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ELEMENTARY OPERATORS, FINITE ASCENT, RANGE CLOSURE AND COMPACTNESS

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Abstract. Given Banach space operators $A_i, B_i \in B(\mathcal{X})$, $1 \leq i \leq 2$, let $\Phi_{\mathbf{AB}} \in B(B(\mathcal{X}))$ denote the elementary operator $\Phi_{\mathbf{AB}}(X) = A_1XB_1 - A_2XB_2$. Then $\Phi_{\mathbf{AB}}$ has finite ascent ≤ 1 for a number of fairly general choices of the operators A_i and B_i. This information is applied to prove some necessary and sufficient conditions for the range of $\Phi_{\mathbf{AB}}$ to be closed and in deciding conditions on the tuples (A_1, A_2) and (B_1, B_2) so that $\Phi^n_{\mathbf{AB}}(X)$ compact for some integer $n \geq 1$ and operator X implies $\Phi_{\mathbf{AB}}(X)$ compact. This generalizes some well known results of Anderson and Foiaş [4], and Yosun [25]. Also considered is the question: What is a necessary and sufficient condition (on the tuples (A_1, A_2), (B_1, B_2) and $\Phi_{\mathbf{AB}}$) for $\Phi^n_{\mathbf{AB}}$ to be compact for some integer $n \geq 1$?

1. Introduction

For a Banach space $\mathcal{X}$, let $B(\mathcal{X})$ denote the algebra of operators, equivalently bounded linear transformations, on $\mathcal{X}$ into itself. Given an operator $T \in B(\mathcal{X})$, the kernel $T^{-1}(0)$ of T is orthogonal, in the sense of G. Birkhoff, to the range $T(\mathcal{X})$ of T, in notation $T^{-1}(0) \perp T(\mathcal{X})$, if $||x|| \leq ||x+y||$ for all $x \in T^{-1}(0)$ and $y \in T(\mathcal{X})$ [8, page 25]. Clearly, $T^{-1}(0) \perp T(\mathcal{X}) \Longrightarrow T^{-1}(0) \cap \overline{T(\mathcal{X})} = \{0\} \Longrightarrow T^{-1}(0) \cap T(\mathcal{X}) = \{0\}$. (Here, as also in the sequel, $\overline{T(\mathcal{X})}$ denotes the closure of $T(\mathcal{X})$.) The range–kernel orthogonality of an operator is related to its ascent. The ascent of $T \in B(\mathcal{X})$, $\text{asc}(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ (if no such n exists then $\text{asc}(T) = \infty$). It is known that $\text{asc}(T) \leq m < \infty$ if and only if $T^{-n}(0) \cap T^m(X) = \{0\}$ for all integers $n \geq m$. Evidently, $T^{-1}(0) \perp T(\mathcal{X})$ implies $\text{asc}(T) \leq 1$.

The range–kernel orthogonality $T^{-1}(0) \perp T(\mathcal{X})$ of Banach space operators has been studied by a number of authors over the past few decades. A classical result of Sinclair [23, Proposition 1] says that "if 0 is in the boundary of the numerical range of a $T \in B(\mathcal{X})$, then $T^{-1}(0) \perp T(\mathcal{X})$". Anderson [3], and Anderson and Foiaş [4], considered the generalized derivation $\delta_{AB} = L_A - R_B \in B(B(\mathcal{H}))$, $\delta_{AB}(X) = AX - XB$, to prove that if $A, B \in B(\mathcal{H})$ are normal (Hilbert space) operators, then $\delta^{-1}_{AB}(0) \perp \delta_{AB}(B(\mathcal{H}))$. These results have since been extended to a variety of elementary operators $\Phi_{\mathbf{AB}}(X) = A_1XB_1 - A_2XB_2$ for a variety of choices of tuples of operators $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ (see [10, 11, 16, 17, 24] for further references). The range–kernel orthogonality of an operator $T \in B(\mathcal{X})$ does not imply that the range $T(\mathcal{X})$ is closed or that $\mathcal{X} = T^{-1}(0) \oplus \overline{T(\mathcal{X})}$; see [4, Example 3.1

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and Theorem 4.1] and [23, Remark 2]. Indeed, range–kernel orthogonality neither implies nor is implied by range closure. Thus, every bounded below operator has closed range and satisfies range-kernel orthogonality, an injective compact quasinilpotent operator (for example, the Volterra integral operator on $L^2(0,1)$) satisfies range–kernel orthogonality but does not have closed range, and no operator T (whether it has closed range or not) with $\infty > \mathrm{asc}(T) \geq 2$ satisfies range-kernel orthogonality. The implication $T^{-1}(0) \perp T(\mathcal{X}) \Longrightarrow \mathrm{asc}(T) \leq 1$ is strictly one way; if $A_i, B_i \in B(\mathcal{H})$, $1 \leq i \leq 2$, are normal Hilbert space operators such that A_1 commutes with A_2 and B_1 commutes with B_2, then $\mathrm{asc}(\Phi_{\mathbf{AB}}) \leq 1$ [13, Theorem 3.4] but $\Phi_{\mathbf{AB}}^{-1}(0) \perp \Phi_{\mathbf{AB}}(B(\mathcal{H}))$ if and only if $(A_1 \oplus B_1^*)^{-1}(0) \cap (A_2 \oplus B_2^*)^{-1}(0) = \{0\}$ [24, Corollary 2.3].

In the following we prove that $\mathrm{asc}(\Phi_{\mathbf{AB}}) \leq 1$ for various choices of the operators A_i and B_i, $1 \leq i \leq 2$. Thus, if $A, B \in B(\mathcal{X})$ are contractions (or, if $A, B \in B(\mathcal{X})$ are normaloid and λ is in the *peripheral spectrum* of $L_A R_B$), then $\mathrm{asc}(L_A R_B - 1) \leq 1$ (resp., $\mathrm{asc}(L_A R_B - \lambda) \leq 1$); if $B \in B(\mathcal{X})$ is a contraction and $A \in B(\mathcal{X})$ is left invertible by a contraction, then $\mathrm{asc}(L_A - R_B) \leq 1$; and if $A_1, B_1^* \in B(\mathcal{H})$ are w–hyponormal (Hilbert space) operators such that $A_1^{-1}(0) \subseteq A_1^{*-1}(0)$ and $B_1^{*-1}(0) \subseteq B_1^{-1}(0)$, $A_2, B_2^* \in B(\mathcal{H})$ are normal operators, A_1 commutes with A_2 and B_1 commutes with B_2, then $\mathrm{asc}(\Phi_{\mathbf{AB}}) \leq 1$. This information is then applied to give some necessary and sufficient conditions for the range of $\Phi_{\mathbf{AB}}$ to be closed (generalizing, in the process, a result of Anderson and Foiaş [4]), and in deciding conditions on the tuples $\mathbf{A}$ and $\mathbf{B}$ so that $\Phi_{\mathbf{AB}}^n(X)$ compact for some integer $n \geq 1$ and operator X implies $\Phi_{\mathbf{AB}}(X)$ compact (this generalizes some results of Yusun [25]). Also considered is the question: What is a necessary and sufficient condition (on $\mathbf{A}$, $\mathbf{B}$ and $\Phi_{\mathbf{AB}}$) for $\Phi_{\mathbf{AB}}^n$ to be compact for some integer $n \geq 1$?

2. Finite ascent, Range-kernel orthogonality

Throughout the following, $\mathcal{X}$ (resp., $\mathcal{H}$) shall denote an infinite dimensional complex Banach space (resp., Hilbert space). For an operator $A \in B(\mathcal{X})$, $L_A \in B(B(\mathcal{X}))$ (resp., $R_A \in B(B(\mathcal{X}))$) shall denote the operator $L_A(X) = AX$ of left multiplication by A (resp., $R_A(X) = XA$ of right multiplication by A). For $A, B \in B(\mathcal{X})$, we shall denote the generalized derivation $L_A - R_B$ by δ_{AB}, the elementary operator $L_A R_B - 1$ by $\triangle_{AB}$, and d_{AB} shall denote either of δ_{AB} and $\triangle_{AB}$. We shall denote the spectrum (the approximate point spectrum) of T by $\sigma(T)$ (resp., $\sigma_a(T)$), and the isolated points of a subset S of $\sigma(T)$ will be denoted by iso (S). The descent $\mathrm{dsc}(T)$ of an operator $T \in B(\mathcal{X})$ is the smallest non-negative integer n such that $T^n(\mathcal{X}) = T^{n+1}(\mathcal{X})$ (if no such n exists, then $\mathrm{dsc}(T) = \infty$). It is well known, see [1, 15], that if both $\mathrm{asc}(T)$ and $\mathrm{dsc}(T)$ are finite then $\mathrm{asc}(T) = \mathrm{dsc}(T) = p < \infty$ for some integer $p \geq 0$, $\mathcal{X} = T^{-p}(0) \oplus T^p(\mathcal{X})$ and $0 \in \mathrm{iso}\, \sigma(T)$. $T \in B(\mathcal{X})$, has *the single-valued extension property* at a complex point λ_0, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T - \lambda) f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. T has SVEP if it has SVEP at every complex λ. Both T and T^* have SVEP at points in the boundary $\partial\sigma(T)$ of the spectrum of T; also $\mathrm{asc}(T) < \infty$ (resp., $\mathrm{dsc}(T) < \infty$) implies T (resp., T^*) has SVEP at 0, and T^* has SVEP implies $\sigma(T) = \sigma_a(T)$ [1, 18].

