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DUAL-SHIFT DECOMPOSITION AND WAVELETS

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Abstract. We introduce the notion of *dual-shift decomposition* of an arbitrary Hilbert space, which is given in terms of two unilateral shifts. After ensuring conditions for the existence of it, such a decomposition is then constructed for the concrete space $\mathcal{L}^2[0,1]$, on which the two unilateral shifts are parts of the dilation-by-2 and the translation-by-1 on $\mathcal{L}^2(\mathbb{R})$. Using multiresolution analysis of wavelet theory it is shown the existence of a Haar-system-type orthonormal basis for $\mathcal{L}^2[0,1]$, which is combined with the dual-shift decomposition to yield a refined decomposition for $\mathcal{L}^2[0,1]$.

1. Introduction

Throughout this paper $\mathcal{H}$ will stand for an arbitrary Hilbert space. By an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself (i.e., a continuous linear transformation of $\mathcal{H}$ into itself). If $A\colon \mathcal{H}\to\mathcal{H}$ is an operator on $\mathcal{H}$, then let $\ker(A)$ denote the kernel of A (i.e., $\ker(A) = A^{-1}(\{0\}) = \{x\in\mathcal{H}\colon Ax = 0\}$), which is a subspace (i.e., a closed linear manifold) of $\mathcal{H}$, and let $\operatorname{ran}(A)$ denote the range of A (i.e., $\operatorname{ran}(A) = A(\mathcal{H})$), which is a linear manifold of $\mathcal{H}$. An isometry is an operator A such that $A^*A = I$, where A^* denotes the adjoint of A, and I stands for the identity operator. Let S and V be unilateral shifts on $\mathcal{H}$. Recall that $\mathcal{H}$ admits a *wandering subspace* decomposition in terms of any unilateral shift, say, $\mathcal{H} = \bigoplus_{k=0}^\infty S^k\ker(S^*) = \bigoplus_{k=0}^\infty V^k\ker(V^*)$ — the symbol $\oplus$ stands for *orthogonal* direct sum. Is it possible to decompose $\mathcal{H}$ into a similar orthogonal decomposition, involving both S and V simultaneously? In other words, when is it true that

$$\mathcal{H} = \bigoplus_{k=1}^\infty S^k\ker(S^*) \oplus \bigoplus_{k=1}^\infty V^k\ker(V^*)\ ?$$

If such a decomposition exists, then we refer to it as a *dual-shift decomposition* of the Hilbert space $\mathcal{H}$. It is worth noticing that we can always get a decomposition similar to the above one, viz., $\mathcal{H} = \bigoplus_{k=0}^\infty S^k\ker(S^*)\oplus\bigoplus_{k=0}^\infty V^k\ker(V^*)$, if we allow the shifts S and V to act on different Hilbert spaces, say, $\mathcal{H}_1$ and $\mathcal{H}_2$, and consider their orthogonal direct sum $\mathcal{H} = \mathcal{H}_1\oplus\mathcal{H}_2$. In fact, since $\mathcal{H} = \mathcal{M}\oplus\mathcal{M}^\perp$ for any subspace $\mathcal{M}$ of $\mathcal{H}$, where $\mathcal{M}^\perp = \mathcal{H}\ominus\mathcal{M}$ denotes the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$, and since $\mathcal{M}$ and $\mathcal{M}^\perp$ are Hilbert spaces, it is enough to consider their wandering subspace decompositions in terms of a unilateral shift S on $\mathcal{M}$, and of a unilateral shift V on $\mathcal{M}^\perp$. However, our dual-shift decomposition requires that S and V are unilateral shifts acting on the same Hilbert space $\mathcal{H}$.

First we establish a sufficient condition for the existence of a dual-shift decomposition for an arbitrary Hilbert space $\mathcal{H}$ in Theorem 1. Then we construct in Theorem 2 such a decomposition for the especial case where $\mathcal{H}$ is the function space $\mathcal{L}^2[0,1]$, with respect to two unilateral shifts, denoted by S and V, which are parts of to the dilation-by-2 and the translation-by-1 bilateral shifts D and T

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on $\mathcal{L}^2(\mathbb{R})$. Let ψ be an orthonormal wavelet in $\mathcal{L}^2[0,1]$, which comes from a scaling function $\phi \in \mathcal{L}^2[0,1]$. We investigate when the orthonormal wavelet functions $\psi_{m,n} = D^m T^n \psi$ lie in $\mathcal{L}^2[0,1]$ and, with the scaling function ϕ, form an orthonormal basis for $\mathcal{L}^2[0,1]$. This is shown by means of the multiresolution analysis (MRA) associated with ϕ and ψ. An example of the above is the Haar system in $\mathcal{L}^2[0,1]$. This system is obtained from the Haar wavelet ψ_H in $\mathcal{L}^2(\mathbb{R})$,

$$\psi_H(t) = \begin{cases} 1, & 0 < t \le \frac{1}{2}, \\ -1, & \frac{1}{2} < t \le 1, \\ 0, & t \in \mathbb{R}\backslash(0,1], \end{cases}$$

restricted to $[0,1]$, $\psi_H \chi_{[0,1]}$, and the associated Haar scaling function $\phi_H = \chi_{[0,1]}$, where $\chi_{[0,1]}$ is the characteristic function of the set $[0,1]$. We derive from the Haar system on $\mathcal{L}^2[0,1]$ similar systems for the subspaces $\mathcal{L}^2[0,\frac{1}{2}]$ and $\mathcal{L}^2[\frac{1}{2},1]$. These Haar systems will finally be combined with the dual-shift decomposition to yield a refined decomposition for the function space $\mathcal{L}^2[0,1]$ in Theorem 3.

2. Dual-Shift Decomposition of $\mathcal{H}$

We begin by supplying an auxiliary result for a pair of isometries on $\mathcal{H}$, which will be required for establishing the dual-shift decomposition of Theorem 1.

Lemma 1. *Let S and V be isometries on a Hilbert space $\mathcal{H}$. The following assertions are pairwise equivalent.*

(a) $\operatorname{ran}(S) = \ker(V^*)$.

(b) $\operatorname{ran}(V) = \ker(S^*)$.

(c) $SS^* + VV^* = I$.

