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A Putnam–Fuglede commutativity property for Hilbert space operators

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Abstract

Given Hilbert space operators $A, B \in B(\mathcal{H})$, define $\delta_{A,B}$ and $\triangle_{A,B}$ in $B(B(\mathcal{H}))$ by $\delta_{A,B}(X) = AX - XB$ and $\triangle_{A,B}(X) = AXB - X$ for each $X \in B(\mathcal{H})$. An operator $A \in B(\mathcal{H})$ satisfies the Putnam–Fuglede properties δ, respectively $\triangle$ (notation: $A \in \mathrm{PF}(\delta)$, respectively $A \in \mathrm{PF}(\triangle)$), if for every isometry $V \in B(\mathcal{H})$ for which the equation $\delta_{A,V^*}(X) = 0$, respectively $\triangle_{A,V^*}(X) = 0$, has a non-trivial solution $X \in B(\mathcal{H})$, the solution X also satisfies $\delta_{A^*,V}(X) = 0$, respectively $\triangle_{A^*,V}(X) = 0$. We prove that an operator $A \in B(\mathcal{H})$ is in $\mathrm{PF}(\triangle)$ if and only if it is in $\mathrm{PF}(\delta)$.

1. Introduction

Given Hilbert space operators $A, B \in B(\mathcal{H})$, let $\delta_{A,B}$ and $\triangle_{A,B} \in B(B(\mathcal{H})))$ denote, respectively, the generalized derivation $\delta_{A,B}(X) = AX - XB$ and the elementary operator $\triangle_{A,B}(X) = AXB - X$ for each $X \in B(\mathcal{H})$. The classical Putnam–Fuglede commutativity theorem says that if A, B are normal (see, e.g., [7, p.84], then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$. A similar result holds for $\triangle_{A,B}$ [12, Theorem 5(i)]: if A, B are normal, then $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$. These are symmetric versions of the Putnam–Fuglede commutativity theorem, which, in general, fail to extend to classes of Hilbert space operators more general than the class of normal operators. For instance, if A and B are subnormal operators, then $\delta_{A,B}(X) = 0$ for some $X \in B(\mathcal{H})$ does not always imply $\delta_{A^*,B^*}(X) = 0$. An asymmetric version of the commutativity theorems, wherein one replaces the pair of operators $\{A, B\}$ by the pair $\{A, B^*\}$, is known to hold for A and B^* belonging to a number of classes of operators which properly contain the class of normal operators. For example, [13, 2], if A is dominant and B^* is hyponormal (i.e., if B is cohyponormal), then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$.

We say that an operator $A \in B(\mathcal{H})$ *satisfies the Putnam–Fuglede property* δ (*resp.*, $\triangle$), $A \in \mathrm{PF}(\delta)$ (*resp.*, $A \in \mathrm{PF}(\triangle)$), *if* (*either trivially* A *is unitary or*) *for every isometry* $V \in B(\mathcal{H})$ *for which the equation* $\delta_{A,V^*}(X) = 0$ (*resp.*, $\triangle_{A,V^*}(X) = 0$) *has a non-trivial solution* $X \in B(\mathcal{H})$, *the solution* X *also satisfies* $\delta_{A^*,V}(X) = 0$ (*resp.*, $\triangle_{A^*,V}(X) = 0$). Property $\mathrm{PF}(\delta)$ gives rise to a characterization of contractions A with $C_{\cdot 0}$ completely non-unitary (shortened, henceforth, to cnu) part [4] (also see [3]): *A contraction* $A \in B(\mathcal{H})$ *has* $C_{\cdot 0}$ *cnu part if and only if* $A \in \mathrm{PF}(\delta)$.