For an operator $T \in B(\mathcal{H})$ with polar decomposition $T = U|T|$, the (first) *Aluthge transform* $\tilde{T}$ of T is the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. We say that the operator *T is w–hyponormal* if

$$|(\tilde{T})^*| \leq |T| \leq |\tilde{T}|.$$

Every hyponormal ($|T^*|^2 \leq |T|^2$) operator is w–hyponormal, and it is easily seen that w–hyponormal operators T are paranormal (i.e., $||Tx||^2 \leq ||T^2x||$ for all unit vectors $x \in \mathcal{H}$); hence $\mathrm{asc}(T) \leq 1$ [9].

Let $A, B \in B(\mathcal{H})$, $\mathcal{H}$ as above a Hilbert space. We say that *the pair (A, B) satisfies the PF–property*, short for the “Putnam-Fuglede commutativity property”, *if* $d_{AB}^{-1}(0) \subseteq d_{A^*B^*}^{-1}(0)$. For w–hyponormal operators $A, B^* \in B(\mathcal{H})$ with $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and ${B^*}^{-1}(0) \subseteq B^{-1}(0)$, d_{AB} satisfies the PF–property [9, Lemma 2.4].

Lemma 2.1. *If $A, B^* \in B(\mathcal{H})$ are w–hyponormal operators with $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and ${B^*}^{-1}(0) \subseteq B^{-1}(0)$, then d_{AB} satisfies the PF-property.*

Lemma 2.2. *If $A \in B(\mathcal{H})$ is a w–hyponormal operator which commutes with a normal operator $B \in B(\mathcal{H})$, then AB is w–hyponormal.*

Proof. Since $AB = BA$ implies $AB^* = B^*A$ (by the classical Putnam-Fuglede theorem), [9, Lemma 2.3] implies AB is w–hyponormal. □

The *numerical range* $W(B(\mathcal{X}), T)$ of $T \in B(\mathcal{X})$ is the set

$$W(B(\mathcal{X}), T) = \{f(T) : f \in B(\mathcal{X})^*, ||f|| = f(I) = 1\},$$

where $B(\mathcal{X})^*$ is the dual space of $B(\mathcal{X})$. $W(B(\mathcal{X}), T)$ is a compact subset of the set $\mathbf{C}$ of complex numbers. If $A, B \in B(\mathcal{X})$ are contractions, then

$$W(B(B(\mathcal{X})), L_AR_B) \subseteq \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}, \text{ and}$$
$$W(B(B(\mathcal{X})), \triangle_{AB}) \subseteq \{\lambda \in \mathbf{C} : |\lambda + 1| \leq 1\}.$$

This implies that $0 \in \partial W(B(B(\mathcal{X})), \triangle_{AB})$, the boundary of $W(B(B(\mathcal{X})), \triangle_{AB})$, and hence [8, Theorem 6, page 27]

$$||\triangle_{AB}(Y) + X|| \geq ||X|| - \sqrt{8||\triangle_{AB}(X)||||Y||}$$

for all $X, Y \in B(\mathcal{X})$. In particular $\triangle_{AB}^{-1}(0) \perp \triangle_{AB}(B(\mathcal{X}))$ and $\mathrm{asc}(\triangle_{AB}) \leq 1$. Consider now a contraction $A \in B(\mathcal{X})$ and an operator $B \in B(\mathcal{X})$ such that B is right invertible by a contraction $B_r \in B(\mathcal{X})$. Then

$$||\triangle_{AB_r}(YB) + X|| \geq ||X|| - \sqrt{8||\triangle_{AB_r}(X)||||YB||}$$

for all $X, Y \in B(\mathcal{X})$. Since $\triangle_{AB_r}(YB) = \delta_{AB}(Y)$, $||YB|| \leq ||B||||Y||$ and $||\triangle_{AB_r}(X)|| \leq ||B_r||||\delta_{AB}(X)|| \leq ||\delta_{AB}(X)||$,

$$||\delta_{AB}(Y) + X|| \geq ||X|| - \sqrt{8||B||||\delta_{AB}(X)||||Y||}$$

for all $X, Y \in B(\mathcal{X})$. Similarly, if $A \in B(\mathcal{X})$ is left invertible by a contraction A_ℓ and $B \in B(\mathcal{X})$ is a contraction, then

$$||\delta_{AB}(Y) + X|| \geq ||X|| - \sqrt{8||A||||\delta_{AB}(X)||||Y||}$$

for $X, Y \in B(\mathcal{X})$. We have proved:

Proposition 2.3. *Let $A, B \in B(\mathcal{X})$. (i) If A, B are contractions, then $\triangle_{AB}^{-1}(0) \perp \triangle_{AB}(B(\mathcal{X}))$ and (hence) $asc(\triangle_{AB}) \leq 1$.*
(ii) If A is a contraction and B is right invertible by a contraction (resp., A is left invertible by a contraction and B is a contraction), then $\delta_{AB}^{-1}(0) \perp \delta_{AB}(B(\mathcal{X}))$ and (hence) $asc(\delta_{AB}) \leq 1$.

$T \in B(\mathcal{X})$ is *normaloid* if $||T|| = r(T)$, where $r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is the spectral radius of T. A more general result, than the one in Proposition 2.3, is possible for the elementary operator $\triangle_{AB}$ in the case in which A, B are normaloid operators. Given a $T \in B(\mathcal{X})$, let

$$\sigma_\pi(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$$

denote the *peripheral spectrum of* T [15, Page 225].

Proposition 2.4. *If $A, B \in B(\mathcal{X})$ are normaloid, then $(L_AR_B-\lambda)^{-1}(0) \perp (L_AR_B-\lambda)(B(\mathcal{X}))$ for all $\lambda \in \sigma_\pi(L_AR_B)$.*

Proof. A, B being normaloid,

$$||(L_AR_B)^n|| = ||L_{A^n}R_{B^n}|| = ||A^n||||B^n|| = (||A||||B||)^n = ||L_AR_B||^n$$

for all integers $n \geq 1$. Hence L_AR_B is normaloid, $r(L_AR_B) = r(A)r(B) = ||A||||B||$, and

$$\sigma_\pi(L_AR_B) = \{\lambda \in \mathbf{C} : \text{there exist } \ \mu \in \sigma_\pi(A), \nu \in \sigma_\pi(B) \ \text{ such that } \ \lambda = \mu\nu\}.$$

If we define the contractions A_1 and B_1 by $A_1 = A/||A||$ and $B_1 = B/||B||$, then $L_{A_1}R_{B_1}$ is a contraction and

$$\sigma_\pi(L_{A_1}R_{B_1}) = \{\lambda \in \mathbf{C} : \text{there exist } \ \mu \in \sigma_\pi(A_1), \nu \in \sigma_\pi(B_1) \ \text{ such that } \ \lambda = \mu\nu, |\mu| = |\nu| = 1\}.$$