Proof. The equivalence between (a) and (b) follows at once since $\ker(A) = \operatorname{ran}(A^*)^\perp$ for every operator A on $\mathcal{H}$, and for every linear manifold $\mathcal{M}$ of $\mathcal{H}$ (recall that $(\mathcal{M}^-)^\perp = \mathcal{M}^\perp$, where $\mathcal{M}^-$ denotes the closure of $\mathcal{M}$, and $\mathcal{M}^{\perp\perp} = \mathcal{M}^-$), and since isometries have a closed range (see e.g., [16, Problem 4.41 and Proposition 5.76]). Suppose any of the equivalent assertions (a) or (b) holds true and take an arbitrary $x = u + v$ in $\mathcal{H} = \operatorname{ran}(S) + \operatorname{ran}(S)^\perp = \ker(V^*) + \operatorname{ran}(V)$ so that $u \in \operatorname{ran}(S) = \ker(V^*)$ and $v \in \operatorname{ran}(V) = \ker(S^*)$. Thus

$$(SS^* + VV^*)x = SS^*u + SS^*v + VV^*u + VV^*v = SS^*Sy + VV^*Vz = Sy + Vz,$$

for some y and z in $\mathcal{H}$ such that $u = Sy$ and $v = Vz$. Therefore $(SS^* + VV^*)x = u + v = x$; that is, assertion (c) holds true. Conversely, suppose (c) holds true. If $u \in \operatorname{ran}(S)$ so that $u = Sy$ for some $y \in \mathcal{H}$, then $SS^*u = SS^*Sy = Sy = u$, and hence $u = SS^*u + VV^*u = u + VV^*u$, so that $u \in \ker(VV^*) = \ker(V^*)$; that is, $\operatorname{ran}(S) \subseteq \ker(V^*)$. On the other hand, if $v \in \ker(V^*)$, then $v = (SS^* + VV^*)V = SS^*v \in \operatorname{ran}(S)$ and so $\ker(V^*) \subseteq \operatorname{ran}(S)$. Hence (c) implies (a). □

Remark 1. It is worth noticing that condition (c) in Lemma 1 is the very Cuntz condition [7], [4, p.53] for the C^* algebra $\mathcal{O}_2$ generated by a couple of isometries on an infinite-dimensional Hilbert space [5] (which is a especial case of partition of the identity). Along these lines, also see, e.g., [1, 2, 6, 9, 10, 14].

Theorem 1. *If S and V are unilateral shifts on a Hilbert space $\mathcal{H}$ such that $SS^* + VV^* = I$, then $\mathcal{H}$ admits the dual-shift decomposition*

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} S^k \ker(S^*) \oplus \bigoplus_{k=1}^{\infty} V^k \ker(V^*).$$

Proof. Recall that a unilateral shift S on a Hilbert space $\mathcal{H}$ is an isometry for which $\mathcal{H}$ admits the orthogonal decomposition [25, 15]

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} S^k \ker(S^*),$$

where the subspace $\ker(S^*)$ is called the *generating wandering subspace* of S, and its dimension is the *multiplicity* of S. Then

$$\operatorname{ran}(S) = S(\mathcal{H}) = \bigoplus_{k=0}^{\infty} SS^k \ker(S^*) = \bigoplus_{k=0}^{\infty} S^k S \ker(S^*).$$

Since the range of any isometry is closed, $\operatorname{ran}(S)$ is a subspace of $\mathcal{H}$, which is clearly S-invariant. Therefore, the restriction of S to its range, $S|_{\operatorname{ran}(S)} \colon \operatorname{ran}(S) \to \operatorname{ran}(S)$, is also a unilateral shift [15, Proposition 6.2] whose wandering subspace is $S\ker(S^*)$. Thus, if $SS^* + VV^* = I$, then it follows from Lemma 1 (since the above expression holds for every unilateral shift, and by uniqueness of the orthogonal decomposition) that this second unilateral shift V together with S are such that the space $\mathcal{H}$ admits the orthogonal decomposition

$$\begin{aligned} \mathcal{H} &= \operatorname{ran}(S) \oplus \operatorname{ran}(S)^{\perp} = \operatorname{ran}(S) \oplus \ker(S^*) = \operatorname{ran}(S) \oplus \operatorname{ran}(V) \\ &= \bigoplus_{k=1}^{\infty} S^k \ker(S^*) \oplus \bigoplus_{k=1}^{\infty} V^k \ker(V^*) \end{aligned}$$

(cf. [16, Theorem 5.25 and Proposition 5.76]). □

3. Dual-Shift Decomposition of $\mathcal{L}^2[0,1]$

The dilation-by-2 operator D on $\mathcal{L}^2(\mathbb{R})$ is defined by

$$g = Df \quad \text{with} \quad g(t) = \sqrt{2}\, f(2t),$$

and its adjoint D^* by

$$g = D^* f \quad \text{with} \quad g(t) = \tfrac{1}{\sqrt{2}}\, f(\tfrac{t}{2}).$$

The translation-by-1 operator T on $\mathcal{L}^2(\mathbb{R})$ is defined by

$$g = Tf \quad \text{with} \quad g(t) = f(t-1),$$

and its adjoint T^* by

$$g = T^* f \quad \text{with} \quad g(t) = f(t+1).$$

Both definitions hold for almost all t in $\mathbb{R}$ with respect to Lebesgue measure. It is well known that $D \colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ and $T \colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ are bilateral shifts (thus unitary operators).

