 4.2 it would suffice to prove that $A \in \mathrm{PF}(\delta)$. Let $X \in \delta^{-1}_{A,V^*}(0)$, where $V \in B(\mathcal{H})$ is an isometry. Decompose A into its normal and pure (i.e., completely nonnormal) parts, $A = A_n \oplus A_p \in B(\mathcal{H}_n \oplus \mathcal{H}_p)$, and V into its unitary and cnu (i.e., into its unitary and unilateral shift) parts $V = V_u \oplus V_c \in B(\mathcal{H}_u \oplus \mathcal{H}_c)$, and let the transformation $X\colon \mathcal{H}_u \oplus \mathcal{H}_c \to \mathcal{H}_n \oplus \mathcal{H}_p$ have the corresponding matrix representation $X = [X_{ij}]^2_{i,j=0}$. Applying the Putnam–Fuglede Theorem for normal A_n and unitary V_u, it then follows that $A_nX_{11} - X_{11}V_u^* = 0$ implies $A_n^*X_{11} - X_{11}V_u = 0$. Considering the equation $A_nX_{12} - X_{12}V_c^* = 0$, the Putnam–Fuglede Theorem for normal A_n and cohyponormal V_c^* [11] implies that $A_n^*X_{12} - X_{12}V_c = 0$ also. Suppose $X_{12} \neq 0$. Then Lemma 4.3 implies that $V_c|_{\ker X_{12}{}^{\perp}}$ is unitary (and unitarily equivalent to $A_n|_{\overline{\mathrm{ran\ X_{12}}}}$). Since this contradicts the fact that V_c is cnu, it follows that $X_{12} = 0$. This leaves us with the equations

$$A_pX_{21} - X_{21}V_u^* = 0 \quad \text{and} \quad A_pX_{22} - X_{22}V_c^* = 0.$$

Suppose $X_{21} \neq 0$, and consider the following restrictions: $A_{p1} = A_p|_{\overline{\mathrm{ran\ X_{21}}}}$ in $B(\overline{\mathrm{ran\ X_{21}}})$, $V_{u1} = V_u|_{\ker X_{21}{}^{\perp}}$ in $B(\ker X_{21}{}^{\perp})$, and $Y = X_{21}|_{\ker X_{21}{}^{\perp}}\colon \ker X_{21}{}^{\perp} \to \mathrm{ran\ X_{12}}$, which is a quasiaffinity in $B(\ker X_{21}{}^{\perp}, \overline{\mathrm{ran\ X_{12}}})$ (i.e. $Y = X_{21}|_{\ker X_{21}{}^{\perp}}$ is injective with dense range). Observe that $V_{u1} = V_u|_{\ker X_{21}{}^{\perp}}$ is unitary, and that

$$X_{21} \in \delta^{-1}_{A_p,V_u^*}(0) \implies Y \in \delta^{-1}_{A_{p1},V_{u1}^*}(0),$$

where the unitary operator V_{u1}^* satisfies (decomposition) property (δ) and A_{p1} satisfies (Dunford's) condition (C), according to assumption (iii). (Recall that condition (C) is inherited by restrictions to (closed) invariant subspaces [30, Proposition 1.2.21].) Applying [30, Proposition 3.7.11] we conclude that the spectrum $\sigma(A_{p1}) = \sigma(V_{u1}^*)$ is contained in the boundary of the unit disc in the complex plane, so that A_{p1} has a unit spectral radius: $r(A_{p1}) = 1$. Assuming, as we did in (i), that A is hereditarily normaloid, A_{p1} is normaloid. Consequently, A_{p1} is a contraction. Since contractive parts of A have $\mathcal{C}_{\cdot 0}$ cnu parts by hypothesis (ii), it follows that $A_{p1} \in \mathcal{C}_{\cdot 0}$. Now apply Lemma 4.1 to conclude $Y \in \delta^{-1}_{A_{p1},V_{u1}^*}(0) \cap \delta^{-1}_{A_{p1}^*,V_{u1}}(0)$, and then Lemma 4.3 to conclude $A_{p1} = A_{p1}|_{\overline{\text{ran Y}}}$ and $V_{u1}^* = V_{u1}^*|_{\ker Y^\perp}$ are unitarily equivalent normal (therefore, unitary and hence C_{11}) operators. This is a contradiction. Hence $X_{21} = 0$ (and so also $Y = 0$). Next we prove that $X_{22} = 0$. If $X_{22} \neq 0$, then letting $A_{p2} = A_p|_{\overline{\text{ran X}_{22}}}$, $V_{c2} = V_c|_{\ker X_{22}{}^\perp}$ and letting $Z = X_{22}|_{\ker X_{22}{}^\perp}\colon \ker X_{22}{}^\perp \to \text{ran X}_{22}$ be the restriction of X_{22} to $\ker X_{22}{}^\perp$, we have, as before, that $Z \in \delta^{-1}_{A_{p2},V_{c2}^*}(0)$. Arguing as above, by using assumptions (i), (ii) and (iii), it is seen that A_{p2} is a contraction with a $\mathcal{C}_{\cdot 0}$ cnu part, and A_{p2} is unitarily equivalent to the unitary V_{c2}^* (and so it must be of class $\mathcal{C}_{11}$), which is another contradiction. Hence $X_{22} = 0$. Therefore, $X = X_{11} \oplus 0$, and so

$$(A_n \oplus A_p)(X_{11} \oplus 0) - (X_{11} \oplus 0)(V_u^* \oplus V_c^*) = 0$$

and

$$(A_n^* \oplus A_p^*)\,(X_{11} \oplus 0) - (X_{11} \oplus 0)\,(V_u \oplus V_c) = 0,$$