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A *part of an operator* $A \in B(\mathcal{H})$ is its restriction to a closed invariant subspace, and we say that a part $A|_M$ *of* A *is reducing if the* (*closed invariant*) *subspace* M *reduces* A. A well-known basic result for Hilbert space contractions reads as follows (see, e.g., [7, Problem 4.10]). If $A \in B(\mathcal{H})$ is a contraction, then every unitary part of A (i.e., every restriction of A to a closed invariant subspace M of A such that $A|_M$ is unitary) is reducing. In other words, *if the restriction of a Hilbert space contraction to an invariant subspace is unitary, the subspace is reducing*. This, however, fails for a general operator $A \in B(\mathcal{H})$.

The purpose of this note is to investigate Putnam–Fuglede properties (shortened to PF-properties) for operators acting on Hilbert spaces. We prove that an operator $A \in B(\mathcal{H})$ satisfies property PF(δ) if and only if it satisfies PF($\triangle$). This, by [11, Theorem 3.2], then implies that a power bounded operator $A \in$ PF($\triangle$) if and only if A is the direct sum of a unitary with a $C_{.0}$ operator. For operators $A \in B(\mathcal{H})$ such that $A^{-1}(0) \subseteq A^{*-1}(0)$, a necessary and sufficient condition for $A \in$ PF(δ) is that the unitary parts of A reduce A. The requirement that $A^{-1}(0) \subseteq A^{*-1}(0)$ is however not necessary (as follows from the fact that every contraction in $B(\mathcal{H})$ with a $C_{.0}$ cnu part satisfies property PF(δ)). We point out that a number of the more commonly considered classes of Hilbert space operators, amongst them hyponormal, p-hyponormal, dominant and k-$*$-paranormal operators, satisfy the PF-property.

2. Main Results

Given an operator $X \in \delta^{-1}_{A,V^*}(0)$ for some operator A and isometry V in $B(\mathcal{H})$, it is clear that $\overline{\mathrm{ran(X)}}$ is invariant under A and $\ker(\mathrm{X})^{\perp}$ is invariant under V: We shall henceforth define the operator A_1 in $B(\overline{\mathrm{ranX}})$ by $A_1 = A|_{\overline{\mathrm{ranX}}}$, the isometry V_1 in $B(\ker^{\perp}\mathrm{X})$ by $V_1 = V|_{\ker(\mathrm{X})^{\perp}}$ and the quasi-affinity $X_1 : \ker(\mathrm{X})^{\perp} \longrightarrow \overline{\mathrm{ranX}}$ by $X_1 x = Xx$ for all $x \in \ker(\mathrm{X})^{\perp}$. Evidently,

$$X \in \delta^{-1}_{A,V^*}(0) \Longrightarrow X_1 \in \delta^{-1}_{A_1,V_1^*}(0).$$

We start with our main result, a theorem proving the equivalence of the property PF(δ) for an operator $A \in B(\mathcal{H})$ with the property PF($\triangle$) for A. The following result from [1] is essential to our proof of this result.

Proposition 2.1 *If* $A, B \in B(\mathcal{H})$, *then the following statements are pairwise equivalent.*

(i) $\mathrm{ran(A)} \subseteq \mathrm{ran(B)}$.

(ii) *There is a* $\mu \geq 0$ *such that* $AA^* \leq \mu^2 BB^*$.

(iii) *There is a bounded operator* C *such that* $A = BC$.

Furthermore, if these conditions hold, then the operator C *may be chosen so that* (a) $||C||^2 = \inf\{\lambda : \mathrm{AA}^* \leq \lambda \mathrm{BB}^*\}$; (b) $\ker(\mathrm{A}) = \ker(\mathrm{C})$; (c) $\mathrm{ran(C)} \subseteq \ker(\mathrm{B})^{\perp}$. (*The operator* C*, under these restrictions, is unique.*)

A particular case of Proposition 2.1 is obtained in the case in which $AA^* = BB^*$: In this case there exists an isometry U such that $A^* = UB^*$. Another result which we shall require in our proof of main result is the following.