Choose a $\lambda_0 = \mu_0\nu_0 \in \sigma_\pi(L_{A_1}R_{B_1})$; let $A_{10} = A_1/\mu_0$ and $B_{10} = B_1/\nu_0$. Then

$$\begin{aligned} &\left\|\frac{\lambda_0}{n}\sum_{i=0}^{n-1}(L_{A_{10}}R_{B_{10}})^i(L_{A_{10}}R_{B_{10}}-1)(Z)\right\| = \left\|\frac{\lambda_0}{n}(L_{A_{10}}^nR_{B_{10}}^n-1)(Z)\ \right\| \\ &= \frac{1}{n}||(L_{A_{10}^n}R_{B_{10}^n}-1)(Z)|| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty \end{aligned}$$

for all $Z \in B(\mathcal{X})$. Set $\lambda_0||A||||B|| = \lambda \in \sigma_\pi(L_AR_B)$. Then $X \in B(\mathcal{X})$ satisfies $(L_{A_{10}}R_{B_{10}})(X) = 0$ if and only if $(L_AR_B)(X) = 0$. An easy calculation shows that $X \in (L_{A_{10}}R_{B_{10}}-1)^{-1}(0)$ implies $X = \frac{1}{n}\sum_{i=0}^{n-1}(L_{A_{10}}R_{B_{10}})^i(X)$. Hence if $X \in (L_AR_B-1)^{-1}(0)$ and $Y = Z/||A||||B||$, then for any $Z \in B(\mathcal{X})$,

$$\begin{aligned} &\left\|X + \frac{\lambda_0}{n}\sum_{i=0}^{n-1}(L_{A_{10}}R_{B_{10}})^i(L_{A_{10}}R_{B_{10}}-1)(Z)\right\| \\ =\ &\left\|\frac{1}{n}\sum_{i=0}^{n-1}(L_{A_{10}}R_{B_{10}})^i(X+\lambda_0(L_{A_{10}}R_{B_{10}}-1)(Z))\right\| \\ \leq\ &||X+\lambda_0(L_{A_{10}}R_{B_{10}}-1)(Z)|| = ||X+(L_{A_1}R_{B_1}-\lambda_0)(Z)|| \\ =\ &||X+(L_AR_B-\lambda)(Y)|| \end{aligned}$$

for all $Y \in B(\mathcal{X})$ and $\lambda \in \sigma_\pi(L_AR_B)$. $\square$

Remark 2.5. The argument of the proof of Proposition 2.4 seemingly does not work for the operator δ_{AB}, even for normal Hilbert space operators A, B. It is easily seen that if $A, B \in B(\mathcal{X})$ are normal, then δ_{AB} is normal, hence $\delta_{AB}^{-1}(0) \perp \delta_{AB}(B(\mathcal{X}))$ and $\mathrm{asc}(\delta_{AB}) \leq 1$. However, δ_{AB} is not normaloid [4]. Recall from [20] that for operators $A, B \in B(\mathcal{H})$, the numerical range $W(B(B(\mathcal{H})), \delta_{AB}) = W(A) - W(B)$, where $W(T) = \{(Tx, x) : x \in \mathcal{H}, ||x|| = 1\}$ denotes the closure of the usual (spatial) numerical range of the operator $T \in B(\mathcal{H})$. Hence: *If* $0 \in \partial(W(A) - W(B))$, *then* $\delta_{AB}^{-1}(0) \perp \delta_{AB}(B(\mathcal{H}))$.

The PF–property implies range-kernel orthogonality: the following proposition is well known for the case in which $d_{AB} = \delta_{AB}$. (Recall: $d_{AB} = \delta_{AB}$ or $\triangle_{AB}$.)

Proposition 2.6. *Let* $A, B \in B(\mathcal{H})$. *If* d_{AB} *satisfies the PF–property, then* $d_{AB}^{-1}(0) \perp d_{AB}(B(\mathcal{H}))$ *(hence* $asc(d_{AB}) \leq 1$*).*

Proof. If $X \in \delta_{AB}^{-1}(0) \subseteq \delta_{A^*B^*}^{-1}(0)$, then $\overline{\mathrm{ran}X}$ reduces A, $\ker^{\perp}X$ reduces B, $A_1 = A|_{\overline{\mathrm{ran}X}}$ and $B_1 = B|_{\ker^{\perp}X}$ are unitarily equivalent normal operators. Furthermore, $\delta_{A_1B_1}(X_1) = 0$, where $X_1 : \ker^{\perp}X \longrightarrow \overline{\mathrm{ran}X}$ is the quasi–affinity defined by setting $X_1x = Xx$ for all $x \in \ker^{\perp}X$. The operators A_1, B_1 being normal, $||X_1|| \leq ||\delta_{A_1B_1}(Y_{11}) + X_1||$ for all $X_1 \in \delta_{A_1B_1}^{-1}(0)$ and (bounded linear) operators $Y_{11} : \ker^{\perp}X \longrightarrow \overline{\mathrm{ran}X}$ [4]. Let $A = A_1 \oplus A_2$ with respect to the decomposition $\mathcal{H} = \overline{\mathrm{ran}X} \oplus \overline{\mathrm{ran}X}^{\perp}$, $B = B_1 \oplus B_2$ with respect to the decomposition $\mathcal{H} = \ker^{\perp}X \oplus \ker X$, $X = X_1 \oplus 0$ $(: \ker^{\perp}X \oplus \ker X \longrightarrow \overline{\mathrm{ran}X} \oplus \overline{\mathrm{ran}X}^{\perp})$, and let $Y = [Y_{ij}]_{i,j=1}^{2}$ $(: \ker^{\perp}X \oplus \ker X \longrightarrow \overline{\mathrm{ran}X} \oplus \overline{\mathrm{ran}X}^{\perp})$ (with Y_{11} as above and some operators Y_{12}, Y_{21} and Y_{22}). A straightforward argument then shows that

$$||X|| = ||X_1|| \leq ||\delta_{A_1B_1}(Y_{11}) + X_1|| \leq ||\delta_{AB}(Y) + X||$$

for all $X \in \delta_{AB}^{-1}(0)$ and $Y \in B(\mathcal{H})$.

Now let $X \in \triangle_{AB}^{-1}(0) \subseteq \triangle_{A^*B^*}^{-1}(0)$. Then $\overline{\mathrm{ran}X}$ reduces A, $\ker^{\perp}X$ reduces B, and $\triangle_{A_1B_1}(X_1) = \triangle_{A_1^*B_1^*}(X_1)$, where the operators A_1, B_1 and the quasi–affinity $X_1 : \ker^{\perp}X \longrightarrow \overline{\mathrm{ran}X}$ are as defined above. Evidently, A_1 and B_1 are quasi–affinities. Since

$$B_1X_1^*(A_1X_1B_1) = B_1X_1^*X_1 \Longleftrightarrow |X_1|^2B_1 = B_1|X_1|^2, \text{ and}$$
$$A_1X_1(B_1X_1^*A_1) = A_1X_1X_1^* \Longleftrightarrow A_1|X_1^*|^2 = |X_1^*|^2A_1,$$

it follows that $A_1U_1B_1 = U_1 = A_1^*U_1B_1^*$, where the unitary operator U_1 is as in the polar decomposition $X_1 = U_1|X_1|$. Hence A_1 and B_1^{-1} are unitarily equivalent normal operators. Since $X_1 \in \triangle_{A_1B_1}^{-1}(0)$ if and only if $X_1 \in \delta_{A_1B_1^{-1}}(0)$,

$$||X_1|| \leq ||\delta_{A_1B_1^{-1}}(Y_{10}) + X_1|| = ||\triangle_{A_1B_1}(Y_{11}) - X_1)||$$

for all $Y_{10} = Y_{11}B_1$. As above, this implies $||X|| \leq ||\triangle_{AB}(Y) - X)||$ for all $X, Y \in B(\mathcal{H})$. $\square$

Combining Lemma 2.1 and Proposition 2.4 we have:

Corollary 2.7. *If* $A, B^* \in B(\mathcal{H})$ *are w–hyponormal operators with* $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ *and* ${B^*}^{-1}(0) \subseteq B^{-1}(0)$, *then* $asc(d_{AB}) \leq 1$.