which completes the proof. $\square$

Remark 4.5 Recall that $A \in B(\mathcal{H})$ is k-paranormal (or k-$*$-paranormal) for some integer $k \geq 1$ if $\|Ax\|^{k+1} \leq \|A^{k+1}x\|\,\|x\|^k$ (or $\|A^*x\|^{k+1} \leq \|A^{k+1}x\|\|x\|^k$) for every $x \in \mathcal{H}$. Every k-$*$-paranormal operator is $(k{+}1)$-paranormal, and k-paranormal operators satisfy the properties that: (a) They are normaloid; (b) every part of a k-paranormal operator is again k-paranormal; (c) k-paranormal contractions have $\mathcal{C}_{\cdot 0}$ cnu parts, and (d) if $AX = XV^*$ for a k-paranormal operator A and unitary V, then ($\{\|XV^n x\|\}$ is a constant sequence for every $x \in \mathcal{H}$, hence) $\|AXx\| = \|Xx\|$ for every $x \in \mathcal{H}$ [34, 15]. Thus, if A is k-paranormal or k-$*$-paranormal and $A_{p1}Y - YV_{u1}^* = 0$ for some quasiaffinity Y and unitary V_{u1}, then A_{p1} is isometric (and this forces $Y = 0$). Again, if A is k-$*$-paranormal, then 0 is a normal eigenvalue of A (i.e., if 0 is in point spectrum $\sigma_p(A)$ of A, then the eigenspace corresponding to 0 is reducing). This, since $0 \in \sigma_p(V_{c2}^*)$ for a unilateral shift V_{c2}, implies that there is no quasiaffinity Z such that $A_{p2}Z - ZV_{c2}^* = 0$ for some pure k-$*$-paranormal operator A_{p2} and unilateral shift V_{c2}. A similar argument does not work for k-paranormal operators A for the reason that $A^{-1}(0) \nsubseteq A^{*-1}(0)$.

5. Paranormal Operators $\notin$ Condition (C)

Hyponormal operators $A \in B(\mathcal{H})$, $AA^* \leq A^*A$, and p-hyponormal operators $A \in B(\mathcal{H})$, $(AA^*)^p \leq (A^*A)^p$, $0 < p \leq 1$, are well known to be hereditarily normaloid and to satisfy (Bishop's) property (β). (Recall: Property (β) implies condition (C).) Furthermore, they have $\mathcal{C}_{\cdot 0}$ cnu contractive parts [12, 26, 34]. Hence hyponormal and p-hyponormal operators $A \in \text{PF}(\delta) \cap \text{PF}(\Delta)$. M-hyponormal operators $A \in B(\mathcal{H})$ (i.e., operator A for which there exists a real $M > 0$ such that $|(A - \lambda I)^*|^2 \leq M|A - \lambda I|^2$ for all complex λ) are not normaloid but satisfy the other two properties above. Dominant operators $A \in B(\mathcal{H})$ (i.e., operators A such that for every complex λ there exists a scalar $M_\lambda > 0$ such that $|(A - \lambda I)^*|^2 \leq M_\lambda|A - \lambda I|^2$) have $\mathcal{C}_{\cdot 0}$ cnu parts and are known not to be normaloid; it is not known if dominant operators satisfy condition (C). Clearly, M-hyponormal operators are dominant operators, and it is known that dominant operators $A \in \text{PF}(\delta)$ [11]. Hence $A \in \text{PF}(\delta) \cap \text{PF}(\Delta)$ for M-hyponormal or dominant operators A.

For an operator $A \in B(\mathcal{H})$ with polar decomposition $A = U|A|$, let $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ in $B(\mathcal{H})$ denote the (first) Aluthge transform of A. $A \in B(\mathcal{H})$ is said to be a w-hyponormal operator if $|\tilde{A}^*| \le |A| \le |\tilde{A}|$. Every w-hyponormal operator is paranormal [4], hence hereditarily normaloid with $\mathcal{C}_{\cdot 0}$ cnu contractive parts. Furthermore, w-hyponormal operators satisfy property (β) [5]. Hence w-hyponormal operators satisfy the PF-property.

Summarizing the above, we have the following result.

Proposition 5.1 *If $A \in B(\mathcal{H})$ is either p-hyponormal $(0 < p \le 1)$ or dominant or w-hyponormal or k-$*$-paranormal (for some integer $k \ge 1$), then $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$.*

Proposition 5.1 fails for k-paranormal operators. Thus let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}} \cup \{f_n\}_{n\in\mathbb{N}}$, and let $A \in B(\mathcal{H})$ be the operator such that $A(e_n) = e_{n+1}$ for all integers $n \in \mathbb{Z}$ and $A(f_n) = x_n e_n + y_n e_{n+1}$ for $n \in \mathbb{N}$, where the real bounded sequences $\{x_n\}$ and $\{y_n\}$ satisfy $x_n = y_n x_{n+1}$ and $y_{n+1} = (x_{n+1}^2 + 1)y_n$ for all $n \in \mathbb{N}$. Then A is a paranormal (i.e., 1-paranormal) operator such that $A \notin \mathrm{PF}(\delta)$ [35, Example 5.2]. Combining this observation with Theorem 4.4 we have the next result.

