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REGULAR LATTICES OF TENSOR PRODUCTS

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Abstract. It is proved that the collection of all regular invariant subspaces of tensor products of operators is a lattice. Further characterization of regular lattices are considered for injective operators. The approach is carried out keeping pace with the results on intrinsic invariant subspaces of direct sums, extending them to regular invariant subspaces of tensor products.

1. Introduction

Let $\mathcal{H}$ be a nonzero Hilbert space (either finite or infinite-dimensional, not necessarily separable). Let $\mathcal{B}[\mathcal{H}]$ denote the normed algebra of all operators on $\mathcal{H}$ (i.e., $\mathcal{B}[\mathcal{H}]$ is the normed algebra of all *bounded* linear transformation of $\mathcal{H}$ into itself), and let $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stand for kernel and range of $A \in \mathcal{B}[\mathcal{H}]$. By a subspace $\mathcal{M}$ of $\mathcal{H}$ we mean a *closed* linear manifold of $\mathcal{H}$, which is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$ (for $\dim \mathcal{H} > 1$), invariant for an operator A (A-invariant) if $A(\mathcal{M}) \subseteq \mathcal{M}$, and reducing for A (or $\mathcal{M}$ reduces A) if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are A-invariant (where $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$ stands for the orthogonal complement of $\mathcal{M}$); equivalently, if $\mathcal{M}$ is invariant for both A and A^* (where $A^* \in \mathcal{B}[\mathcal{H}]$ denotes the adjoint of $A \in \mathcal{B}[\mathcal{H}]$). An operator A is reducible if it has a nontrivial reducing subspace, otherwise it is called irreducible. If $\mathcal{M}$ is A-invariant, then let $A|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$ denote the restriction of A to $\mathcal{M}$. If $\{\mathcal{H}_i\}_{i=1}^m$ is a finite collection of Hilbert spaces, then their orthogonal direct sum, denoted by $\bigoplus_{i=1}^m \mathcal{H}_i$, is the Hilbert space made up of all m-tuples $x = \{x_i\}_{i=1}^m$, denoted by $x = \bigoplus_{i=1}^m x_i$, with each x_i in $\mathcal{H}_i$, where sum and scalar multiplication are defined componentwise, whose inner product is given by $\langle x, y\rangle = \sum_{i=1}^m \langle x_i, y_i\rangle$ for every $x = \bigoplus_{i=1}^m x_i$ and $y = \bigoplus_{i=1}^m y_i$ in $\bigoplus_{i=1}^m \mathcal{H}_i$. If $\{A_i\}_{i=1}^m$ is a finite collection of operators with A_i in $\mathcal{B}[\mathcal{H}_i]$, then their direct sum, denoted by $\bigoplus_{i=1}^m A_i$, is the operator in $\mathcal{B}[\bigoplus_{i=1}^m \mathcal{H}_i]$ given by $\bigoplus_{i=1}^m A_i x = \bigoplus_{i=1}^m A_i x_i$ for every $x = \bigoplus_{i=1}^m x_i$ in $\bigoplus_{i=1}^m \mathcal{H}_i$. Recall that $\big\| \bigoplus_{i=1}^m A_i \big\| = \max_{1 \le i \le m} \|A_i\|$, and $\big(\bigoplus_{i=1}^m A_i\big)\big(\bigoplus_{i=1}^m A_i'\big) = \big(\bigoplus_{i=1}^m A_i A_i'\big)$ if $\{A_i'\}_{i=1}^m$ is a collection of m operators with each A_i' also in $\mathcal{B}[\mathcal{H}_i]$.

We consider the concept of tensor product space in terms of the single tensor product of two vectors as a conjugate bilinear functional on the Cartesian product of a pair of nonzero Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ (see, e.g., [2], [10], [11] and [12]; for an abstract approach see, e.g., [1], [9] and [14].) The single tensor product of a pair of vectors (x, y), with x in $\mathcal{H}$ and y in $\mathcal{K}$, is a conjugate bilinear functional $x \otimes y \colon \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ defined by $(x \otimes y)\,(u, v) = \langle x\,; u\rangle\,\langle y\,; v\rangle$ for every $(u, v) \in \mathcal{H} \times \mathcal{K}$. The tensor product space $\mathcal{H} \otimes \mathcal{K}$ is the completion of the inner product space consisting of all (finite) sums of single tensors $x_k \otimes y_k$ with $x_k \in \mathcal{H}$ and $y_k \in \mathcal{K}$, which is a Hilbert space with respect to the inner product