Given operator $A, B \in B(\mathcal{H})$, if $\delta_{A,B}(X) = 0 \Longrightarrow \delta_{A^*,B^*}(X) = 0$ for all $X \in B(\mathcal{H})$, then $\delta_{A,B}(AX) = 0 \Longrightarrow \delta_{A^*,B^*}(AX) = 0$, and hence $(A^*A - AA^*)X = X(B^*B - BB^*) = AXX^* - XX^*A = X^*XB - BX^*X = 0$. Again, if $\triangle_{A,B}(X) = 0 \Longrightarrow \triangle_{A^*,B^*}(X) = 0$

for all $X \in B(\mathcal{H})$, then $\triangle_{A,B}(AX) = 0 \Longrightarrow \triangle_{A^*,B^*}(AX) = 0$ and $\triangle_{A,B}(XB) = 0 \Longrightarrow \triangle_{A^*,B^*}(XB) = 0$, and it follows that $(A^*A - AA^*)XB^* = A^*X(B^*B - BB^*) = AXX^* - XX^*A = X^*XB - BX^*X = 0$. We have the folowing result.

Lemma 2.2 *Let $A, B \in B(\mathcal{H})$.* (i) *If $X \in \delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$, then $\overline{\mathrm{ran(X)}}$ reduces A, $\ker(\mathrm{X})^\perp$ reduces B, and $A|_{\overline{\mathrm{ran(X)}}}$ and $B|_{\ker(\mathrm{X})^\perp}$ are unitarily equivalent normal operators.* (ii) *If $X \in \triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$, then $\overline{\mathrm{ran(X)}}$ reduces A, $\ker(\mathrm{X})^\perp$ reduces B, and $A|_{\overline{\mathrm{ran(X)}}}$ and $(B|_{\ker(\mathrm{X})^\perp})^{-1}$ are unitarily equivalent normal operators.*

Proof. The proof is clear except perhaps for the fact that $B|_{\ker(\mathrm{X})^\perp}$ is invertible. To see this, let $A_1 = A|_{\overline{\mathrm{ran(X)}}}$, $B_1 = B|_{\ker(\mathrm{X})^\perp}$, and define the quasi-affinity $X_1 : \ker(\mathrm{X})^\perp \to \overline{\mathrm{ran(X)}}$ by setting $X_1x = Xx$ for all $x \in \ker(\mathrm{X})^\perp$. Let X_1 have the polar decomposition $X_1 = U|X_1|$; then U is a unitary. Since $\overline{\mathrm{ran(X)}}$ reduces A and $\ker(\mathrm{X})^\perp$ reduces B, $AXB = X$ implies $(A_1UB_1 - U)|X_1| = 0 \Longleftrightarrow A_1UB_1 = U$. But then B_1 is left invertible and normal; hence B_1 is invertible. □

The following version of the (asymmetic) Putnam–Fuglede theorem is well known (and follows easily, for example, from the main result of [2]).

Lemma 2.3 *$AX = XB$ implies $A^*X - XB^* = 0$ for unitary operators A and isometries B^* in $B(\mathcal{H})$.*

Theorem 2.4 *$A \in \mathrm{PF}(\delta) \Longleftrightarrow \mathrm{A} \in \mathrm{PF}(\triangle)$.*

Proof. (a) $A \in \mathrm{PF}(\triangle) \Longrightarrow \mathrm{A} \in \mathrm{PF}(\delta)$.

Let $X \in \delta^{-1}_{A,V^*}(0)$, V an isometry. Then $A|X^*|^2A^* = |X^*|^2$, and it follows from an application of Proposition 2.1 that there exists an isometry $U \in B(\mathcal{H})$ such that

$$|X^*| = U|X^*|A^* \Longleftrightarrow A|X^*|U^* = |X^*|.$$

This, since $A \in \mathrm{PF}(\triangle)$, implies that $A^*|X^*|U = |X^*|$. An application of Lemma 2.2 now implies that $\overline{\mathrm{ran(X)}}$ reduces A and $A_1 = A|_{\overline{\mathrm{ran(X)}}}$ is unitary. Decompose $A \in B(\overline{\mathrm{ran(X)}} \oplus \overline{\mathrm{ran(X)}}^\perp)$ by $A = A_1 \oplus A_2$, $V \in B(\mathcal{H}_u \oplus \mathcal{H}_c)$ into its unitary and cnu (= completely non-unitary) parts by $V = V_u \oplus V_c$, and let X have the corresponding matrix representation $X = [X_{ij}]^2_{i,j=1}$. Then