Corollary 5.2 *Paranormal operators do not necessarily satisfy condition (C).*

Proof. Suppose to the contrary that paranormal operators have condition (C). Then the normaloid property of paranormal operators coupled with the facts that parts of a paranormal operator are paranormal and a cnu paranormal contraction is of the class $\mathcal{C}_{\cdot 0}$ implies by Theorem 4.4 that $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$. This is a contradiction. □

Recall, [19], that every subscalar Banach space operator satisfies property (β): Paranormal operators are not subscalar. A paranormal operator $A \in B(\mathcal{H})$ is said to be totally paranormal if $\|(A - \lambda I)x\|^2 \le \|(A - \lambda I)^2 x\|$ for every $\lambda \in \mathbb{C}$ and every unit vector $x \in \mathcal{H}$. Contractions belonging to a subclass $\mathcal{S} \subseteq B(\mathcal{H})$ may possess property $\mathrm{PF}(\delta)$ without operators (not necessarily contraction operators) in $\mathcal{S}$ possessing property $\mathrm{PF}(\delta)$. Thus paranormal contractions; that is, contractions $A \in B(\mathcal{H})$ such that $\|Ax\|^2 \le \|A^2x\|$ for every unit vector $x \in \mathcal{H}$, have $\mathcal{C}_{\cdot 0}$ cnu part, and so satisfy $\mathrm{PF}(\delta)$ [33, 12, 15, 34]. However, there exists a paranormal operator $A \in B(\mathcal{H})$ such that A does not satisfy property $\mathrm{PF}(\delta)$ [35, Example 5.2]. An examination of [35, Example 5.2] shows that paranormal operator A of the example has the property that the unitary subspaces (i.e., closed subspaces of the operator such that its restriction to the subspace is unitary) of the operator are not reducing. (In general, the eigenspaces — more generally, normal subspaces — of a paranormal operator are not reducing.) If one assumes that 0 is a normal eigenvalue (i.e., if $\ker A \subseteq \ker A^*$) of the a paranormal operator A, then a necessary and sufficient condition for $A \in \mathrm{PF}(\delta)$ is that the unitary subspaces of A reduce A. Totally paranormal operators satisfy condition (C) [27]. As a consequence we have the following result.

Proposition 5.3 *If an $A \in B(\mathcal{H})$ is totally paranormal, then $A \in \mathrm{PF}(\delta) \cap \mathrm{PF}(\Delta)$.*

6. k-quasi-(P) operators

Pagacz [34, Proposition 4.1] proves that a cnu (p, k)-quasihyponormal contraction is of class $\mathcal{C}_{\cdot 0}$. Here an $A \in B(\mathcal{H})$ is a (p, k)-quasihyponormal operator for some integer $k \ge 1$ and a real number $0 < p \le 1$ if ${A^*}^k(|A|^{2p} - |A^*|^{2p})A^k \ge 0$. It is clear that, if a (p, k)-quasihyponormal operator has dense range, then it is p-hyponormal. The restriction of a (p, k)-quasihyponormal operator to a closed invariant subspace is again (p, k)-quasihyponormal, and every (p, k)-quasihyponormal operator $A \in B(\mathcal{H})$ has an upper triangular matrix representation $A = \left(\begin{smallmatrix} A_1 & A_3 \\ 0 & A_2 \end{smallmatrix}\right)$ in $B(\overline{\mathrm{ran}\, \mathrm{A^k}} \oplus {A^*}^{k^{-1}}(0))$ such that A_1 is p-hyponormal and A_2 is k-nilpotent [22]. Let $(\mathbf{P})$ denote a class of operators in $B(\mathcal{H})$

defined by a positivity condition. Generalizing the definition of (p,k)-quasihyponormal operators, we say that an operator A in $(\mathbf{P})$ is k-quasi-$(\mathbf{P})$ for some integer $k \geq 1$ if [14]: (i) Every restriction of A to a closed invariant subspace is again k-quasi-$(\mathbf{P})$; (ii) A has dense range implies A is in $(\mathbf{P})$; (iii) A has an upper triangular matrix representation $A = \left(\begin{smallmatrix} A_1 & A_3 \\ 0 & A_2 \end{smallmatrix}\right)$ with respect to some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$ such that $A_1 \in (\mathbf{P}) \cap B(\mathcal{H}_1)$ and $A_2 \in B(\mathcal{H}_2)$ is k-nilpotent. If contractions in $(\mathbf{P})$ satisfy the PF-property, then also k-quasi-$(\mathbf{P})$ contractions satisfy the PF-property .

Proposition 6.1 *The k-quasi-$(\mathbf{P})$ contraction $A \in B(\mathcal{H})$ has $\mathcal{C}_{\cdot 0}$ cnu part if and only if (the contraction) $A_1 \in B(\mathcal{H}_1)$ has $\mathcal{C}_{\cdot 0}$ cnu part.*

Proof. If we let $S = \delta^k_{A_1,-A_2}(A_3)$, then

$$A^{n+k} = \begin{pmatrix} A_1^{n+k} & A_1^n S \\ 0 & 0 \end{pmatrix}$$

and

$$\|A_1^{*n}(A_1^{*k})x\| \leq \|A^{*n+k}(x \oplus y)| \leq (\|A_1^{*k}\| + \|S^*\|)\|A_1^{*n}x\| \leq (1+2^k)\|A_1^{*n}x\|$$

for every $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$. The proof follows. □

The following corollary is immediate from Proposition 6.1.