Proposition 2.6 does not extend to operators $\Phi_{\mathbf{AB}}(X) = \sum_{i=1}^{n} A_i X B_i$, $\mathbf{A} = (A_1, \cdots, A_n)$ and $\mathbf{B} = (B_1, \cdots, B_n)$ n-tuples of mutually commuting operators in $B(\mathcal{H})$, such that $\Phi_{\mathbf{AB}}(X) = 0$ implies $\Phi_{\mathbf{A^*B^*}}(X) = \sum_{i=1}^{n} A_i^* X B_i^* = 0$. Thus, if A_i and $B_i \in B(\mathcal{H})$ are normal operators for all $1 \leq i \leq n$, and A_i commutes with A_j and B_i commutes with B_j for all $1 \leq i, j \leq n$, then $\Phi_{\mathbf{AB}}^{-(n-1)}(0) = \Phi_{\mathbf{A^*B^*}}^{-(n-1)}(0)$ [21, Theorem 5]. (Here we thank the referee for pointing out (i) an error in the original statement $\Phi_{\mathbf{AB}}^{-1}(0) = \Phi_{\mathbf{A^*B^*}}^{-1}(0)$ and (ii) that [22] has a counter example showing that this equality may fail for $n > 2$.) If $n > 2$ then there is a $\Phi_{\mathbf{AB}}$ such that $\mathrm{asc}(\Phi_{\mathbf{AB}}) > 1$ [21]. Obviously, such an operator $\Phi_{\mathbf{AB}}$ does not satisfy the range-kernel orthogonality property. Recall from [24, Theorem 2.4] that if $n = 2$, $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ are commuting tuples of normal operators, then $\Phi_{\mathbf{AB}}^{-1}(0) \perp \Phi_{\mathbf{AB}}(B(\mathcal{H}))$ if and only if $(A_1 \oplus B_1)^{-1}(0) \cap (A_2 \oplus B_2)^{-1}(0) = \{0\}$. Consequently, $\Phi_{\mathbf{AB}}^{-1}(0) \perp \Phi_{\mathbf{AB}}(B(\mathcal{H}))$ may fail even in the case in which $n = 2$. In the following we consider commuting tuples (A_1, A_2) and (B_1, B_2) such that A_2, B_2 are normal and A_1, B_1^* are w–hyponormal with $A_1^{-1}(0) \subseteq A_1^{*-1}(0)$ and $B_1^{*-1}(0) \subseteq B_1^{-1}(0)$ to prove that $\mathrm{asc}(\Phi_{\mathbf{AB}}) \leq 1$. We remark here that one can prove the range-kernel orthogonality for such an operator $\Phi_{\mathbf{AB}}$ under the additional hypothesis that $(A_1 \oplus B_1^*)^{-1}(0) \cap (A_2 \oplus B_2^*)^{-1}(0) = \{0\}$: We leave the detail to the reader, see however the proof of [11, Theorem 2.7].

Proposition 2.8. *Let $A_1, B_1^* \in B(\mathcal{H})$ be two w–hyponormal operators, and let $A_2, B_2 \in B(\mathcal{H})$ be two normal operators. Define $\Phi_{\mathbf{AB}} \in B(B(\mathcal{H}))$ by $\Phi_{\mathbf{AB}}(X) = A_1 X B_1 - A_2 X B_2$. If A_1 commutes with A_2, B_1 commutes with B_2, $A_1^{-1}(0) \subseteq A_1^{*-1}(0)$ and $B_1^{*-1}(0) \subseteq B_1^{-1}(0)$, then $\mathrm{asc}(\Phi_{\mathbf{AB}}) \leq 1$.*

Proof. If we let $\mathcal{H} \oplus \mathcal{H} = \mathcal{H}_0$, $A = A_1 \oplus A_2^*$, $B = B_1 \oplus B_2^*$, $X = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \in B(\mathcal{H}_0)$ for a $Y \in B(\mathcal{H})$, and define $\phi_{AB} \in B(B(\mathcal{H}_0))$ by $\phi_{AB}(Z) = AZA^* - BZB^*$, then A is w–hyponormal with $A^{-1}(0) \subseteq A^{*-1}(0)$, B is normal, A and B commute and $Y \in \Phi_{\mathbf{AB}}^{-1}(0)$ if and only if $X \in \phi_{AB}^{-1}(0)$. Consequently, to prove $\mathrm{asc}(\Phi_{\mathbf{AB}}) \leq 1$ it would suffice to prove $\mathrm{asc}(\phi_{AB}) \leq 1$. To simplify notation and for convenience, in the following let $A \in B(\mathcal{H})$ be a w–hyponormal operator which satisfies $A^{-1}(0) \subseteq A^{*-1}(0)$ and which commutes with the normal operator $B \in B(\mathcal{H})$. Then either (a) $B^{-1}(0) = \{0\}$, or $(b)(i)$ $B^{-1}(0) \neq \{0\} = A^{-1}(0)$, or $(b)(ii)$ $B^{-1}(0) \neq \{0\}$, $A^{-1}(0) \neq \{0\}$ and $A^{-1}(0) \neq B^{-1}(0)$, or $(b)(iii)$ $B^{-1}(0) = A^{-1}(0) \neq \{0\}$. In the following we start by proving that $\mathrm{asc}(\phi_{AB}) \leq 1$ if (a) holds, and then prove that the proof reduces to this case if any of the other three conditions holds.

Assume $B^{-1}(0) = \{0\}$. For a natural number n, let $\Gamma_n = \{\lambda \in \mathbf{C} : |\lambda| \leq 1/n\}$, and let $E_B(\Gamma_n)$ denote the corresponding spectral projection. Set $I - E_B(\Gamma_n) = P_n$; then $P_n \to I$ in the strong topology. Since A, B commute implies A, B^* commute, $P_n\mathcal{H}$ reduces both A and B. Hence $A = A_{1n} \oplus A_{2n}$ and $B = B_{1n} \oplus B_{2n}$ (with respect to the decomposition $\mathcal{H} = (I - P_n)\mathcal{H} \oplus P_n\mathcal{H}$), where A_{in} are w–hyponormal with $A_{in}^{-1}(0) \subseteq A_{in}^{*}{}^{-1}(0)$ $(i = 1, 2)$, B_{1n} is normal and B_{2n} is invertible normal. For an $X \in \phi_{AB}^{-1}(0)$, let $P_n X P_n = X_n$; then $X_n \longrightarrow X$ weakly (even, strongly). If we now set $B_{2n}^{-1} A_{2n} = C_n$, then Lemma 2.2 implies that C_n is w–hyponormal with $C_n^{-1}(0) \subseteq C_n^{*-1}(0)$. Since

$$\begin{aligned} P_n \phi_{AB}(X) P_n &= P_n(AXA^* - BXB^*)P_n = A_{2n}(P_n X P_n)A_{2n}^* - B_{2n}(P_n X P_n)B_{2n}^* \\ &= A_{2n} X_n A_{2n}^* - B_{2n} X_n B_{2n}^* = B_{2n}(C_n X_n C_n^* - X_n)B_{2n}^*, \end{aligned}$$

$X_n \in \triangle^{-1}_{C_n C_n^*}(0)$. Hence, Lemma 2.1,

$$||X_n|| \leq ||\triangle_{C_n C_n^*}(T_n) + X_n||$$

for all $T_n \in B(P_n\mathcal{H})$. Choosing $T_n = B_{2n} Z_n B_{2n}^*$, we have

$$||X_n|| \leq ||\phi_{A_{2n}B_{2n}}(Z_n) + X_n||$$

for all $Z_n = P_n Z P_n \in B(P_n\mathcal{H})$. Trivially, $||\phi_{A_{2n}B_{2n}}(Z_n) + X_n|| \leq ||\phi_{AB}(Z) + X||$. Hence, since $||X_n|| \longrightarrow ||X||$,

$$||X|| \leq ||\phi_{AB}(Z) + X|| \text{ for all } X \in \phi_{AB}^{-1}(0) \text{ and } Z \in B(\mathcal{H}).$$

This implies $\text{asc}(\phi_{AB}) \leq 1$ in the case in which $B^{-1}(0) = \{0\}$.

Suppose now that $(b)(i)$ is satisfied. Decompose $\mathcal{H}$ by $\mathcal{H} = \ker^{\perp} B \oplus \ker B$. Then $B = B_1 \oplus 0$ and $A = A_1 \oplus A_2$ (recall: A commutes with B). Letting $X \in \phi_{AB}^{-1}(0)$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$, we then have

$$\phi_{AB}(X) = \begin{bmatrix} \phi_{A_1B_1}(X_{11}) & L_{A_1}R_{A_2^*}(X_{12}) \\ L_{A_2}R_{A_1^*}(X_{21}) & L_{A_2}R_{A_2^*}(X_{22}) \end{bmatrix} = 0.$$

Since A_1 and A_2 are injective, $X_{12} = X_{21} = X_{22} = 0$. Thus, $\phi_{AB}(X) = 0$ if and only if $\phi_{A_1B_1}(X_{11}) = 0$, where B_1 is injective.