We now construct in Theorem 2 a dual-shift decomposition for the function space $\mathcal{L}^2[0,1]$, viewed as a subspace of the function space $\mathcal{L}^2(\mathbb{R})$. Such a construction involves two unilateral shifts on $\mathcal{L}^2[0,1]$ obtained from the bilateral shifts D and T on $\mathcal{L}^2(\mathbb{R})$. To see how D behaves on $\mathcal{L}^2[0,1]$, observe that

$$\int_0^1 |f(t)|^2 dt = \int_0^{\frac{1}{2}} |\sqrt{2}f(2t)|^2 dt \quad \text{for every} \quad f \in \mathcal{L}^2[0,1].$$

Indeed, the restriction of an isometry is again an isometry, and this shows that, when $D\colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ is restricted to $\mathcal{L}^2[0,1]$, it acts as an isometry into $\mathcal{L}^2[0,\frac{1}{2}]$. The subspace $\mathcal{L}^2[0,1]$ of $\mathcal{L}^2(\mathbb{R})$ is D-invariant (since $D(f\chi_{[0,1]}) \subseteq \mathcal{L}^2[0,\frac{1}{2}] \subseteq \mathcal{L}^2[0,1]$) for every $f \in \mathcal{L}^2(\mathbb{R})$, and since every $f \in \mathcal{L}^2[\alpha,\beta]$ is identified with $f\chi_{[\alpha,\beta]}$ for some $f \in \mathcal{L}^2(\mathbb{R})$). So we may identify $\mathcal{L}^2[0,\frac{1}{2}]$ with $\{f \in \mathcal{L}^2[0,1]\colon f(t) = 0 \text{ a.e. on } [\frac{1}{2},1]\}$. Thus the part of D on $\mathcal{L}^2[0,1]$ (i.e., the restriction $D|_{\mathcal{L}^2[0,1]}$ of D to the invariant subspace $\mathcal{L}^2[0,1]$) is an isometry whose range is $\mathcal{L}^2[0,\frac{1}{2}]$. Set

$$S = D|_{\mathcal{L}^2[0,1]}\colon \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]; \;\; \text{an isometry with } \; \operatorname{ran}(S) = \mathcal{L}^2[0,\tfrac{1}{2}].$$

It is worth recalling that $\mathcal{L}^2(\alpha,\beta) = \mathcal{L}^2(\alpha,\beta] = \mathcal{L}^2[\alpha,\beta) = \mathcal{L}^2[\alpha,\beta]$, since equality in these spaces is interpreted in terms of classes of equivalence (i.e., a.e. with respect to Lebesgue measure).

Proposition 1. *The operator $S\colon \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ is a unilateral shift on $\mathcal{L}^2[0,1]$, with wandering subspace $\mathcal{L}^2[\frac{1}{2},1]$. Thus $\mathcal{L}^2[0,1]$ admits the decomposition*

$$\mathcal{L}^2[0,1] = \bigoplus_{k=0}^{\infty} S^k \mathcal{L}^2[\tfrac{1}{2},1].$$

Proof. It is readily verified that this part S of the bilateral shift D is a unilateral shift [15, Lemma 2.14]. Moreover, $\ker(S^*) = \operatorname{ran}(S)^\perp = \mathcal{L}^2[0,\frac{1}{2}]^\perp = \mathcal{L}^2[\frac{1}{2},1]$. □

Corollary 1. *Set $S' = S|_{\mathcal{L}^2[0,\frac{1}{2}]} = D|_{\mathcal{L}^2[0,\frac{1}{2}]}$. This is a unilateral shift on $\mathcal{L}^2[0,\frac{1}{2}]$ with wandering subspace $\mathcal{L}^2[\frac{1}{4},\frac{1}{2}]$. Moreover, $\ker(S^*) = \mathcal{L}^2[\frac{1}{2},1]$, and*

$$\mathcal{L}^2[0,\tfrac{1}{2}] = \bigoplus_{k=1}^{\infty} S^k \mathcal{L}^2[\tfrac{1}{2},1] = \bigoplus_{k=0}^{\infty} S^k \mathcal{L}^2[\tfrac{1}{4},\tfrac{1}{2}].$$

Proof. That $S' = D|_{\mathcal{L}^2[0,\frac{1}{2}]}$ is a unilateral shift on $\mathcal{L}^2[0,\frac{1}{2}]$ follows by the same argument in the proof of Proposition 1. Now observe from Proposition 1 that

$$\mathcal{L}^2[0,\tfrac{1}{2}] = S(\mathcal{L}^2[0,1]) = \bigoplus_{k=0}^{\infty} S^{k+1} \mathcal{L}^2[\tfrac{1}{2},1] = \bigoplus_{k=1}^{\infty} S^k \mathcal{L}^2[\tfrac{1}{2},1].$$

Since $S(\mathcal{L}^2[\frac{1}{2},1]) = \mathcal{L}^2[\frac{1}{4},\frac{1}{2}]$, it follows that

$$\bigoplus_{k=0}^{\infty} S^{k+1} \mathcal{L}^2[\tfrac{1}{2},1] = \bigoplus_{k=0}^{\infty} S^k S(\mathcal{L}^2[\tfrac{1}{2},1]) = \bigoplus_{k=0}^{\infty} S^k \mathcal{L}^2[\tfrac{1}{4},\tfrac{1}{2}],$$

which completes the proof. □

Next we construct a second unilateral shift which, together with S, will yield a dual-shift decomposition for $\mathcal{L}^2[0,1]$. Recall that

$$DT^2 = TD$$

or, equivalently, since T is unitary,

$$DT = TDT^*,$$

which means that DT is unitarily equivalent to the bilateral shift D of infinite multiplicity through the unitary operator T (cf. [8, 17]), and therefore DT is also a bilateral shift on $\mathcal{L}^2(\mathbb{R})$ of infinite multiplicity (see e.g., [15, Proposition 2.10]). Again, to see how DT behaves on $\mathcal{L}^2[0,1]$, observe that

$$\int_0^1 |f(t)|^2 dt = \int_{\frac{1}{2}}^1 |\sqrt{2} f(2t-1)|^2 dt.$$

This shows that DT (which is an isometry on $\mathcal{L}^2(\mathbb{R})$, since composition of isometries is again an isometry) acts as an isometry of $\mathcal{L}^2[0,1]$ into $\mathcal{L}^2[\frac{1}{2},1]$ (i.e., when restricted to $\mathcal{L}^2[0,1]$, $DT\colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ acts as an isometry into $\mathcal{L}^2[\frac{1}{2},1]$). Using the same argument put forward in the case of the bilateral shift D, the subspace $\mathcal{L}^2[0,1]$ of $\mathcal{L}^2(\mathbb{R})$ is (DT)-invariant as well. Also, as in the case of the unilateral shift $S = D|_{\mathcal{L}^2[0,1]}$, we identify $\mathcal{L}^2[\frac{1}{2},1]$ with $\{f \in \mathcal{L}^2[0,1]\colon f(t) = 0 \text{ a.e. on } [0,\frac{1}{2}]\}$. Thus the part of DT on $\mathcal{L}^2[0,1]$ (i.e., the restriction $(DT)|_{\mathcal{L}^2[0,1]}$ of DT to the invariant subspace $\mathcal{L}^2[0,1]$) is an isometry whose range is $\mathcal{L}^2[\frac{1}{2},1]$. Set

$$V = (DT)|_{\mathcal{L}^2[0,1]}\colon \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]; \;\text{ an isometry with } \operatorname{ran}(V) = \mathcal{L}^2[\tfrac{1}{2},1].$$