Corollary 6.2 *The k-quasi-$(\mathbf{P})$ contraction $A \in B(\mathcal{H})$ satisfies the PF-property if and only if the contraction $A_1 \in (\mathbf{P}) \cap B(\mathcal{H}_1)$ satisfies the PF-property.*

It is well known that the operator $A = A_1 \oplus A_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ has SVEP if and only if A_1 and A_2 have SVEP; SVEP for A_1 and A_2 implies SVEP for $A = \left(\begin{smallmatrix} A_1 & A_3 \\ 0 & A_2 \end{smallmatrix}\right)$ in $B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and then $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$. Considering the operator $A = A_1 \oplus A_2$ in $B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and a closed subset Ω of the complex plane, if A has SVEP then the local spectral subspace $X_A(\Omega)$ satisfies

$$\begin{aligned} X_A(\Omega) &= X_A(\Omega \cap \sigma(A)) = X_A(\{\Omega \cap \sigma(A_1)\} \cup \{\Omega \cap \sigma(A_2)\}) \\ &= X_{A_1}(\Omega \cap \sigma(A_1))) \oplus X_{A_2}(\Omega \cap \sigma(A_2)) = X_{A_1}(\Omega) \oplus X_{A_2}(\Omega). \end{aligned}$$

Thus $X_A(\Omega))$ is closed if and only if $X_{A_i}(\Omega)$ is closed for $i = 1, 2$: Equivalently, A satisfies condition (C) if and only if A_i satisfy condition (C) for $i = 1, 2$. Evidently, nilpotent operators satisfy condition (C).

Lemma 6.3 *If a k-quasi-$(\mathbf{P})$ operator $A \in B(\mathcal{H})$ satisfies condition (C), then $A_1 \in (\mathbf{P})$ satisfies condition (C). Conversely, if $A_1 \in (\mathbf{P})$ satisfies condition (C) and $z^k|_{\sigma(A)}$ is injective, then $A \in$$k$-quasi-$(\mathbf{P})$ satisfies condition (C).*

Proof. That A_1 satisfies condition (C) is immediate from the fact that if A satisfies condition (C), then so does its restriction to an invariant subspace. Conversely, suppose that A_1 satisfies condition (C). Recall from [1, Corollary 2.6] that if R and S are Banach space operators such that RS and SR are well defined, then RS satisfies condition (C) if and only if SR satisfies condition (C). Since

$$A^k = \begin{pmatrix} A_1^k & \delta^{k-1}_{A_1,-A_2}(A_3) \\ 0 & 0 \end{pmatrix},$$

and since A_1 satisfies condition (C) implies A_1^k satisfies condition (C) for every $k \geq 1$,

$$A^k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \delta^{k-1}_{A_1,-A_2}(A_3) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1^k & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies condition (C) if and only if $\left(\begin{smallmatrix} A_1^k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ satisfies condition (C). Thus, A^k satisfies condition (C). The hypothesis that the polynomial z^k is injective on $\sigma(A)$ now implies that A satisfies condition (C) [28, Remark 3.3.10]. $\square$

It is well known that p-hyponormal operators $(0 < p \leq 1)$, M-hyponormal operators and w-hyponormal operators satisfy property (β), and hence also condition (C). Thus, k-quasi-p-hyponormal operators (equivalently, (p,k)-quasihyponormal operators), k-quasi-M-hyponormal operators and k-quasi-w-hyponormal operators satisfy condition (C) (indeed, property (β)). M hyponormal operators are not normaloid; if A does not have a dense range, then the operators belonging to neither of the classes k-quasi-p-hyponormal and k-quasi-w-hyponormal are normaloid. Nevertheless we have the following. Let $(\mathbf{P_1})$ denote the class of operators $A \in B(\mathcal{H})$ which are p-hyponormal or w-hyponormal or M-hyponormal or dominant.

Theorem 6.4 *Operators $A \in k$-quasi $(\mathbf{P_1})$ satisfy the PF-property.*

Proof. If we decompose A into its normal and pure parts, V into its unitary and cnu parts, and let $X \in \delta_{A,V^*}^{-1}(0)$ have the (corresponding) matrix representation $X = [X_{ij}]_{i,j=1}^2$, then the argument of the proof of Theorem 4.4 shows that

$$A_n^* X_{11} - X_{11} V_u = 0 \qquad \text{and} \qquad X_{12} = 0.$$

Furthermore, in the notation of the proof of Theorem 4.4,

$$A_{p1} Y - Y V_{u1}^* = 0 = A_{p2} Z - Z V_{c2}^*.$$

Since Y, Z are quasiaffinities and V_{u1}^*, V_{c2}^* have dense range, A_{p1} and A_{p2} have dense range. By hypothesis, both A_{p1} and A_{p2} are k-quasi-$(\mathbf{P_1})$ operators; hence A_{pi} in $(\mathbf{P_1})$ for $i = 1, 2$. Applying Proposition 5.1 it now follows that $Y = Z = 0$; hence $X = X_{11} \oplus 0$ and $X \in \delta_{A^*,V}^{-1}(0)$. $\square$

General k-quasiparanormal operators are neither normaloid nor do they satisfy condition (C); they do not satisfy the PF-property. It is straightforward to verify that $A \in B(\mathcal{H})$ is $*$-paranormal if and only if $A^{*2}A^2 - 2\lambda AA^* + \lambda^2 \geq 0$ for all real λ, and then $A \in k$-quasi-$*$-paranormal if and only if $A^{*k}(A^{*2}A^2 - 2\lambda AA^* + \lambda^2)A^k \geq 0$. Theorem 6.4 extends to k-quasi-$*$-paranormal operators, as follows upon combining the argument of the proof of Theorem 6.4 with the facts (observed in Remark 4.5) that A_{p1} is isometric and 0 is a normal eigenvalue of a $*$-paranormal operator.