If $(b)(ii)$ is satisfied, then we may assume without loss of generality that $B^{-1}(0) \not\subseteq A^{-1}(0)$. Decompose $\mathcal{H}$ by $\mathcal{H} = \ker^{\perp} B \oplus (\ker B \ominus \ker A_{22}) \oplus \ker A_{22}$, where $A_{22} = A|_{\ker B}$. Then $B = B_1 \oplus 0 \oplus 0$, $A = A_1 \oplus A_2 \oplus 0$, B_1 and A_2 are injective, and A_1 is w–hyponormal with $A_1^{-1}(0) \subseteq A_1^{*-1}(0)$. Letting $X \in B(\mathcal{H})$ have the representation $X = [X_{ij}]_{i,j=1}^3$, we have $X \in {\phi_{AB}^n}^{-1}(0)$ if and only if

$$\phi_{AB}^n(X) = \begin{bmatrix} \phi_{A_1B_1}^n(X_{11}) & (L_{A_1}R_{A_2^*})^n(X_{12}) & 0 \\ (L_{A_2}R_{A_1^*})^n(X_{21}) & (L_{A_2}R_{A_2^*})^n(X_{22}) & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Since w–hyponormal operators have ascent less than or equal to one,

$$(L_{A_1}R_{A_2^*})^n(X_{12}) = 0 \Longleftrightarrow L_{A_1}^n X_{12} = 0 \Longleftrightarrow L_{A_1}(X_{12}) = 0 \Longleftrightarrow L_{A_1}R_{A_2^*}(X_{12}) = 0,$$
$$(L_{A_2}R_{A_1^*})^n(X_{21}) = 0 \Longleftrightarrow R_{A_1^*}^n X_{21} = 0 \Longleftrightarrow R_{A_1^*}(X_{21}) = 0 \Longleftrightarrow L_{A_2}R_{A_1^*}(X_{21}) = 0 \text{ and}$$
$$(L_{A_2}R_{A_2^*})^n(X_{22}) = 0 \Longleftrightarrow X_{22} = 0.$$

Hence $\text{asc}(\phi_{AB}) \leq 1 \Longleftrightarrow \text{asc}(\phi_{A_1B_1}) \leq 1$, where $B_1^{-1}(0) = \{0\}$.

Finally, if $(b)(iii)$ is satisfied, then upon letting A_2 and $B = B_1 \oplus 0$, where A_1 and B_1 are injective. Letting $X = [X_{ij}]_{i,j=1}^2$ it is then seen that $X \in \phi_{AB}^{-1}(0)$ if and only if $X_{11} \in \phi_{A_1B_1}^{-1}(0)$. □

3. Range Closure

An operator $T \in B(\mathcal{X})$ is *polar at* $\lambda \in \sigma(T)$ if $\text{asc}(T - \lambda) = \text{dsc}(T - \lambda) < \infty$. Clearly, if T is polar at λ, then $\lambda \in \text{iso}\, \sigma(T)$. We say that T *is polaroid if it is polar at every* $\lambda \in$ *iso* $\sigma(T)$. T is *left polar at* $\lambda \in \sigma_a(T)$ (resp., *right polar at* $\lambda \in \sigma_s(T)$, $\sigma_s(T) = \sigma_a(T^*)$ the surjectivity spectrum of T) if $\text{asc}(T - \lambda) = d < \infty$ and $(T - \lambda)^{d+1}(\mathcal{X})$ is closed (resp., $\text{dsc}(T - \lambda) = d < \infty$ and $T^d(\mathcal{X})$ is closed). It can be seen that if T is left polar at $\lambda \in \sigma_a(T)$ (resp., T is right polar at $\lambda \in \sigma_s(T)$), then $\lambda \in \text{iso}\, \sigma_a(T)$ (resp., $\lambda \in \text{iso}\, \sigma_s(T)$). We say that T *is left polaroid (resp., right polaroid) if it is left polar at every* $\lambda \in$ *iso* $\sigma_a(T)$ *(resp., if it is right polar* *at every $\lambda \in$ iso $\sigma_s(T)$).* Evidently, T is polar at λ if and only if it is both left and right polar at λ. If $\mathcal{X} = \mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$, then T is left polar at $\lambda \in$ iso $\sigma_a(T)$ (resp., right polar at $\lambda \in$ iso $\sigma_s(T)$) if and only if there exist T-invariant closed subspaces M_1 and M_2 of $\mathcal{H}$ such that $\mathcal{H} = M_1 \oplus M_2$, $(T-\lambda)|_{M_1}$ is d-nilpotent (for some integer $d \geq 1$) and $(T-\lambda)|_{M_2}$ is bounded below (resp., $(T-\lambda)|_{M_1}$ is d-nilpotent and $(T-\lambda)|_{M_2}$ is surjective) [2, Theorem 3.4]. It is easily seen that T is right polar at $\lambda \in$ iso $\sigma_s(T)$ if and only if T^* is left polar at $\lambda \in$ iso $\sigma_a(T^*)$. The polaroid property (resp., the left polaroid property) transfers from $A, B \in B(\mathcal{X})$ to $L_A R_B$ and $L_A - R_B$ (resp., from $A, B^* \in B(\mathcal{H})$ to $L_A R_B$ and $L_A - R_B$).

Proposition 3.1. *(i) If $A, B \in B(\mathcal{X})$ are polaroid, then $L_A R_B, L_A - R_B \in B(B(\mathcal{X}))$ are polaroid.*
(ii) If $A, B^* \in B(\mathcal{H})$ are left polaroid, then $L_A R_B, L_A - R_B \in B(B(\mathcal{H}))$ are left polaroid.

Proof. See [7, Lemma 4.7] and [5, Theorem 3.6] for a proof of (i), and see [6, Theorem 3.4] for the proof of the case $L_A R_B$ of (ii). To prove the case $L_A - R_B$ of (ii), start by observing that $L_A - R_B$ is left polar at $\lambda \in$ iso $\sigma_a(L_A - R_B)$ if and only if $L_{A-\lambda} - R_B$ is left polar at $0 \in$ iso $\sigma_a(L_A - R_B - \lambda)$, and that L_A is left polar at $\lambda \in$ iso $\sigma_a(A)$ if and only if $L_{A-\lambda}$ is left polar at $0 \in$ iso $\sigma_a(A-\lambda)$. Hence to prove the result it would suffice to consider $0 \in$ iso $\sigma_a(L_A - R_B)$. Let $0 \in$ iso $\sigma_a(L_A - R_B) =$ iso $(\sigma_a(A) - \sigma_s(B)) = ($iso $\sigma_a(A) -$ iso $\sigma_s(B)) \setminus \text{acc}\sigma_a(L_A - R_B)$ (where $\text{acc}\sigma_a(\cdot)$ denotes the accumulation points of $\sigma_a(\cdot)$). Then there exist finite sequences $(\alpha) = \{\alpha_i\}_{i=1}^n \subseteq$ iso $\sigma_a(A)$ and $(\beta) = \{\beta_i\}_{i=1}^n \subseteq$ iso $\sigma_s(B)$ such that $\alpha_i - \beta_i = 0$ for all $1 \leq i \leq n$. The operator A and B^* being left polaroid (Hilbert space) operators, there exist A-invariant (closed) subspaces M_i and B-invariant (closed) subspaces N_i, $i = 1, 2$, such that the following holds:

$$
\begin{aligned}
&\mathcal{H} = M_1 \oplus M_2 = N_1 \oplus N_2, M_1 = \oplus_{i=1}^n M_{1i} = \oplus_{i=1}^n H_0(A-\alpha_i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \oplus_{i=1}^n (A-\alpha_i)^{-d_i}(0) \ \text{ and} \\
&N_1 = \oplus_{i=1}^n N_{1i} = \oplus_{i=1}^n H_0(B-\beta_i) = \oplus_{i=1}^n (B-\beta_i)^{-c_i}(0) \text{ for some integers} \\
&\qquad\qquad\qquad\qquad\qquad\qquad 1 \leq c_i, d_i (1 \leq i \leq n), \\
&A_1 = A|_{M_1} = \oplus_{i=1}^n A|_{M_{1i}} = \oplus_{i=1}^n A_{1i}, \quad B_1 = B|_{N_1} = \oplus_{i=1}^n B|_{N_{1i}} = \oplus_{i=1}^n B_{1i}, \\
&A_2 = A|_{M_2} \ \text{ and } \ B_2 = B|_{N_2}, \sigma_a(A_{1i}) = \{\alpha_i\} \ \text{ and } \ \sigma_s(B_{1i}) = \{\beta_i\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \ 1 \leq i \leq n, \\
&\sigma_a(A_2) = \sigma_a(A) \setminus \{\alpha_1, \cdots, \alpha_n\} \ \text{ and } \ \sigma_s(B_2) = \sigma_s(B) \setminus \{\beta_1, \cdots, \beta_n\}.
\end{aligned}
$$

Furthermore, $L_A - R_B$ is bounded below on its closed invariant subspaces $B(N_i, M_j)$, $1 \leq i, j \leq 2$ with $i = j \neq 1$, and $B(N_{1t}, M_{1k})$, $1 \leq t \neq k \leq n$. If we let

$$
\begin{aligned}
&E_1 = \oplus_{i=1}^n B(N_{1i}, M_{1i}) \ \text{ and} \\
&E_2 = \oplus_{1 \leq i,j \leq 2,\ i=j \neq 1} B(N_i, M_j) \oplus \{\oplus_{1 \leq t \neq k \leq n} B(N_{1t}, M_{1k})\},
\end{aligned}
$$

then $B(\mathcal{H}) = E_1 \oplus E_2$, $(L_A - R_B)|_{E_1}$ is nilpotent and $(L_A - R_B)|_{E_2}$ is bounded below. Hence $L_A - R_B$ is left polar at 0. □

$T \in B(\mathcal{X})$ is *simply polar at $\lambda \in$ iso $\sigma(T)$* (resp.,*left simply polar at $\lambda \in$ iso $\sigma_a(T)$*) if $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) = 1$ (resp., if T is left polar at λ with $\text{asc}(T-\lambda) = 1$). Evidently, T left simply polar at λ implies $T - \lambda$ has closed range;

conversely, if $\mathrm{asc}(T-\lambda) \leq 1$ and $(T-\lambda)(\mathcal{X})$ is closed, then (either $\lambda \notin \sigma_a(T)$ or) T is left simply polar at λ. Recall from [12, Corollary 3.3 and Lemma 3.1] that if T is left simply polar at λ and T^* has SVEP at λ, then $T-\lambda$ is Drazin invertible with Drazin index one (i.e., T is simply polar at λ).