Proposition 2. *The operator $V\colon \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ is a unilateral shift on $\mathcal{L}^2[0,1]$, with wandering subspace $\mathcal{L}^2[0,\frac{1}{2}]$. Thus $\mathcal{L}^2[0,1]$ admits the decomposition*

$$\mathcal{L}^2[0,1] = \bigoplus_{k=0}^{\infty} V^k \mathcal{L}^2[0,\tfrac{1}{2}].$$

Proof. The same argument in the proof of Proposition 1 applies to this case. □

Corollary 2. *Set $V' = V|_{\mathcal{L}^2[\frac{1}{2},1]} = (DT)|_{\mathcal{L}^2[\frac{1}{2},1]}$. This is a unilateral shift on $\mathcal{L}^2[\frac{1}{2},1]$ with wandering subspace $\mathcal{L}^2[\frac{1}{2},\frac{3}{4}]$. Moreover,* $\ker(V^*) = \mathcal{L}^2[0,\frac{1}{2}]$*, and*

$$\mathcal{L}^2[\tfrac{1}{2},1] = \bigoplus_{k=1}^{\infty} V^k \mathcal{L}^2[0,\tfrac{1}{2}] = \bigoplus_{k=0}^{\infty} V^k \mathcal{L}^2[\tfrac{1}{2},\tfrac{3}{4}].$$

Proof. This follows by Proposition 2 as Corollary 1 follows by Proposition 1. □

Theorem 2. *The space $\mathcal{L}^2[0,1]$ admits the following dual-shift decomposition with respect to the unilateral shifts S and V.*

$$\begin{aligned}\mathcal{L}^2[0,1] = \mathcal{L}^2[0,\tfrac{1}{2}] \oplus \mathcal{L}^2[\tfrac{1}{2},1] &= \bigoplus_{k=1}^{\infty} S^k \mathcal{L}^2[\tfrac{1}{2},1] \oplus \bigoplus_{k=1}^{\infty} V^k \mathcal{L}^2[0,\tfrac{1}{2}] \\ &= \bigoplus_{k=0}^{\infty} S^k \mathcal{L}^2[\tfrac{1}{4},\tfrac{1}{2}] \oplus \bigoplus_{k=0}^{\infty} V^k \mathcal{L}^2[\tfrac{1}{2},\tfrac{3}{4}].\end{aligned}$$

Proof. Since $\operatorname{ran}(V) = \mathcal{L}^2[\frac{1}{2},1]$ with $\ker(V^*) = \operatorname{ran}(V)^\perp = \mathcal{L}^2[0,\frac{1}{2}]$ and $\operatorname{ran}(S) = \mathcal{L}^2[0,\frac{1}{2}]$ with $\ker(S^*) = \operatorname{ran}(S)^\perp = \mathcal{L}^2[\frac{1}{2},1]$, it follows that $\operatorname{ran}(V) = \ker(S^*)$, and so $SS^* + VV^* = I$ by Lemma 1. Thus Theorem 2 is a consequence of Theorem 1 together with Corollaries 1 and 2. □

Corollary 3. *The space* $\mathcal{L}^2[0,1]$ *admits the following orthogonal decomposition*

$$\mathcal{L}^2[0,1] = \bigoplus_{k=0}^{\infty} \mathcal{L}^2[\tfrac{1}{2^{k+2}}, \tfrac{1}{2^{k+1}}] \oplus \bigoplus_{k=0}^{\infty} \mathcal{L}^2[1-\tfrac{1}{2^{k+1}}, 1-\tfrac{1}{2^{k+2}}].$$

Proof. $S^k\mathcal{L}^2[\frac{1}{4},\frac{1}{2}] = \mathcal{L}^2[\frac{1}{2^{k+2}}, \frac{1}{2^{k+1}}]$ and $V^k\mathcal{L}^2[\frac{1}{2},\frac{3}{4}] = \mathcal{L}^2[1-\frac{1}{2^{k+1}}, 1-\frac{1}{2^{k+2}}]$. □

4. Application to Wavelets

The dual-shift decomposition will be applied in Theorem 3 to yield a refined decomposition of $\mathcal{L}^2[0,1]$ in terms of a Haar system. To begin with, we consider the following classical setup for an arbitrary Hilbert space $\mathcal{H}$, which will be required in the sequel. Let $\mathbb{Z}$ denote the set of all integers. Recall that a bilateral shift U on $\mathcal{H}$ is a unitary operator for which there exists a *generating wandering subspace* $\mathcal{W}$ (which is not unique) such that, for every pair of *distinct* integers $m, n \in \mathbb{Z}$,

$$U^m\mathcal{W} \perp U^n\mathcal{W},$$

and, since it is generating, $\mathcal{H}$ admits the orthogonal decomposition [25, 15, 17]

$$\mathcal{H} = \bigoplus_{m=-\infty}^{\infty} U^m\mathcal{W}.$$

An alternate definition of bilateral shifts is given below (see e.g., [3, 17]). This is actually the Lax–Phillips [18] definition of outgoing and incoming subspaces for a unitary operator U.

Definition 1. A *bilateral shift* $U\colon \mathcal{H} \to \mathcal{H}$ is a unitary operator for which there is subspace $\mathcal{V}$ of $\mathcal{H}$ (called *outgoing* subspace) satisfying the following conditions.

(i) $U\mathcal{V} \subset \mathcal{V}$,

(ii) $\bigcap_{m\in\mathbb{Z}} U^m\mathcal{V} = \{0\}$,

(iii) $\big(\bigcup_{m\in\mathbb{Z}} U^m\mathcal{V}\big)^- = \mathcal{H}$.