$$A_1X_{11} = X_{11}V_u \Longrightarrow A_1^*X_{11} = X_{11}V_u^*$$

(apply Lemma 2.3 or the classical Putnam–Fuglede theorem [7]). If $X_{12} \neq 0$, then Lemma 2.3 applied to $A_1X_{12} = X_{12}V_c^*$ implies $A_1^*X_{12} = X_{12}V_c$, and hence (by Lemma 2.2) $\ker(\mathrm{X}_{12})^\perp$ reduces V_c and $V_c|_{\ker(\mathrm{X}_{12})^\perp}$ is unitary. This being a contradiction, $X_{12} = 0$. Considering now $A_2X_{21} = X_{21}V_c^*$, the equality $A_2|X_{21}^*|^2A_2^* = |X_{21}^*|^2$ implies the existence of an isometry W such that $W|X_{21}^*|A_2^* = |X_{21}^*|$. Equivalently:

$$A_2|X_{21}^*|W^* = |X_{21}^*| \Longleftrightarrow (A_1 \oplus A_2)\begin{pmatrix} 0 & 0 \\ |X_{21}^*| & 0 \end{pmatrix}(W^* \oplus I) = \begin{pmatrix} 0 & 0 \\ |X_{21}^*| & 0 \end{pmatrix}.$$

Since $A \in \mathrm{PF}(\triangle)$, we conclude that $A_2^*|X_{21}^*|W = |X_{21}^*|$. Consequently, $\overline{\mathrm{ran(X_{21})}}$ reduces A_2 and $A_2|_{\overline{\mathrm{ran(X_{21})}}}$ is unitary. But then $A_2X_{21} = X_{21}V_c^*$ implies $A_2^*X_{21} = X_{21}V_c$, and this forces V_c to have a unitary part. This contradiction implies $X_{21} = 0$. Finally, we consider

$A_2X_{22} = X_{22}V_c^*$. Arguing as above we have in this case that $A_2|X_{22}^*|^2A_2^* = |X_{22}^*|^2$ implies the existence of an isometry E such that $A_2|X_{22}^*|E^* = |X_{22}^*|$,

$$(A_1 \oplus A_2)\begin{pmatrix} 0 & 0 \\ 0 & |X_{22}^*| \end{pmatrix}(E^* \oplus I) = \begin{pmatrix} 0 & 0 \\ 0 & |X_{22}^*| \end{pmatrix},$$

and hence (since $A \in \mathrm{PF}(\triangle)$) $A_2^*X_{22} = X_{22}V_c$. But then V_c has a unitary part, a contradiction. Hence $X_{22} = 0$, $X = X_{11} \oplus 0$, and

$$AX = XV^* \Longrightarrow A_1X_{11} = X_{11}V_u^* \Longrightarrow A_1^*X_{11} = X_{11}V_u \Longrightarrow A^*X = XV,$$

i.e., $A \in \mathrm{PF}(\delta)$.

(b) $\mathrm{PF}(\delta) \Longrightarrow \mathrm{PF}(\triangle)$.