Let Ψ_{AB} denote either of the operators **(a)** $\triangle_{AB}$ with $A, B \in B(\mathcal{H})$ contractions, or **(b)** δ_{AB} with $A, B^{-1} \in B(\mathcal{H})$ contractions, or **(c)** $L_A R_B - \lambda$ with $A, B \in B(\mathcal{H})$ normaloid and $\lambda \in \sigma_\pi(L_A R_B)$.

Theorem 3.2. *If $A, B^* \in B(\mathcal{H})$ are left polaroid, then the following conditions are pairwise equivalent:*
(i) $0 \in$ *iso* $\sigma_a(\Psi_{AB})$
(ii) Ψ_{AB} *is left polar at* 0.
(iii) Ψ_{AB} *has closed range.*
(iv) There exist finite sequences $(\alpha) = \{\alpha_i\}_{i=1}^n \subseteq$ *iso* $\sigma_a(A)$ *and* $(\beta) = \{\beta_i\}_{i=1}^n \subseteq$ *iso* $\sigma_s(B)$ *such that, for all* $1 \leq i \leq n$, $\alpha_i\beta_i - 1 = 0$ *if* Ψ_{AB} *is as in* **(a)**, $\alpha_i - \beta_i = 0$ *if* Ψ_{AB} *is as in* **(b)**, *and* $\alpha_i\beta_i - \lambda = 0$ *if* Ψ_{AB} *is as in* **(c)**.
(v) $B(\mathcal{H}) = {\Psi_{AB}}^{-1}(0) \oplus \Psi_{AB}(B(\mathcal{H}))$.
(vi) $0 \in$ *iso* $\sigma(\Psi_{AB})$.

Proof. It is straightforward to see that if an invertible operator T is left polar at a point $\lambda(\neq 0)$, then T^{-1} is left polar at λ^{-1}. Thus if, in case **(b)**, B^* is left polaroid, then $C^* = B^{*-1}$ is left polaroid. Since $(L_A R_{B^{-1}} - 1)^{-1}(0) = (L_A - R_B)^{-1}(0)$, $(L_A R_{B^{-1}} - 1)(B(\mathcal{X})) = (L_A - R_B)(B(\mathcal{X}))$ and $0 \in \mathrm{iso}\, \sigma_a(L_A - R_B) \iff 0 \in \mathrm{iso}\, \sigma_a(L_A R_{B^{-1}} - 1)$ in the case in which $A, B^{-1} \in B(\mathcal{H})$ are contractions, the proof of the theorem for the case in which Ψ_{AB} is as in **(b)** follows from that for case **(a)**. Observe that if Ψ_{AB} is the operator of either of the cases **(a)** and **(b)**, then $asc(\Psi_{AB}) \leq 1$ and $0 \in \partial\sigma(\Psi_{AB})$; furthermore, this follows from our hypothesis that A and B^* are left polaroid, Ψ_{AB} is left simply polaroid.

The implications $(v) \Longrightarrow (vi) \Longrightarrow (i)$ are evident, and the implication $(i) \Longrightarrow (ii)$ is a straightforward consequence of the fact that Ψ_{AB} is left polaroid (hence, left polar at $0 \in \mathrm{iso}\, \sigma_a(\Psi_{AB})$). Again, $(ii) \Longrightarrow (iii)$ is evident, and if (iii) is satisfied, then ($asc(\Psi_{AB}) \leq 1$ and $\Psi_{AB}(B(\mathcal{X}))$ is closed imply) Ψ_{AB} is left simply polar at 0 and hence $0 \in \mathrm{iso}\, \sigma_a(\Psi_{AB})$. An argument similar to that used in the proof of Proposition 3.1 now gives the existence of the sequences $(\alpha) = \{\alpha_i\}_{i=1}^n \subseteq \mathrm{iso}\, \sigma_a(A)$ and $(\beta) = \{\beta_i\}_{i=1}^n \subseteq \mathrm{iso}\, \sigma_s(B)$ such that, for all $1 \leq i \leq n$, $\alpha_i\beta_i - 1 = 0$ if Ψ_{AB} is as in **(a)**, and $\alpha_i\beta_i - \lambda = 0$ if Ψ_{AB} is as in **(c)**. The implication $(iv) \Longrightarrow (i)$ being evidently true (since $\mathrm{iso}\, \sigma_a(L_A R_B) \subseteq \mathrm{iso}\, \sigma_a(A) \mathrm{iso}\, \sigma_s(B)$), we have $(iii) \Longrightarrow (iv) \Longrightarrow (i)$. The point $0 \in \partial\sigma(\Psi_{AB})$ implies Ψ^*_{AB} has SVEP at 0. Hence, as remarked upon above, Ψ_{AB} is polar at 0 (and then $0 \in \mathrm{iso}\, \sigma(\Psi_{AB})$). Thus $(ii) \Longrightarrow (v)$, and the proof is complete. □

Remark 3.3. (a). Given normal operators $A, B \in B(\mathcal{X})$, both δ_{AB} and δ^*_{AB} have SVEP everywhere (thus $\sigma(\delta_{AB}) = \sigma(\delta^*_{AB}) = \sigma_a(\delta_{AB})$), $\mathrm{asc}(\delta_{AB} - \lambda) \leq 1$ for all complex λ and δ_{AB} is (simply) polaroid. (Observe, however, that δ_{AB} is not normaloid [4, Example 5.6].) The argument of the proof of Theorem 3.2 gives us the following generalization of [4, Theorem 3.3]. *If $A, B \in B(\mathcal{X})$ are polaroid, $asc(\delta_{AB}) \leq 1$ and δ^*_{AB} has SVEP at 0, then the conditions (i) to (vi) of Theorem 3.2 are mutually equivalent with Ψ_{AB} replaced by δ_{AB}.*

(b). We do not know if Proposition 3.1 (ii) extends to left polaroid operators in $B(\mathcal{X})$ (and hence whether one can replace $B(\mathcal{H})$ by $B(\mathcal{X})$ in Theorem 3.2).

It is however straightforward to see that if $A, B \in B(\mathcal{X})$ are polaroid, then the conditions $(ii), (iii), (iv), (v)$ and (vi) of Theorem 3.2 are pairwise equivalent.

(c). If $A, B^* \in B(\mathcal{H})$ are left polaroid and $d_{AB}^{-1}(0) \subseteq d_{A^*B^*}^{-1}(0)$, then conditions (i) to (iv) of Theorem 3.2 are mutually equivalent with Ψ_{AB} replaced by d_{AB}; if also d_{AB}^* has SVEP at 0, then all six conditions of the theorem are equivalent. Trivially, the operator $\Phi_{\mathbf{AB}}$ of Proposition 2.8 has closed range if and only if $\Phi_{\mathbf{AB}}$ is left polar at 0. We have not been able to prove a result similar to Theorem 3.2 for $\Phi_{\mathbf{AB}}$: However, as we shall see in the following, a satisfactory version of Theorem 3.2 is possible for the operators $\Phi_{\mathbf{AB}}$ if we restrict ourselves to separable Hilbert spaces $\mathcal{H}$ and operators $\Phi_{\mathbf{AB}} \in B(\mathcal{C}_p)$, where $\mathcal{C}_p = \mathcal{C}_p(\mathcal{H})$, $1 < p < \infty$, denotes the von Neumann–Schatten p-class.