Equivalently, a bilateral shift can also be defined in terms of its adjoint $U^* = U^{-1}$ as follows. A *bilateral shift* $U\colon \mathcal{H} \to \mathcal{H}$ is a unitary operator for which there is subspace $\mathcal{V}'$ of $\mathcal{H}$ (called *incoming* subspace) satisfying the following conditions.

(i′) $\mathcal{V}' \subset U^*\mathcal{V}'$,

(ii′) $\bigcap_{m\in\mathbb{Z}} U^m\mathcal{V}' = \{0\}$,

(iii′) $\big(\bigcup_{m\in\mathbb{Z}} U^m\mathcal{V}'\big)^- = \mathcal{H}$.

Proposition 3. *If* $\mathcal{V}$ *is an outgoing subspace of a unitary operator* U*, then* $\mathcal{V} = \bigoplus_{m=0}^{\infty} U^m\mathcal{W}$*, and* $\mathcal{H} = \bigoplus_{m=-\infty}^{\infty} U^m\mathcal{W}$*, where* $\mathcal{W} = \mathcal{V} \ominus U\mathcal{V}$ *is a generating wandering subspace for* U*. Hence,* U *is a bilateral shift operator on* $\mathcal{H}$*. Moreover,* $\mathcal{V}$ *is a nonreducing invariant subspace of* U*. Conversely, if a subspace* $\mathcal{M}$ *of* $\mathcal{H}$ *is invariant but nonreducing for a bilateral shift* U*, then there is a wandering subspace* $\mathcal{W}$ *for* U *so that* $\mathcal{M} = \bigoplus_{m=0}^{\infty} U^m\mathcal{W}$*.*

Proof. [13, 18]. □

Corollary 4. *Let* $\mathcal{V}$ $(\mathcal{V}')$ *be a subspace of* $\mathcal{H}$ *and set* $\mathcal{V}_m = U^m\mathcal{V}$ $(\mathcal{V}'_m = U^{*m}\mathcal{V}')$ *for every* $m \in \mathbb{Z}$*, where* U *is a unitary operator on* $\mathcal{H}$*. Then,*

(i) $\mathcal{V}_{m+1} \subset \mathcal{V}_m \quad (\mathcal{V}'_m \subset \mathcal{V}'_{m+1})$ *for every* $m \in \mathbb{Z}$,

(ii) $\bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \{0\} \qquad \big(\bigcap_{m\in\mathbb{Z}} \mathcal{V}'_m = \{0\}\big)$,

(iii) $\big(\bigcup_{m\in\mathbb{Z}} U^m\mathcal{V}_m\big)^- = \mathcal{H} \qquad \big(\big(\bigcup_{m\in\mathbb{Z}} U^m\mathcal{V}'_m\big)^- = \mathcal{H}\big)$,

if and only if $\mathcal{V}$ *is an outgoing* ($\mathcal{V}'$ *is an incoming*) *subspace for* U*.*

Proof. Straightforward by Definition 1 and Proposition 3. □

Now, for the concrete case of $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$, consider the MRA definition [24].

Definition 2. A function $\phi \in \mathcal{L}^2(\mathbb{R})$ and a family $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ of subspaces of $\mathcal{L}^2(\mathbb{R})$ are called a *scaling function* and a *multiresolution analysis* — MRA (respectively, and with respect to each other) if the following conditions hold true.

(i) $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the subspace $\mathcal{V}_0(\phi)$,

(ii) $\mathcal{V}_m(\phi) \subset \mathcal{V}_{m+1}(\phi)$ (or, $\mathcal{V}_{m+1}(\phi) \subset \mathcal{V}_m(\phi)$) for every $m \in \mathbb{Z}$,

(iii) $\bigcap_{m\in\mathbb{Z}} \mathcal{V}_m(\phi) = \{0\}$,

(iv) $\big(\bigcup_{m\in\mathbb{Z}} \mathcal{V}_m(\phi)\big)^- = \mathcal{L}^2(\mathbb{R})$,

(v) $v \in \mathcal{V}_m(\phi) \Leftrightarrow v(2\,\cdot) \in \mathcal{V}_{m+1}(\phi)$ (or $v \in \mathcal{V}_m(\phi) \Leftrightarrow v(\tfrac{1}{2}\,\cdot) \in \mathcal{V}_{m+1}(\phi)$) for every $m \in \mathbb{Z}$.

Definition 2(v) is related to the bilateral shift D (i.e., to the dilation-by-2 operator) by the expression $\mathcal{V}_{m+1}(\phi) = D^*\mathcal{V}_m(\phi)$ (or $\mathcal{V}_m(\phi) = D\,\mathcal{V}_{m+1}(\phi)$) for every $m \in \mathbb{Z}$ [19], while Definition 2(i) is native only to MRA, and has nothing to do with the fact that D is a bilateral shift (for a detailed discussion along this line, and also for generalizations, see e.g., [20, 21, 22, 23]).

Corollary 5. *A* MRA *is a sequence of decreasingly-nested subspaces* $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ *of* $\mathcal{L}^2(\mathbb{R})$ (i.e., $\mathcal{V}_m(\phi) \subset \mathcal{V}_{m+1}(\phi)$) *generated from an incoming subspace* $\mathcal{V}_0(\phi)$ *for the bilateral shift* D *by* $\mathcal{V}_m(\phi) = D^{*m}\mathcal{V}_0(\phi)$ *for every* $m \in \mathbb{Z}$*, where* $\mathcal{V}_0(\phi)$ *is, in turn, generated by a scaling function* $\phi \in \mathcal{L}^2(\mathbb{R})$ (i.e., $\mathcal{V}_0(\phi) = \overline{\text{span}}\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$)*.*

Proof. Straightforward from Corollary 4 and Definition 2. □

Consider again the dilation-by-2 and the translation-by-1 operators D and T on $\mathcal{L}^2(\mathbb{R})$, and recall the following of definition of an orthonormal wavelet in $\mathcal{L}(\mathbb{R})$.