The proof in this case is similar to the case above, so we shall be economical at times. Suppose that $AXV^* = X$ for some isometry $V \in B(\mathcal{H})$ and $X \in B(\mathcal{H})$. Then $A|X^*|^2A^* = |X^*|^2$, and hence there exists an isometry U such that $|X^*|A^* = U|X^*| \Longleftrightarrow A|X^*| = |X^*|U^*$. Since $A \in \mathrm{PF}(\delta)$, $A^*|X^*| = |X^*|U$, and hence $\overline{\mathrm{ran(X)}}$ reduces A and $A_1 = A|_{\overline{\mathrm{ran(X)}}}$ is unitary. Let (as above) $A = A_1 \oplus A_2$, $V = V_u \oplus V_c$ and let X have the corresponding matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then: $A_1X_{11} = X_{11}V_u^* \Longrightarrow A_1^*X_{11} = X_{11}V_u$; $A_1X_{12} = X_{12}V_c^* \Longrightarrow A_1^*X_{12} = X_{12}V_c \Longrightarrow V_c$ has a unitary part, implies $X_{12} = 0$. Again, $A_2X_{21} = X_{21}V_c^*$ implies $A_2|X_{21}^*|^2A_2^* = |X_{21}^*|^2$, and hence there exists an isometry W such that

$$\begin{aligned} |X_{21}^*|A_2^* = W|X_{21}^*| \quad &\Longleftrightarrow \quad A_2|X_{21}^*| = |X_{21}^*|W^* \\ &\Longrightarrow \quad A\begin{pmatrix} 0 & 0 \\ |X_{21}^*| & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ |X_{21}^*| & 0 \end{pmatrix}(W^* \oplus I). \end{aligned}$$

Since $A \in \mathrm{PF}(\delta)$, it follows that $A_2^*|X_{21}^*| = |X_{21}^*|W$, and hence that $\overline{\mathrm{ran(X_{21})}}$ reduces A_2 and $A_2|_{\overline{\mathrm{ran(X_{21})}}}$ is unitary. Finally (an argument similar to the one above shows that) $A_2X_{22} = X_{22}V_c$ implies $A_2|X_{22}^*|^2A_2^* = |X_{22}^*|^2$, and this in turn implies $\overline{\mathrm{ran(X_{22})}}$ reduces A_2 and $A_2|_{\overline{\mathrm{ran(X_{22})}}}$ is unitary. Considering the equation $(A_2|_{\overline{\mathrm{ran(X_{22})}}})Z = Z(V_c|_{\mathrm{ker(X_{22})^\perp}})^*$, where $Z : \mathrm{ker(X_{22})}^\perp \to \overline{\mathrm{ran(X_{22})}}$ is the quasiaffinity defined by setting $Zx = X_{22}x$ for all $x \in \mathcal{H}_c$, it follows that the cnu contraction V_c has a unitary part: This contradiction implies that $X_{22} = 0$. To complete the proof, consider now

$$(A_1 \oplus A_2)\begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix}(V_u^* \oplus V_c^*) = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix}.$$

Since $A_1$and V_u are unitary, $A_1X_{11}V_u^* = X_{11} \Longrightarrow A_1^*X_{11}V_u = X_{11}$. The equation $A_2X_{21}V_u^* = X_{21} \Longrightarrow A_2X_{21} = X_{21}V_u$; since $A_2|_{\overline{\mathrm{ran(X_{21})}}}$ is unitary, we have $A_2^*X_{21} = X_{21}V_u^* \Longrightarrow A_2^*X_{21}V_u = X_{21}$. Hence $AXV^* = X$ implies

$$(A_1^* \oplus A_2^*)\begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix}(V_u \oplus V_c) = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix},$$

i.e., $A^*XV = X$. $\square$

Recall that an operator $A \in B(\mathcal{H})$ is a $C_{0.}$ operator if $\mathrm{inf_{n\geq 0}||A^nx||} = 0$ for all $x \in \mathcal{H}$, $A \in C_{1.}$ if $\mathrm{inf_{n\geq 0}||A^nx||} > 0$ for all $0 \neq x \in \mathcal{H}$, $A \in C_{.0}$ if $A^* \in C_{0.}$, $A \in C_{.1}$ if $A^* \in C_{1.}$

and $A \in C_{\alpha\beta}$ if $A \in C_{\alpha.} \cap C_{.\beta}$ for all $\alpha, \beta = 0, 1$. The operator $A \in B(\mathcal{H})$ is power bounded if $\{A^n\}_{n=0}^{\infty}$ is a bounded sequence in $B(\mathcal{H})$. Evidently, A is power bounded if and only if A^* is power bounded. Recall from [6] that every power bounded operator $A \in B(\mathcal{H})$ has an upper triangular matrix representation $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, with respect to some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$, such that $A_{11} \in C_{0.}$ and $A_{22} \in C_{1.}$.