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{C}_p = \mathcal{C}_p(\mathcal{H})$, $1 < p < \infty$, denote the von Neumann–Schatten p-class. Then $\mathcal{C}_p$ is a reflexive Banach space with norm $||X||_p = \big(\sum_j s_j^p(X)\big)^{1/p}$, where $s_j(X)$ are the singular values of $X \in \mathcal{C}_p$. The dual space of $\mathcal{C}_p$ is the space $\mathcal{C}_{p'}$, where $1/p + 1/p' = 1$.

Theorem 3.4. *If $\Phi_{\mathbf{AB}} \in B(\mathcal{C}_p)$ is the operator $\Phi_{\mathbf{AB}}(X) = A_1XB_1 - A_2XB_2$, where A_i and B_i, $i = 1, 2$, are the operators of Proposition 2.8, then the following conditions are mutually equivalent:*

(i) $\Phi_{\mathbf{AB}}(\mathcal{C}_p)$ is closed.
(ii) $\Phi_{\mathbf{AB}}$ is left simply polar at 0.
(iii) $\Phi_{\mathbf{AB}}$ is simply polar at 0.
(iv) $\mathcal{C}_p = \Phi_{\mathbf{AB}}^{-1}(0) \oplus \Phi_{\mathbf{AB}}(\mathcal{C}_p)$.

Proof. Since $\operatorname{asc}(\Phi_{\mathbf{AB}}) \leq 1$, Proposition 2.8, the implications $(iv) \Longrightarrow (i) \Longrightarrow (ii)$ and $(iii) \Longrightarrow (iv)$ are evidently true. To prove the implication $(ii) \Longrightarrow (iii)$, we prove in the following that the adjoint operator $\Phi_{\mathbf{AB}}^*$ has SVEP at 0. This would then imply (by [12, Corollary 3.3 and Lemma 3.1]) that $\Phi_{\mathbf{AB}}$ is simply polar whenever it is simply left polar at 0. Let $X \in \mathcal{C}_p$ and $Y \in \mathcal{C}_p'$. Then $\operatorname{tr}(\Phi_{\mathbf{AB}}(X)Y) = \operatorname{tr}((A_1XB_1 - A_2XB_2)Y) = \operatorname{tr}(X(B_1YA_1 - B_2YA_2))$, where tr(.) denotes the trace functional. Hence $\Phi_{\mathbf{AB}}^*$, the adjoint operator of the operator $\Phi_{\mathbf{AB}}$, is the operator $\Phi_{\mathbf{AB}}^* \in B(\mathcal{C}_{p'})$, $\Phi_{\mathbf{AB}}^*(Y) = B_1YA_1 - B_2YA_2$. We prove next that $\operatorname{asc}(\Phi_{\mathbf{AB}}^*) \leq 1$. For $Y \in \mathcal{C}_{p'}$ such that $\Phi_{\mathbf{AB}}^{*2}(Y) = 0$, set $\Phi_{\mathbf{AB}}^*(Y) = Z$. Then, since $0 = \operatorname{tr}(X\Phi_{\mathbf{AB}}^*(Z)) = \operatorname{tr}(\Phi_{\mathbf{AB}}(X)Z)$ for all $X \in \mathcal{C}_p$, we must have $Z = 0$, i.e., we must have $\operatorname{asc}(\Phi_{\mathbf{AB}}^*) \leq 1$. Consequently, $\Phi_{\mathbf{AB}}^*$ has SVEP at 0. This completes the proof. □

4. Compactness

Let $\Theta_{AB} \in B(B(\mathcal{X}))$ denote either of the elementary operators Ψ_{AB} of Theorem 3.2 (but without the left polaroid hypothesis on A and B^*) and the operator $\Phi_{\mathbf{AB}}$ of Proposition 2.8 but with $A_1, B_1^* \in B(\mathcal{H})$ hyponormal (thus the hypothesis $A_1^{-1}(0) \subseteq A_1^{*-1}(0)$ and $B_1^{*-1}(0) \subseteq B_1^{-1}(0)$ is redundant). Recall from [25, Theorem 3] that if an $A \in B(\mathcal{H})$ is left invertible by a contraction and B is a contraction, then $(\delta_{AB}^n)^{-1}(0) \cap \mathcal{K}(\mathcal{H}) = \delta_{AB}^{-1}(0) \cap \mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H}) \subset B(\mathcal{H})$ denotes the (two sided) ideal of compact operators. This is an easy consequence of our results. Let, for convenience, Υ_{AB} denote either of the operators Θ_{AB} and the operator d_{AB} of Proposition 2.6 (recall: $d_{AB} = \delta_{AB}$ or $\triangle_{AB}$ and $d_{AB}^{-1}(0) \subseteq d_{A^*B^*}^{-1}(0)$). Since $\operatorname{asc}(\Upsilon_{AB}) \leq 1$, $X \in \mathcal{K}(\mathcal{X}) \cap {\Upsilon_{AB}^n}^{-1}(0)$ *for some integer $n \geq 1$ if and only if*

$X \in \mathcal{K}(\mathcal{X}) \cap \Upsilon_{AB}^{-1}(0)$. The next problem that we consider is the characterization of the operators $A, B \in B(\mathcal{X})$ (or, A_i and $B_i \in B(\mathcal{H})$, $1 \leq i \leq 2$) such that the operator $\Omega_{AB}^n = (L_A - R_B)^n$ or $(L_A R_B - \lambda)^n$ or $\Phi_{\mathbf{AB}}^n$ is compact for some integer $n \geq 1$ implies Ω_{AB} is compact? We start however with the following generalization of [25, Theorem 6] to the operator Γ_{AB}. Recall [1] that the (Fredholm) *essential spectrum* $\sigma_e(T)$ of $T \in B(\mathcal{X})$ is the set $\sigma_e(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm}\}$.

Theorem 4.1. *Let $\Gamma_{AB} = \Theta_{AB}$, or δ_{AB} with $A, B \in B(\mathcal{X})$ normal. If $\Gamma_{AB}^n(X)$ is compact for some integer $n \geq 1$ and operator $X \in B(\mathcal{X})$, then $\Gamma_{AB}(X)$ is compact.*

Proof. If $\Pi : B(\mathcal{X}) \longrightarrow B(\mathcal{X}) \setminus \mathcal{K}(\mathcal{X})$, denotes the *Calkin map*, then, given a $T \in B(\mathcal{X})$, $\sigma(\Pi(T))$ is the (Fredholm) *essential spectrum* $\sigma_e(T)$ of T, the essential norm $||T||_e = ||\Pi(T)||$ satisfies $||T||_e \leq ||T||$ and the essential spectral radius $r_e(T) = r(\Pi(T)) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$ satisfies $r_e(T) \leq r(T)$ [19, Section 19].

It is known that if A, B are normal, then δ_{AB} is normal and has finite ascent ≤ 1. Since A, B normal implies $\Pi(A), \Pi(B)$ normal, $\delta_{\Pi(A)\Pi(B)}$ is normal. Hence if $\Gamma_{AB} = \delta_{AB}$ with A, B normal, then $\text{asc}(\delta_{\Pi(A)\Pi(B)}) \leq 1$. If $\Gamma_{AB} = \delta_{AB}$, where $B \in B(\mathcal{X})$ is a contraction and $A \in B(\mathcal{X})$ is left invertible by a contraction, then the left inverse A_ℓ of A and the operator B being contractions the operators $\Pi(A_\ell)$ and $\Pi(B)$ are contractions and it follows from the argument of the proof of Proposition 2.3 that $\text{asc}(\delta_{\Pi(A)\Pi(B)}) \leq 1$. If $\Gamma_{AB} = \triangle_{AB}$, A and $B \in B(\mathcal{X})$ contractions, then $\text{asc}(\triangle_{\Pi(A)\Pi(B)}) \leq 1$. Now let $\Gamma_{AB} = L_A R_B - \lambda$, where A, B are normaloid and $\lambda \in \sigma_\pi(L_A R_B)$. Then, since $\sigma_e(L_A R_B) = \sigma_e(A)\sigma(B) \bigcup \sigma(A)\sigma_e(B)$, either $|\lambda| > r_e(L_A R_B)$ or $|\lambda| = r_e(L_A R_B)$ (in which case $||L_A R_B|| = r(L_A R_B) = r_e(L_A R_B) \leq ||AB||_e \leq ||L_A R_B||$). In either case $\text{asc}(L_{\Pi(A)} R_{\Pi(B)} - \lambda) \leq 1$. Finally, if $\Gamma_{AB} = \Phi_{\mathbf{AB}}$, then the operators $\Pi(A_1)$, $\Pi(B_1^*)$ are hyponormal, the operators $\Pi(A_2), \Pi(B_2)$ are normal, $\Pi(A_1)$ commutes with $\Pi(A_2)$ and $\Pi(B_1)$ commutes with $\Pi(B_2)$. Conclusion: $\text{asc}(\Gamma_{\Pi(A)\Pi(B)}) \leq 1$. To conclude the proof, suppose now that $\Gamma_{AB}^n(X)$ is compact. Then $\Gamma_{\Pi(A)\Pi(B)}^n(\Pi(X)) = 0$ implies $\Gamma_{\Pi(A)\Pi(B)}(\Pi(X)) = 0$, and this in turn implies that $\Gamma_{AB}(X)$ is compact. □

A proof of the following theorem for the case in which $\Omega_{AB} = L_A - R_B$ appears in [14, Proposition 4]; our proof however is different from that in [14].