Definition 3. An element ψ of $\mathcal{L}^2(\mathbb{R})$ is an *orthonormal wavelet* if

$$\|\psi\| = 1 \quad \text{and} \quad \psi(\cdot - m) \perp \psi(\cdot - n) \ \text{ for every } \ m, n \in \mathbb{Z} \ \text{ such that } \ m \neq n$$

(i.e., $\psi(\cdot - m)$ is orthogonal to $\psi(\cdot - n)$ for every $m \neq n$ in $\mathbb{Z}$), and the subspace

$$\mathcal{W}(\psi) = \overline{\text{span}}\{\psi(\cdot - m)\}_{m\in\mathbb{Z}}$$

is a generating wandering subspace of the unitary operator D.

It follows at once from this definition that, corresponding to an orthonormal wavelet ψ, the space $\mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m=-\infty}^{\infty} D^m \mathcal{W}(\psi).$$

For every $m, n \in \mathbb{Z}$, set

$$\psi_{m,n} = D^m T^n \psi.$$

The set $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is a double indexed orthonormal basis for $\mathcal{L}^2(\mathbb{R})$, referred to as a *wavelet orthonormal basis* for $\mathcal{L}^2(\mathbb{R})$, and each $\psi_{m,n}$ is called a *wavelet* (*orthonormal*) *function* generated from the wavelet ψ. Let $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ be a MRA with scaling function ϕ such that (cf. Definition 2)

$$\mathcal{V}_m(\phi) \subset \mathcal{V}_{m+1}(\phi).$$

Let $\mathcal{W}_m = \mathcal{V}_{m+1}(\phi) \ominus \mathcal{V}_m(\phi)$ be the orthogonal complement in $\mathcal{V}_{m+1}(\phi)$ of $\mathcal{V}_m(\phi)$, so that, for every $m \in \mathbb{Z}$,

$$\mathcal{V}_{m+1}(\phi) = \mathcal{V}_m(\phi) \oplus \mathcal{W}_m.$$

It can be verified [24] that there exists a wavelet ψ such that, for every $m \in \mathbb{Z}$,

$$\mathcal{W}_m(\psi) = \mathcal{W}_m = D^m \mathcal{W}_0 = \mathcal{W}_0(\psi),$$

with

$$\mathcal{W}_0(\psi) = \mathcal{W}_0 = \mathcal{W}(\psi) = \overline{\operatorname{span}}\{\psi(\cdot - m)\}_{m\in\mathbb{Z}},$$

so that $\mathcal{W}_0(\psi)$ is a generating wandering subspace of the bilateral shift D. Thus

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m=-\infty}^{\infty} D^m \mathcal{W}_0(\psi) = \mathcal{V}_0(\phi) \oplus \bigoplus_{m=0}^{\infty} \mathcal{W}_m(\psi).$$

Lemma 2. *If an orthonormal wavelet ψ lies in $\mathcal{L}^2[0,1]$, then the wavelet functions $\psi_{m,n-1} = D^m T^{n-1}\psi$ lie in $\mathcal{L}^2[0,1]$ if and only if $m \geq 0$ and $1 \leq n \leq 2^m$.*

Proof. Consider the above setup. If an orthonormal wavelet ψ in $\mathcal{L}^2(\mathbb{R})$ lies in $\mathcal{L}^2[0,1]$ (considered as a subspace of $\mathcal{L}^2(\mathbb{R})$), then it is plain that

$$\psi \in \mathcal{L}^2[0,1] \quad \Longrightarrow \quad T^{n-1}\psi \in \mathcal{L}^2[n-1,n] \quad \text{for every} \quad n \geq 1.$$

This, in turn, implies that

$$\psi_{m,n-1} = D^m T^{n-1}\psi \in \mathcal{L}^2[\tfrac{n-1}{2^m}, \tfrac{n}{2^m}] \quad \text{for every} \quad m \geq 0,\ n \geq 1.$$

Therefore, $\psi_{m,n-1}$ lies in $\mathcal{L}^2[0,1]$ if and only if $m \geq 0$ and $1 \leq n \leq 2^m$. □

The orthonormal set $\{\psi_{m,n-1}\}_{m\geq 0\,,\,1\leq n\leq 2^m}$ of Lemma 2 is not a basis for $\mathcal{L}^2[0,1]$ — it does not span $\mathcal{L}^2[0,1]$. This is shown in the next lemma. Let $\phi \in \mathcal{L}^2[0,1]$ be a scaling function which results in an orthonormal wavelet $\psi \in \mathcal{L}^2[0,1]$. Let $\mathcal{Z}_m(\psi)$ be the finite-dimensional subspace of $\mathcal{L}^2[0,1]$ defined, for each $m \geq 0$, by

$$\mathcal{Z}_m(\psi) = \operatorname{span}\{\psi_{m,n-1}\}_{1\leq n\leq 2^m} = \operatorname{span}\{D^m T^{n-1}\psi\}_{1\leq n\leq 2^m}.$$

Lemma 3. *The set $\{\phi\,,\, D^m T^{n-1}\psi\}_{m\geq 0\,,\,1\leq n\leq 2^m}$ is an orthonormal basis for the Hilbert space $\mathcal{L}^2[0,1]$, so that*

$$\mathcal{L}^2[0,1] = \operatorname{span}\{\phi\} \oplus \bigoplus_{m=0}^{\infty} \mathcal{Z}_m(\psi),$$

Proof. Let $\{\mathcal{V}_m(\phi)\}_{m\in\mathbb{Z}}$ be the MRA with scaling function ϕ which results in the wavelet ψ. Recall that

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{V}_0(\phi) \oplus \bigoplus_{m=0}^{\infty} \mathcal{W}_m(\psi).$$

Let $P\colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ be the orthogonal projection onto the subspace $\mathcal{L}^2[0,1]$ (i.e., $\operatorname{ran}(P) = \mathcal{L}^2[0,1]$), so that

$$\mathcal{L}^2[0,1] = P(\mathcal{L}^2(\mathbb{R})) = P(\mathcal{V}_0(\phi)) \oplus \bigoplus_{m=0}^{\infty} P(\mathcal{W}_m(\psi)).$$