Pagacz [11, Theorem 3.2] has proved that

> *a power bounded operator satisfies property* PF(δ) *if and only if it is the direct sum of a unitary with a* $C_{.0}$ *operator*.

Along the same line we have the following corollaries of the previous theorem.

Corollary 2.5 *A power bounded operator satisfies property* PF($\triangle$) *if and only if it is the direct sum of a unitary with a* $C_{.0}$ *operator.*

Corollary 2.6 *A contraction satisfies property* PF($\triangle$) *if and only if it has a* $C_{.0}$ *cnu part.*

Contractions belonging to a subclass $\mathcal{S} \subset B(\mathcal{H})$ may possess property PF(δ) without operators (not necessarily contraction operators) in $\mathcal{S}$ possessing property PF(δ). Thus paranormal contractions, i.e., contractions $A \in B(\mathcal{H})$ such that $||Ax||^2 \leq ||A^2x||$ for all unit vectors x in $\mathcal{H}$, have $C_{.0}$ cnu part, and so satisfy PF(δ) [9, 4, 5, 10]. However, there exists a paranormal operator $A \in B(\mathcal{H})$ such that A does not satisfy property PF(δ) [11, Example 5.2]. An examination of [11, Example 5.2] shows that paranormal operator A of the example has the property that the unitary subspaces (i.e., closed subspaces of the operator such that its restriction to the subspace is unitary) of the operator are not reducing. (In general, the eigenspaces — more generally, normal subspaces — of a paranormal operator are not reducing.) If one assumes that 0 is a normal eigenvalue (i.e., if $\ker(A) \subseteq \ker(A^*)$) of the paranormal operator A, then a necessary and sufficient condition for $A \in$ PF(δ) is that the unitary subspaces of A reduce A.

Theorem 2.7 *Let* $A \in B(\mathcal{H})$*. If the eigenspace corresponding to the eigenvalue* 0 *of* A *reduces* A*, then a necessary and sufficient condition for* $A \in$ PF(δ) *(hence also* $A \in$ PF($\triangle$)*) is that the unitary subspaces of* A *reduce* A.

Before going onto to proving the theorem we remark that the condition "the eigenspace corresponding to the eigenvalue 0 of A reduces A" is in no way necessary for $A \in$ PF(δ). Consider for example a contraction $A \in B(\mathcal{H})$ with a $C_{.0}$ cnu part, when it is seen that $A \in$ PF(δ) irrespective of whether 0 is a normal eigenvalues of A (or not).

Proof. To see the necessity of the condition, consider unitary operators U_1 and U_2 in $B(\mathcal{H}_1)$ such that $U_1X_1 = X_1U_2$ for some $X_1 \in B(\mathcal{H}_1)$. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $X = X_1 \oplus 0$ in $B(\mathcal{H})$, $V^* = U_2^* \oplus U^*$ for some unilateral shift $U \in B(\mathcal{H}_2)$ and define the operator $A \in B(\mathcal{H})$ by $A = \begin{pmatrix} U_1 & E_1 \\ 0 & E_2 \end{pmatrix}$ for some operators $E_1 \in B(\mathcal{H}_2, \mathcal{H}_1)$ and $E_2 \in B(\mathcal{H}_2)$. Then $AX - XV^* = 0 = A^*X - XV$ if and only if $E_1 = 0$. To prove sufficiency, assume that $AX = XV^*$, $X \in B(\mathcal{H})$, for some isometry $V \in B(\mathcal{H})$. Define A_1, X_1 and V_1 as before. Decompose A_1 into its normal and pure (i.e., completely non-normal) parts by $A_1 = A_{1n} \oplus A_{1p}$, V_1 into its unitary and cnu parts by $V_1 = V_{1u} \oplus V_{1c}$ (so that V_{1c} is a unilateral shift), and let X_1 have the corresponding matrix representation $X_1 = [X_{ij}]_{i,j=1}^2$. We prove that X_{22} acts on the trivial space $\{0\}$. If $X_{12} \neq 0$, then the