Theorem 4.2. *(a) The following conditions are mutually equivalent.*

(i) d_{AB}^n is compact for some integer $n \geq 1$.
(ii) $A - \alpha$ and $B - \beta$ are nilpotent for some scalars α, β such that $\alpha = \beta$ if $d_{AB} = \delta_{AB}$ and $\alpha = 1/\beta$ if $d_{AB} = \triangle_{AB}$.
(iii) d_{AB} is nilpotent.

(b) If $\Phi_{\mathbf{AB}}$ is the operator of Proposition 2.8 but with A_1, B_1^* hyponormal, then $\Phi_{\mathbf{AB}}^n$ is compact for some integer $n \geq 1$ if and only if one of the following conditions is satisfied.
(i) $\Phi_{\mathbf{AB}} = 0$.
(ii) A_1 and A_2, or B_1 and B_2, are compact normal operators

Proof. (a) *Case $d_{AB} = \delta_{AB}$*. If δ_{AB}^n is compact, then $\sigma_e(\delta_{AB}^n) = \{0\}$ implies (by the spectral mapping theorem for the Fredholm essential spectrum that) $\sigma_e(\delta_{AB}) = \{0\}$ (i.e., δ_{AB} is a Riesz operator). Since $\sigma_e(\delta_{AB}) = \{\sigma_e(A) - \sigma(B)\} \bigcup \{\sigma(A) - \sigma_e(B)\}$, $\sigma(A) = \sigma_e(A) = \{\alpha\} = \sigma_e(B) = \sigma(B)$ for some scalar α and $\sigma(\delta_{AB}) = \sigma(A) - \sigma(B) = \{0\} = \sigma_e(\delta_{AB})$. The hypothesis $\delta_{AB}^n = \sum_{i=0}^n (-1)^{n-i} \begin{pmatrix} n \\ i \end{pmatrix} L_{A^{n-i}} R_{B^i}$

is compact implies also that A and B are *algebraic operators*. (Observe that A^i (resp. B^i), $0 \leq i \leq n$, linearly independent implies B^i (resp., A^i) compact for all $0 \leq i \leq n$, and this contradicts the fact that the identity operator is not compact.) Hence A and B are polaroid: indeed, since $\sigma(A) = \sigma(B) = \{\alpha\}$, $A - \alpha$ and $B - \alpha$ are nilpotent [1, Theorem 3.83]. Hence δ_{AB} is polaroid (by Proposition 3.1), consequently nilpotent (since $\delta_{AB} = \{0\}$). This proves $(i) \Longrightarrow (ii) \Longrightarrow (iii)$. The implication $(iii) \Longrightarrow (i)$ being evident, the proof follows.
Case $d_{AB} = \triangle_{AB}$. The proof is similar, so we shall be brief. The hypothesis $\triangle^n_{AB}$ is compact implies A, B are algebraic, $\sigma(A) = \sigma_e(A) = \{\alpha\}$, $\sigma(B) = \sigma_e(B) = \{1/\alpha\}$ and $\sigma(\triangle_{AB}) = \sigma_e(\triangle_{AB}) = \{0\}$. Thus $(A - \alpha)$ and $(B - 1/\alpha)$ are nilpotent. Consequently, $\triangle_{AB}$ is (polar, indeed) nilpotent.

(b) The hypotheses imply (by Theorem 4.1) that $\Phi^n_{\mathbf{AB}}(X)$ is compact if and only if $\Phi_{\mathbf{AB}}(X)$ compact for all $X \in B(\mathcal{X})$; equivalently, $\Phi^n_{\mathbf{AB}}$ is compact if and only if $\Phi_{\mathbf{AB}}$ is compact. Suppose that $\Phi_{\mathbf{AB}}$ is compact. We have two possibilities: (a) B_1 and B_2 (resp., A_1 and A_2) are linearly independent; (b) B_1 and B_2 (resp., A_1 and A_2) are linearly dependent. Suppose to start with that B_1 and B_2 (similarly, A_1 and A_2) are linearly independent. Then A_1 and A_2 (resp., B_1 and B_2) are compact [14, Theorem 2]. Since a compact hyponormal operator (indeed, a compact paranormal operator) is normal, A_1 and A_2 (resp., B_1 and B_2) are compact normal operators. Thus, if (a) holds, then either A_1 and A_2 (or B_1 and B_2) are compact normal operators, or, if A_1 (resp., B_1) fails to be either normal or compact, then B_1 and B_2 (resp., A_1 and A_2) are linearly dependent. Consider next (b). If $B_2 = \alpha B_1$ for some scalar α, then B_1 and B_2 are commuting normal operators such that $\Phi_{\mathbf{AB}} = L_{(\alpha A_1 - A_2)} R_{B_1}$ is compact. Since A_1 and A_2 commute,

$$\begin{aligned}\sigma_e(\Phi_{\mathbf{AB}}) &= \{\sigma_e(\alpha A_1 - A_2)\sigma(B_1)\} \bigcup \{\sigma(\alpha A_1 - A_2)\sigma_e(B_1)\} \\ &= \{[\alpha\sigma_e(A_1) - \sigma_e(A_2)]\sigma(B_1)\} \bigcup \{[\alpha\sigma(A_1) - \sigma(A_2)]\sigma_e(B_1)\} = \{0\};\end{aligned}$$

hence either $\sigma(B_1) = \{0\}$, or, $\alpha\sigma(A_1) - \sigma(A_2) = \{0\}$. Since B_1 is normal, $\sigma(B_1) = \{0\}$ implies $B_1 = 0$ (implies $\Phi_{\mathbf{AB}} = 0$). So assume $\alpha\sigma(A_1) - \sigma(A_2) = \{0\}$. Then $\sigma(A_1) = \{\beta/\alpha\}$ and $\sigma(A_2) = \{\beta\}$ for scalar β. Normal and hyponormal (indeed, paranormal) operators are known to be simply polaroid. Hence $A_1 = (\beta/\alpha)I$ and $A_2 = \beta I$, and then $\Phi_{\mathbf{AB}} = 0$. A similar argument works for the case in which A_1 and A_2 are linearly dependent to prove that $\Phi_{\mathbf{AB}} = 0$. □

Remark 4.3. (a).The argument of the proof of Theorem 4.2(a) extends to prove that if $d_{AB} = (L_A R_B - \lambda)^n$ is compact for some integer $n \geq 1$ and scalar λ, then the equivalences $(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii)$ Theorem 4.2(a) hold with $\alpha\beta = \lambda$. Evidently, if $\operatorname{asc}(d_{AB}) \leq 1$, then d_{AB} is compact if and only if $d_{AB} = 0$.

(b). For an operator $T \in B(\mathcal{X})$, T^n is Riesz if and only if T is Riesz [1, Theorem 3.113]. Hence the operator d^n_{AB} is Riesz if and only if d_{AB} is Riesz. Suppose that d_{AB} is Riesz. Then $\sigma_e(d_{AB}) = \{0\}$, $\sigma(A) = \sigma(B) = \{\alpha\}$ for some scalar α if $d_{AB} = \delta_{AB}$, and $\sigma(A) = \{\alpha\}$ and $\sigma(B) = \{1/\alpha\}$ for some scalar $\alpha \neq 0$ if $d_{AB} = \triangle_{AB}$. Clearly, $\sigma(d_{AB}) = \{0\}$ and $d_{AB} = Q$ is a quasinilpotent. If we now assume that A and B are polaroid, then d_{AB} is polaroid, and hence nilpotent. Conclusion: *If $A, B \in B(\mathcal{X})$ are polaroid operators, then d^n_{AB} is a Riesz operator for some integer $n \geq 1$ if and only if d_{AB} is a nilpotent operator.* The same conclusion is valid for $d_{AB} = L_A R_B - \lambda$.

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