7. Dynamics of power bounded operators in PF(δ)

For a Banach space operator $T \in B(\mathcal{X})$ and a (nontrivial) vector $x \in \mathcal{X}$, the orbit of x under T is the set $\mathrm{Orb}(T,x) = \{x, Tx, T^2x, ...\}$; the operator T is supercyclic, with a supercyclic vector x, if the set of scalar multiples of the elements of $\mathrm{Orb}(T,x)$ is dense in $\mathcal{X}$. It is immediate that if $T \in B(\mathcal{X})$ has a supercyclic vector, then $\mathcal{X}$ is separable and T has a dense range. Recall from [3, Theorem 2.2] that if $T \in B(\mathcal{X})$ is a power bounded operator (i.e., if $\sup_n \|T^n\| < \infty$) which is supercyclic, then $T \in \mathcal{C}_{0\cdot}$. We prove below that this observation describes in a natural way the non-supercyclicity of power bounded operators in $A \in B(\mathcal{H}) \cap \mathrm{PF}(\delta)$.

For an operator $A \in B(\mathcal{H})$, let $S(\delta_{A,V^*})$ denote the set

$$S(\delta_{A,V^*}) = \big\{V \in B(\mathcal{H}) : V \text{ isometric}, \delta_{A,V^*}^{-1}(0) \neq \{0\}\big\}.$$

It is then immediate from $X \in \delta^{-1}_{A,V^*}(0)$ implies $X \in \delta^{-1}_{A^*,V}(0)$ for an operator A in $B(\mathcal{H}) \cap \mathrm{PF}(\delta)$ that if $S(\delta_{A,V^*}) \neq \varnothing$, then A is the direct sum of a unitary operator and a (possibly trivial) operator. If the operator $A \in B(\mathcal{H}) \cap \mathrm{PF}(\delta)$ is power bounded, then $A = A_u \oplus A_c$, where A_u is unitary and A_c is a (possibly trivial) $\mathcal{C}_{\cdot 0}$ operator [35].

Theorem 7.1 *If an operator $A \in B(\mathcal{H}) \cap \mathrm{PF}(\delta)$ is a power bounded operator such that $S(\delta_{A,V^*}) \neq \varnothing$, then neither A nor A^* is supercyclic.*

Proof. Evidently, A is power bounded if and only if A^* is power bounded. If A (resp., A^*) is supercyclic, then $A \in \mathcal{C}_{0\cdot}$ (resp., $A \in \mathcal{C}_{\cdot 0}$) and so A can not have a $\mathcal{C}_{11}$, in particular a unitary, part. Thus, if also $A \in \mathrm{PF}(\delta)$, then $A \in \mathcal{C}_{00}$ if A is supercyclic and $A \in \mathcal{C}_{\cdot 0}$ if A^* is supercyclic. But then the implication $AX - XV^* = 0 \Longrightarrow A^*X - XV = 0$ holds if and only if $X = 0$ for all $V \in B(\mathcal{H})$. Thus, for an $A \in \mathrm{PF}(\delta)$, A or A^* is supercyclic only if $S(\delta_{A,V^*}) = \varnothing$. Contrapositively, for an $A \in \mathrm{PF}(\delta)$, A and A^* are not supercyclic whenever $S(\delta_{A,V^*}) \neq \varnothing$. $\square$

Considering contractions $A \in B(\mathcal{H})$ which are either p-hyponormal $(0 < p \leq 1)$ or w-hyponormal or dominant or paranormal or $*$-paranormal, it is trivially verified that the direct sum of A with a unitary is again a contraction of the same class as A. Such contractions A satisfy property $\mathrm{PF}(\delta)$ [12, 15, 25, 26, 33, 34, 35], and if A is not cnu, then $S(\delta_{A,V^*}) \neq \varnothing$. Hence:

Operators $A \in B(\mathcal{H})$ which are either p-hyponormal or w-hyponormal or dominant or paranormal or $*$-paranormal fail (*in general*) *to be supercyclic.*

Indeed more is true. An operator $A \in B(\mathcal{H})$ is n-finitely supercyclic for some integer $n \geq 1$ if there is an n-dimensional subspace $\mathcal{M}$ of $\mathcal{H}$ such that $\mathrm{Orb(A,\mathcal{M})} = \bigcup\{\mathcal{M}, \mathrm{A}(\mathcal{M}), \mathrm{A}^2(\mathcal{M}), ...\}$ is dense in $\mathcal{H}$. Recall from [20] that n-supercyclicity does not imply $(n-1)$-supercyclicity. We assume in the following theorem that our contractions $A \in B(\mathcal{H})$ are nontrivial in the sense that their unitary part (whenever it exists) is not an isometry on a finite dimensional subspace of $\mathcal{H}$.

Theorem 7.2 *If an operator $A \in B(\mathcal{H}) \cap \mathrm{PF}(\delta)$ is a contraction, then*

(i) *A n-supercyclic implies $S(\delta_{A,V^*}) = \varnothing$,*

(ii) *$S(\delta_{A,V^*}) \neq \varnothing$ implies neither of A and A^* is n-supercyclic.*

Proof. (i) Decomposing A into its unitary and cnu parts by $A = A_u \oplus A_c$ it is seen that A n-supercyclic implies A_u is m-supercyclic for some $m \leq n$ [7, Proposition 2.3]. Since no unitary on an $(m+1)$-dimensional Hilbert space can be m-supercyclic ([20, Theorem 4.9] and [7, Corollary 2.2]), we must have $A = A_c$. This implies $S(\delta_{A,V^*}) = \varnothing$.