Putnam–Fuglede theorem for normal A_{1n} and (purely) hyponormal V_{1c} [2, 13] applied to $A_{1n}X_{12} - X_{12}V_{1c}^* = 0$ implies $A_{1n}^*X_{12} - X_{12}V_{1c} = 0$, and hence that $V_{1c}|_{\ker^\perp(X_{12})}$ is unitary, a contradiction. Hence $X_{12} = 0$. This since X_1 is a quasi–affinity, implies that X_{22} is injective. Since $A_{1p}X_{22} = X_{22}V_{1c}^*$, A_{1p} is injective and $0 \in \sigma_p(V_{1c}^*)$, we must have $X_{22} = 0$. This, since X_1 is a quasi-affinity, implies X_{22} acts on the trivial space. Consequently, $A_1 = A_{1n}$ and $V_1 = V_{1u}$, which then leads to the implication $A_1X_1 = X_1V_1^* \Longleftrightarrow A_1^*X_1 = X_1V_1$. Hence A_1 is unitary. But then A_1 reduces A (so that $A = A_1 \oplus A_2$), and $AX - XV^* = 0$ implies $A^*X - XV = 0$. $\square$

An interesting consequence of the above theorem is the following.

Corollary 2.8 *An injective operator in $B(\mathcal{H})$ such that its unitary parts are reducing satisfies property* PF(δ).

Remark 2.9 (i). A number of the more commonly studied classes of Hilbert space operator are known to satisfy the property that their normal subspaces are reducing. Thus, if $A \in B(\mathcal{H})$ is either hyponormal ($AA^* \leq A^*A$) or p-hyponormal ($(AA^*)^p \leq (A^*A)^p$, $0 < p \leq 1$) or dominant (for every complex λ there exists a scalar $M_\lambda > 0$ such that $|(A-\lambda)^*|^2 \leq M_\lambda|A-\lambda|^2$) or k-$*$-paranormal ($||A^*x||^k \leq ||A^kx||$ for each unit vector $x \in \mathcal{H}$ and integer $k \geq 1$), then normal subspaces of A are reducing. For all such classes of operators, A satisfies properties PF(δ) and PF($\triangle$). *Paranormal operators do not satisfy this property, and hence properties* PF(δ) *and* PF($\triangle$) (*in general*) *fail for paranormal operators* (cf. [11]).

(ii). *If the cnu part A_c of a contraction $A \in B(\mathcal{H})$ is not a $C_{.0}$ contraction, then there exists an isometry $V \in B(\mathcal{H})$ and a non-trivial solution $X \in B(\mathcal{H})$ of $\delta_{A,V^*}(X) = 0$ such that $\delta_{A^*,V}(X) \neq 0$.* To see this one argues as follows: If we let $A = A_u \oplus A_c$ with respect to some decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$ of $\mathcal{H}$, where A_u is unitary, then $A_c^* \notin C_{0.}$ implies that $A_c^nA_c^{*n}$ converges strongly to a non-trivial operator $T \geq 0$ such that $T^{\frac{1}{2}}A_c^* = V_cT^{\frac{1}{2}}$ for some isometry V_c [8, Proposition 1]. Define X_c by $X_cx = T^{\frac{1}{2}}x$ for all $x \in \mathcal{H}_c$. Then $A_cX_c = X_cV_c^*$. Now set $I_{\mathcal{H}_u} \oplus X_c = X$ and $A_u \oplus V_c = V$. Then $AX = XV^*$ but $A^*X \neq XV$ (for if then A_c has a unitary part – a contradiction).

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