Recall that $\mathcal{V}_0(\phi) = \overline{\operatorname{span}}\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$, $\phi(\cdot - n) \in \mathcal{L}^2[0,1]$ if and only if $n = 0$, and $\mathcal{Z}_m \in \mathcal{L}^2[0,1]$ for every $m \geq 0$. Therefore, $P(\mathcal{V}_0(\phi)) = P(\overline{\operatorname{span}}\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}) = \operatorname{span}\{\phi\}$, and $P(\mathcal{W}_m(\psi)) = P(\overline{\operatorname{span}}\{\psi(\cdot - m)\}_{m\in\mathbb{Z}}) = \operatorname{span}\{\psi_{m,n-1}\}_{1\leq n\leq 2^m} = Z_m(\psi)$, according to Lemma 2. □

Lemma 3 gives a simple proof of the interesting fact that the orthogonal complement in $\mathcal{L}^2[0,1]$ of the set of orthonormal wavelet functions $\psi_{m\,n-1}$, generated from an orthonormal wavelet ψ in $\mathcal{L}^2[0,1]$ is the subspace spanned by the associated scaling function ϕ — also in $\mathcal{L}^2[0,1]$. An example of this is the Haar system.

The most well-known, and the very first, orthonormal wavelet in $\mathcal{L}^2(\mathbb{R})$ is the Haar wavelet ψ_H [24], defined by

$$\psi_H(t) = \begin{cases} 1, & 0 < t \leq \frac{1}{2}, \\ -1, & \frac{1}{2} < t \leq 1, \\ 0, & t \in \mathbb{R}\backslash(0,1]. \end{cases}$$

Let $\chi_{[0,1]} \in \mathcal{L}^2(\mathbb{R})$ denote the characteristic function of the closed interval $[0,1]$. For every f in $\mathcal{L}^2(\mathbb{R})$ we shall identify the product $f\chi_{[0,1]} \in \mathcal{L}^2(\mathbb{R})$ with the restriction $f|_{[0,1]} \in \mathcal{L}^2[0,1]$ of f to $[0,1]$, and write $f\chi_{[0,1]} \in \mathcal{L}^2[0,1]$. For the particular case of the Haar wavelet ψ_H, we shall write ψ_H for $\psi_H|_{[0,1]} = \psi_H\chi_{[0,1]}$ (a suggestive abuse of notation). Recall that the Haar system [11, 12] is the set $\{h_1\,,\, h_{2^m+n}\}_{\,m\geq 0\,,\,1\leq n\leq 2^m}$ of functions in $\mathcal{L}^2[0,1]$ defined a.e. in $[0,1]$ (i.e., for almost all $t \in [0,1]$ with respect to Lebesgue measure) by

$$h_1 = \chi_{[0,1]},$$

and, for every $m \geq 0$ and $1 \leq n \leq 2^m$,

$$h_{2^m+n}(t) = \sqrt{2}^m \big(h_1(2^{m+1}t - 2n + 2) - h_1(2^{m+1}t - 2n + 1)\big).$$

The Haar system $\{h_1\,,\, h_{2^m+n}\}_{\,m\geq 0\,,\,1\leq n\leq 2^m}$ is a (Schauder) basis for the Banach spaces $\mathcal{L}^p[0,1]$ for every $p \geq 1$. In particular, it is an orthonormal basis for the Hilbert space $\mathcal{L}^2[0,1]$. (It was shown in [1] that the Haar system is also eigenbasis of the Time Operator of Statistical Physics).

Lemma 4. *The Haar system $\{h_1,\, h_{2^m+n}\}_{m\geq 0\,,\,1\leq n\leq 2^m}$ in $\mathcal{L}^2[0,1]$ is the orthonormal basis for $\mathcal{L}^2[0,1]$ consisting of the constant function $\chi_{[0,1]}$ and the Haar wavelet orthonormal functions $\psi^H_{m,n-1} = D^mT^{n-1}\psi_H$ for $m \geq 0$ and $1 \leq n \leq 2^m$, so that $\psi^H_{0,1}$ is the Haar wavelet ψ_H.*

Proof. The functions h_{2^m+n} can be rewritten in terms of the operators D and T as follows. First recall that $DT^2 = TD$, which means that $DT^{2m} = T^mD$ for every $m \in \mathbb{Z}$ [17] or, equivalently, $DT^{2n-2} = T^{n-1}D$ for every $n \in \mathbb{Z}$. Therefore,

$$\begin{aligned} h_{2^m+n} &= \tfrac{1}{\sqrt{2}}\big((D^{m+1}T^{2n-2}\,h_1) - (D^{m+1}T^{2n-1}\,h_1)\big) \\ &= \tfrac{1}{\sqrt{2}}\big(D^mT^{n-1}(D - DT)\big)\chi_{[0,1]} \end{aligned}$$

for every $m \ge 0$ and $1 \le n \le 2^m$. Thus, for $m = 0$, and so $n = 1$, it follows that h_2 is precisely the Haar wavelet on $[0,1]$:

$$h_2(t) = \tfrac{1}{\sqrt{2}}(D - DT)\chi_{[0,1]}(t) = \psi_H(t) = \begin{cases} 1, & 0 < t \le \frac{1}{2}, \\ -1, & \frac{1}{2} < t \le 1. \end{cases}$$

Therefore, we can infer that h_{2^m+n} can be written in terms of ψ_H as

$$h_{2^m+n} = D^mT^{n-1}\psi_H \quad \text{for every} \quad m \ge 0 \text{ and } 1 \le n \le 2^m.$$

Thus, by Lemma 2, $\{h_1,\, h_{2^m+n}\}_{m\ge0\,,\,1\le n\le 2^m}$ consists of $\chi_{[0,1]}$ (which is the constant function 1 on $[0,1]$) and $D^mT^{n-1}\psi_H$ for $m \ge 0$ and $1 \le n \le 2^m$. Now, the Haar scaling function ϕ_H associated with the Haar wavelet ψ_H is the characteristic function $\chi_{[0,1]}$ [24]. Hence, by Lemma 3, the Haar system

$$\big\{\phi_H = \chi_{[0,1]}\;,\; \psi^H_{m,n-1} = D^mT^{n-1}\psi_H\big\}_{m\ge0\,,\,1\le n\le 2^m}$$

is an orthonormal basis for $\mathcal{L}^2[0,1]$. □

Corollary 6. *Set* $\psi^H_{m,n-1} = D^mT^{n-1}\psi_H$ *for every* $m \ge 0$ *and* $1 \le n \le 2^m$.