(ii) If $S(\delta_{A,V^*}) \neq \varnothing$, then the hypothesis $A \in \mathrm{PF}(\delta)$ implies that $A = A_u \oplus A_c$ (with respect to some decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$ of $\mathcal{H}$), where A_u is unitary and $A_c \in \mathcal{C}_{\cdot 0}$ is cnu. Trivially, the spectrum $\sigma(A_u)$ is a subset of $\partial(\mathbb{D})$, the boundary of the unit disc in the complex plane. The operator A_c being a contraction, $\|A_c\| \leq 1$. If $\|A_c\| < 1$ or $r(A_c) < \|A_c\| = 1$, then $r(A_c) < 1$, and so there exists a number $\rho < 1$ such that $\sigma(A_c) \subseteq \{\lambda : 0 < |\lambda| < \rho\}$. If, instead, $r(A_c) = \|A_c\| = 1$, then (A_c is normaloid and) there exists a λ (in the *peripheral spectrum* of A_c [21, p.225]) such that $|\lambda| = 1$, $\mathrm{asc(A_c - \lambda)} \leq 1$ and $\dim(A_c^* - \overline{\lambda})^{-1}(0) > 0$ [21, Proposition 54.2]. (Here, the *ascent* $\mathrm{asc(A_c - \lambda)}$ of A_c at λ is the least nonnegative integer t such that $(A_c - \lambda)^{-(t+1)}(0) = (A_c - \lambda)^{-t}(0)$.) If $x \in \mathcal{H}_c$ is an eigenvector corresponding to $\overline{\lambda}$ in the point spectrum of A_c^*, then

$$\|(A_c - \lambda)x\|^2 = \|A_c x\|^2 - \|x\|^2 \leq 0.$$

Hence the eigenspace M corresponding to the eigenvalue λ of A_c reduces A_c and $A_c|_M$ in unitary. This contradiction implies that $r(A) \neq 1$. Conclusion: The spectrum

$$\sigma(A) = \sigma(A_u) \cup \sigma(A_c) \subseteq \big\{\lambda : \rho < |\lambda| = 1\big\} \cup \big\{\lambda : |\lambda| < \rho < 1\big\}.$$

Hence A is not n-supercyclic [20, Proposition 4.5]. Since a similar argument works in the case in which $A \in \mathrm{PF}(\delta)$ and A^* is n-supercyclic (observe that $A \in \mathrm{PF}(\delta)$ implies $A^* = A_u^* \oplus A_c^*$), the proof follows. □

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Erratum/Addendum

to the paper "PF property and property (β) for paranormal operators", Rend. Circ. Mat. Palermo 63 (2014), 129–140

by B.P. Duggal and C.S. Kubrusly

The proof of Theorem 7.2(ii) – *If* $A \in B(\mathcal{H}) \cap \mathrm{PF}(\delta)$ *is a contraction, then* $S(\delta_{A,V^*}) \neq \varnothing$ *implies* A *is not* n*-supercyclic* – of our paper of the title is incomplete (in that it fails to consider the case $\alpha(A_c^* - \overline{\lambda}) = 0$). We provide here additional argument to complete the proof, and prove an analogue of the result for weakly supercyclic operators.

We follow the notation and terminology of the paper of the title. Thus $B(\mathcal{H})$ denotes the algebra of bounded linear operators on an infinite dimensional complex Hilbert space, $\delta_{A,B} \in B(B(\mathcal{H}))$ denotes the generalized derivation $\delta_{A,B}(X) = AX - XB$, and an operator $A \in B(\mathcal{H})$ satisfies the *Putnam–Fuglede property* δ, denoted $A \in \mathrm{PF}(\delta)$, if whenever the equation $AX = XV^*$ holds for some isometry V and operator $X \in B(\mathcal{H})$, then also $A^*X = XV$. An operator $A \in B(\mathcal{H})$ is *n-supercyclic* for some $n \in \mathbb{N}$ if $\mathcal{H}$ has an n-dimensional subspace M with dense orbit $\mathrm{Orb}_M(A) = \bigcup_{m\in\mathbb{N}} A^m M$; a 1-supercyclic operator is supercyclic, and we say that A is *weakly supercyclic* if there exists a vector $x \in \mathcal{H}$, with M the corresponding one dimensional subspace generated by x, such that $\mathrm{Orb}_M(A)$ is weakly dense (i.e., dense in the weak topology) in $\mathcal{H}$. It is clear that if an operator $A \in B(\mathcal{H})$ is n-supercyclic or weakly supercyclic, then $\mathcal{H}$ is separable (see [1, 2, 5, 7] for more information).

For an operator $A \in B(\mathcal{H})$, let $S(\delta_{A,V^*}) = \{V \in B(\mathcal{H}) : V \text{ isometric}, \delta_{A,V^*}^{-1}(0) \neq \{0\}\}$, and let $S_D(\delta_{A,V^*})$ denote the set of those isometries $V \in B(\mathcal{H})$ for which there exists an $X \in B(\mathcal{H})$ with dense range such that $\delta_{A,V^*}(X) = 0$. Clearly, if $A \in B(\mathcal{H})$ is a contraction, $S(\delta_{A,V^*}) \neq \varnothing$ and $A \in PF(\delta)$, then A is the direct sum of a unitary with some (possibly trivial) operator, and if also $S_D(\delta_{A,V^*}) \neq \varnothing$, then A is unitary.