(a) *The Haar system for* $\mathcal{L}^2[0,\frac{1}{2}]$ *is the set of orthonormal functions*

$$\big\{\chi_{[0,\frac{1}{2}]}\;,\; \psi^H_{m,n-1}\chi_{[0,\frac{1}{2}]}\big\}_{m\ge1\,,\,1\le n\le 2^{m-1}}.$$

(b) *The Haar system for* $\mathcal{L}^2[\frac{1}{2},1]$ *is the set of orthonormal functions*

$$\big\{\chi_{[\frac{1}{2},1]}\;,\; \psi^H_{m,n-1}\chi_{[\frac{1}{2},1]}\big\}_{m\ge1\,,\,2^{m-1}+1\le n\le 2^m}.$$

Proof. A consequence of Lemmas 3 and 4, since $\mathcal{L}^2[0,1] = \mathcal{L}^2[0,\frac{1}{2}] \oplus \mathcal{L}^2[\frac{1}{2},1]$. □

Returning to the dual-shift decomposition of $\mathcal{L}^2[0,1]$ in Theorem 2.

Theorem 3. *Every* $f \in \mathcal{L}^2[0,1]$ *admits the orthogonal expansion*

$$\begin{aligned} f \;=\; & \sum_{k=1}^{\infty}\alpha_{k,0,0}\,S^k\chi_{[\frac{1}{2},1]} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{2^{m-1}}\alpha_{k,m,n}\,S^k\psi^H_{m,n-1}\chi_{[\frac{1}{2},1]} \\ & + \sum_{k=1}^{\infty}\beta_{k,0,0}\,V^m\chi_{[0,\frac{1}{2}]} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=2^{m-1}+1}^{2^m}\beta_{k,m,n}\,V^k\psi^H_{m,n-1}\chi_{[0,\frac{1}{2}]}. \end{aligned}$$

Proof. By Theorem 2, each $f \in \mathcal{L}^2[0,1]$ admits the orthogonal expansion

$$f = \sum_{k=1}^{\infty}S^kg_k + \sum_{k=1}^{\infty}V^kh_k,$$

where $g_k \in \mathcal{L}^2[\frac{1}{2}, 1]$ and $h_k \in \mathcal{L}^2[0, \frac{1}{2}]$ for every $k \geq 1$, with $\sum_{k=1}^{\infty} \|g_k\|^2 < \infty$ and $\sum_{k=1}^{\infty} \|h_k\|^2 < \infty$. The Fourier series expansion of the functions g_k and h_k in terms of the orthonormal bases for $\mathcal{L}^2[\frac{1}{2}, 1]$ and for $\mathcal{L}^2[0, \frac{1}{2}]$ comprising the Haar systems given in Corollary 6 (which come from Lemmas 2, 3, and 4) leads to

$$g_k = \alpha_{k,0,0}\, \chi_{[\frac{1}{2},1]} + \sum_{m=1}^{\infty} \sum_{n=1}^{2^{m-1}} \alpha_{k,m,n}\, \psi^H_{m,n-1} \chi_{[\frac{1}{2},1]},$$

$$h_k = \beta_{k,0,0}\, \chi_{[0,\frac{1}{2}]} + \sum_{m=1}^{\infty} \sum_{n=2^{m-1}+1}^{2^m} \beta_{k,m,n}\, \psi^H_{m,n-1} \chi_{[0,\frac{1}{2}]}.$$

Thus we get the claimed orthogonal expansion for f, since each S^k and each V^k is a *continuous linear* transformation. □

Remark 2. Observe that Lemma 1 and Theorem 1 deal with an abstract Hilbert space. However our construction starting with Proposition 1 throughout the paper deals with the concrete functions space $\mathcal{L}^2[0, 1]$. It is plain that the closed domain $[0, 1]$ can be replaced with any compact subset of $\mathbb{R}$; in particular, with any compact set with no isolated points (i.e., with any perfect bounded set — for instance, any finite intersection of closed nondegenerate bounded intervals). It does not matter whether the argument of the functions are interpreted as "time", or "frequency", or "space". Along these lines, consider the Shannon wavelet $\psi_S \in \mathcal{L}^2(\mathbb{R})$ defined as the characteristic function of the set $[-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}] \subset \mathbb{R}$ (see, e.g., [14]),

$$\psi_S(\omega) = \chi_{[-\frac{1}{2},-\frac{1}{4}]\cup[\frac{1}{4},\frac{1}{2}]}(\omega)$$

for every $\omega \in \mathbb{R}$. As before, we may identify the $\psi_S \in \mathcal{L}^2(\mathbb{R})$ with the restriction of it to $[-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$, so that $\psi_S \colon [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}] \to \mathbb{R}$ in $\mathcal{L}^2[-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$. This will lead to a similar example, where the previous Haar wavelet ψ_H is replaced with the Shannon wavelet ψ_S, yielding similar results (on a different domain) regarding our concrete construction of Section 4; in particular, for the especial examples considered from Lemma 4 to Theorem 3.

5. Conclusion

We have introduced the concept of a dual-shift decomposition of a Hilbert space. In particular, we have constructed such a decomposition for the function space $\mathcal{L}^2[0, 1]$ by using the two bilateral shifts of wavelet theory: the dilation-by-2 and translation-by-1 operators. Moreover, we have derived Haar-like systems on $\mathcal{L}^2[0, \frac{1}{2}]$ and $\mathcal{L}^2[\frac{1}{2}, 1]$ from the celebrated Haar system on $\mathcal{L}^2[0, 1]$ and, simultaneously, we have shown that a multiresolution analysis (MRA) of wavelet theory is actually generated from an outgoing or an incoming subspace (a concept of the Lax–Phillips scattering theory) of the bilateral shift dilation-by-2 operator. Moreover, using the MRA, we have shown that any wavelet living in $\mathcal{L}^2[0, 1]$ together with its scaling function do indeed generate a Haar-like system for the space $\mathcal{L}^2[0, 1]$.

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