Theorem 7.2 *Let A be a contraction in $B(\mathcal{H}) \cap PF(\delta)$.*

(a) *If A is n-supercyclic, then $S(\delta_{A,V^*}) = \varnothing$.*

(b) *If A is weakly supercyclic, then either*

(b_1) *$S(\delta_{A,V^*}) = \varnothing$, or*

(b_2) *$S(\delta_{A,V^*}) \neq \varnothing$ and A is a unitary (hence $S_D(\delta_{A,V^*}) \neq \varnothing$).*

Proof. If $S(\delta_{A,V^*}) \neq \varnothing$ and $A \in PF(\delta)$, then there exists a decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$ of $\mathcal{H}$ such that $A = A|_{\mathcal{H}_u} \oplus A|_{\mathcal{H}_c} = A_u \oplus A_c$, where A_u is unitary and A_c is completely non-unitary (indeed, a $C_{.0}$ contraction). (The component A_c may be absent if there exists an $X \in B(\mathcal{H})$ with dense range satisfying $\delta_{A,V^*}(X) = 0$.) Evidently, in such a case, $\sigma(A_u)$ is contained in the boundary $\partial\mathbf{D}$ of the unit disc in the complex plane, $||A_c|| \leq 1$ and the spectral radius $r(A_c)$ of A_c satisfies $r(A_c) \leq 1$.

The proof here is similar to that of [3, Theorem 7.2(ii)]. However, as earlier stated, the argument of the proof of [3, Theorem 7.2(ii)] is incomplete: We provide below the missing argument and reproduce here a complete proof for the reader's convenience.

(a) Suppose a contraction $A \in B(\mathcal{H}) \cap PF(\delta)$ is n-supercyclic and $S(\delta_{A,V^*}) \neq \varnothing$. Then, as observed above, $A = A_u \oplus A_c$, where A_u is unitary, $\sigma(A_u) \subseteq \{\lambda : |\lambda| = 1\} = \partial\mathbf{D}$ (= the boundary of the unit disc $\mathbf{D}$ in the complex plane). If $r(A_c) < 1$, then there exists a real number $\rho < 1$ such that $\sigma(A_c) \subseteq \{\lambda : |\lambda| \leq \rho\}$. Consequently, $\sigma(A) \subseteq \partial\mathbf{D} \cup \{\lambda : |\lambda| \leq \rho\}$, and this by [5, Proposition 4.5] is a contradiction. Hence $r(A_c) = 1$, which then implies that A_c is normaloid. Letting $\sigma_\pi(A_c) = \{\lambda \in \sigma(A_c) : |\lambda| = r(A_c)\}$ denote the *peripheral spectrum* of A_c, [6, Proposition 54.2] implies that $\mathrm{asc}(A_c - \lambda) \leq 1$ and $\beta(A_c - \lambda) = \mathrm{dim}(\mathcal{H}_c/(\mathrm{A_c}-\lambda)(\mathcal{H}_c)) > 0$ for all $\lambda \in \sigma_\pi(A_c)$. Since $\alpha(A_c^* - \overline{\lambda}) = \mathrm{dim}(\mathrm{A}_c^* - \overline{\lambda}) \leq \beta(\mathrm{A_c} - \lambda)$, either $\alpha(A_c^* - \overline{\lambda}) = 0$ or $\alpha(A_c^* - \overline{\lambda}) > 0$. We prove that neither of these alternatives is feasible, leading us to conclude that our hypothesis $r(A_c) = 1$ is not possible. Recall the (easily proved) fact that the eigenvalues μ of a contraction in $B(\mathcal{H})$ such that $|\mu| = 1$ are normal (i.e., the corresponding eigenspace is reducing). Hence, if $\alpha(A_c^* - \overline{\lambda}) > 0$, then A_c has a unitary direct summand — a contradiction. Consequently, $\alpha(A_c^* - \overline{\lambda}) = 0$. We claim that $\overline{\lambda} \notin \mathrm{iso}\sigma(A_c^*)$. If $\overline{\lambda} \in \mathrm{iso}\sigma(A_c^*)$, then $\mathcal{H}_c = P(\mathcal{H}_c) \oplus (I - P)(\mathcal{H}_c)$, where P is the Riesz projection associated with $\overline{\lambda}$ [6, Theorem 49.1]. Evidently, $\mathrm{dim}(\mathrm{P}(\mathcal{H}_c)) = \infty$

(for if $\dim(\mathrm{P}(\mathcal{H}_c)) < \infty$, then $\overline{\lambda}$ is a pole, hence an eigenvalue). Since the adjoint of an n-supercyclic operator can not have an infinite dimensional invariant subspace [2], we have a contradiction. This leaves us with the case $\overline{\lambda} \in \sigma_a(A_c^*)$ and $\overline{\lambda} \notin \mathrm{iso}\sigma(A_c^*)$. Then, given $\epsilon > 0$, every ϵ-neighbourhood of $\overline{\lambda}$ contains an element of the point spectrum of A_c^*. Since the adjoint of an n-supercyclic operator has at most (counting multiplicity) n eigenvalues [2], again we have a contradiction. Hence $\alpha(A_c^* - \overline{\lambda}) \neq 0$, which leads to yet another contradiction. Therefore, if a contraction $A \in B(\mathcal{H}) \cap PF(\delta)$ is n-supercyclic, then $S(\delta_{A,V^*}) = \varnothing$. (Equivalently, if $S(\delta_{A,V^*}) \neq \varnothing$, then a contraction $A \in B(\mathcal{H}) \cap PF(\delta)$ is not n-supercyclic.)

(b) The proof here is similar to that of the first part, and we shall use freely the relevant parts of the argument above in our proof below. Suppose that $A \in B(\mathcal{H})$ is (w-s) (short for "weakly supercyclic"). If $S(\delta_{A,V^*}) \neq \varnothing$, and $A \in PF(\delta)$, then $A = A_u \oplus A_c$. Since the compression of a (w-s) operator to the orthogonal complement of an invariant subspace of the operator is again (w-s), A_c is a (w-s) contraction. If $r(A_c) < 1$, then there exists a real number $0 < \rho < 1$ such that $\sigma(A) \subseteq \partial\mathbf{D} \cup \{\lambda : |\lambda| \leq \rho\}$, and this by [1, Lemma 3.3] is a contradiction. We prove next that $r(A_c) \neq 1$, which then leads us to conclude that either our hypothesis $S(\delta_{A,V^*}) \neq \varnothing$ is false or A_c acts on the trivial space 0 (and hence A is a unitary). Suppose then that $r(A_c) = 1$. Then, see above, $\alpha(A_c^* - \overline{\lambda}) > 0$ for a $\lambda \in \sigma_\pi(A_c)$ implies A_c has a unitary summand, and hence we must have that $\alpha(A_c^* - \overline{\lambda}) = 0$. If $\sigma_\pi(A_c) \subset \mathrm{iso}\sigma(A_c)$, then, upon letting P denote the Riesz projection associated with the spectral set $\sigma_\pi(A_c)$, we have $\mathcal{H}_c = P(\mathcal{H}_c) \oplus (I - P)(\mathcal{H}_c)$. Recall now the fact that an invariant subspace of a (w-s) operator has co-dimension 1 or ∞. (This is a simple consequence of the fact that if T is a (w-s) operator on a finite dimensional space, then it is supercyclic, and there are no supercyclic operators on a finite dimensional space of dimension greater than one.) Hence $\dim((\mathrm{I} - \mathrm{P})(\mathcal{H}_c))$ is either 1 or ∞. In either case $\sigma(A_c|_{(I-P)(\mathcal{H}_c)}) \neq \varnothing$ and $|\mu| < 1$ for every $\mu \in \sigma(A_c|_{(I-P)(\mathcal{H}_c)})$. Since this contradicts [1, Lemma 3.3] for A_c, we must have that there is a $\lambda \in \sigma_\pi(A_c)$ such that $\lambda \notin \mathrm{iso}\sigma(A_c)$. But then $\overline{\lambda} \notin \mathrm{iso}\sigma_a(A_c^*)$, and hence, given $\epsilon > 0$, every ϵ-neighbourhood of $\overline{\lambda}$ contains an element of the point spectrum of A_c^*. Hence ($\sigma_\pi(A_c) = \varnothing$ forcing thereby that) $r(A_c) \neq 1$, and $\sigma(A) \subseteq \partial\mathbf{D} \cup \{\lambda : |\lambda| \leq \rho\}$ for some positive number $\rho < 1$. But then (in view of [1, Lemma 3.3]) we must have either that A is (w-s) and $S(\delta_{A,V^*}) = \varnothing$ or A_c acts on the trivial space 0 (so that $\sigma(A_c) = \varnothing$, A is unitary and $\delta_{A,V^*}(I) = 0$, where V is the isometry $V = A^*$). □

Let $\triangle_{A,B} \in B(B(\mathcal{H}))$ denote the *elementary operator* $\triangle_{A,B}(X) = AXB - X$. Then an operator $A \in B(\mathcal{H})$ is said to satisfy property PF($\triangle$) if whenever the equation $AXV^* - X = 0$ holds for some isometry V and operator $X \in B(\mathcal{H})$, then $A^*XV^* - X = 0$. It is known, [4, Theorem 2.4], that $A \in \mathrm{PF}(\delta) \iff \mathrm{A} \in \mathrm{PF}(\triangle)$. If we let $S(\triangle_{A,V^*}) = \{V \in B(\mathcal{H}) : V \text{ isometric}, \triangle_{A,V^*}^{-1}(0) \neq \{0\}\}$, then $S(\triangle_{A,V^*}) \neq \varnothing \iff S(\delta_{A,V^*}) \neq \varnothing$; furthermore, there exists an isometry V and an operator X with dense range satisfying $\delta_{A,V^*}(X) = 0$ if and only if there exists an isometry W and an operator Y with dense range such that $AYW^* = Y$ (see the proof [4, Theorem 2.4]. Hence:

Corollary 1 *Let A be a contraction in $B(\mathcal{H}) \cap \mathrm{PF}(\triangle)$.*

(a) *If $S(\triangle_{A,V^*}) \neq \varnothing$, then A is not n-supercyclic.*

(b) *If A is weakly supercyclic, then either*

(b$_1$) *$S(\triangle_{A,V^*}) = \varnothing$, or,*

(b$_2$) *$S(\triangle_{A,V^*}) \neq \varnothing$ and A is unitary.*

Contractions belonging to a subclass $\mathcal{S} \subset B(\mathcal{H})$ may possess property PF(δ) (hence also property PF($\triangle$)) without operators (not necessarily contraction operators) in $\mathcal{S}$ possessing the property [4]. However, since an $A \in B(\mathcal{H})$ is n-supercyclic (or, weakly supercyclic) implies rA is n-supercyclic (resp., weakly supercyclic) for every non-zero scalar r, the theorem (above) has an analogue for (general) operators in $B(\mathcal{H})$.

Corollary 2 *Let $A \in B(\mathcal{H})$ be such that the contraction operator $B = \frac{1}{\|A\|}A \in \mathrm{PF}(\delta)$.*
(a) *If $S(\delta_{B,V^*}) \neq \varnothing$, then A is not n-supercyclic.*
(b) *If A is weakly supercyclic, then either*
(b$_1$) *$S(\delta_{B,V^*}) = \varnothing$, or,*
(b$_2$) *$S(\delta_{B,V^*}) \neq \varnothing$ and A is a scalar multiple of a unitary operator.*

References

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