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$$\big\langle \textstyle\sum_k x_k \otimes y_k \,;\, \sum_\ell w_\ell \otimes z_\ell \big\rangle = \sum_k \sum_\ell \langle x_k \,;\, w_\ell\rangle \, \langle y_k \,;\, z_\ell\rangle$$

for every $\sum_k x_k \otimes y_k$ and $\sum_\ell w_\ell \otimes z_\ell$ in $\mathcal{H} \otimes \mathcal{K}$. The *tensor product* of two operators A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$ is the transformation $A \otimes B \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ defined by

$$(A \otimes B) \textstyle\sum_k x_k \otimes y_k = \sum_k A x_k \otimes B y_k \quad \text{for every} \quad \sum_k x_k \otimes y_k \in \mathcal{H} \otimes \mathcal{K},$$

which is an operator in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. For an expository paper on tensor product along these lines the reader is referred to [5]. The tensor product of a pair of Hilbert spaces and of a pair of operators can be naturally extended to a finite collection of Hilbert spaces and to a finite collection of operators, as follows. For any integer $m \geq 2$, let $\{\mathcal{H}_i\}_{i=1}^m$ be a finite collection of Hilbert spaces. The single tensor product of an m-tuple of vectors $(x_1, \cdots, x_m)$, with each x_i in $\mathcal{H}_i$, is the conjugate multilinear functional $\bigotimes_{i=1}^m x_i \colon \prod_{i=1}^m \mathcal{H}_i \to \mathbb{C}$ defined by $\big(\bigotimes_{i=1}^m x_i\big)(u_1, \cdots, u_m) = \prod_{i=1}^m \langle x_i \,;\, u_i\rangle$ for every $(u_1, \cdots, u_m) \in \prod_{i=1}^m \mathcal{H}_i$. The tensor product space $\bigotimes_{i=1}^m \mathcal{H}_i$ is the completion of the inner product space of all (finite) sums of single tensor products $\bigotimes_{i=1}^m x_{i,k}$ with $x_{i,k} \in \mathcal{H}_i$, which is again a Hilbert space with respect to the inner product

$$\big\langle \textstyle\sum_k \bigotimes_{i=1}^m x_{i,k} \,;\, \sum_\ell \bigotimes_{i=1}^m w_{i,\ell} \big\rangle = \sum_k \sum_\ell \prod_{i=1}^m \langle x_{i,k} \,;\, w_{i,\ell}\rangle$$

for every $\sum_k \bigotimes_{i=1}^m x_{i,k}$ and $\sum_\ell \bigotimes_{i=1}^m w_{i,\ell}$ in $\bigotimes_{i=1}^m \mathcal{H}_i$. The tensor product of a finite collection $\{A_i\}_{i=1}^m$ of operators, with each A_i in $\mathcal{B}[\mathcal{H}_i]$, is given by

$$\big(\textstyle\bigotimes_{i=1}^m A_i\big) \sum_k \bigotimes_{i=1}^m x_{i,k} = \sum_k \bigotimes_{i=1}^m A_i x_{i,k} \quad \text{for every} \quad \sum_k \bigotimes_{i=1}^m x_{i,k} \in \bigotimes_{i=1}^m \mathcal{H}_i.$$

This defines an operator in $\mathcal{B}[\bigotimes_{i=1}^m \mathcal{H}_i]$ with the following properties: $\big(\bigotimes_{i=1}^m A_i\big)^* = \bigotimes_{i=1}^m A_i^*$, and $\big(\bigotimes_{i=1}^m A_i\big)^{-1} = \bigotimes_{i=1}^m A_i^{-1}$ if each A_i is invertible. Also $\big\|\bigotimes_{i=1}^m A_i\big\| = \prod_{i=1}^m \|A_i\|$, and $\big(\bigotimes_{i=1}^m A_i\big)\big(\bigotimes_{i=1}^m A_i'\big) = \big(\bigotimes_{i=1}^m A_i A_i'\big)$ in $\mathcal{B}[\bigotimes_{i=1}^m \mathcal{H}_i]$ if $\{A_i'\}_{i=1}^m$ is a collection of m operators with each A_i' in $\mathcal{B}[\mathcal{H}_i]$.

The (orthogonal) direct sum $\bigoplus_{i=1}^m \mathcal{M}_i$ of subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ is a subspace of the direct sum space $\bigoplus_{i=1}^m \mathcal{H}_i$. (Indeed, $\bigoplus_{i=1}^m \mathcal{M}_i \cong \big(\sum_{i=1}^m \mathcal{M}_i\big)^- = \sum_{i=1}^m \mathcal{M}_i$ — see, e.g., [4, Theorem 5.10 and Proposition 5.24]). The tensor product counterpart is also readily verified. Indeed, this comes from the fact that if $\{e_{i,\gamma_i}\}_{\gamma_i \in \Gamma_i}$ is an orthonormal basis for each $\mathcal{H}_i$, then $\big\{\bigotimes_{i=1}^m e_{i,\gamma_i}\big\}_{(\gamma_1, \cdots, \gamma_m) \in \prod_{i=1}^m \Gamma_i}$ is an orthonormal basis for $\bigotimes_{i=1}^m \mathcal{H}_i$ (see, e.g., [14, Theorem 3.12(b)]). Thus the tensor product $\bigotimes_{i=1}^m \mathcal{M}_i$ of subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ is a subspace of the tensor product space $\bigotimes_{i=1}^m \mathcal{H}_i$.

2. Regular subspaces

It is clear that the direct sum $\bigoplus_{i=1}^m \mathcal{M}_i$ of subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ is invariant (reducing) for the direct sum $\bigoplus_{i=1}^m A_i$ of operators A_i on $\mathcal{H}_i$ if and only if each $\mathcal{M}_i$ is invariant (reducing) for A_i. Here the tensor product counterpart is quite different, as it will be seen in the forthcoming Lemma 1.

We borrow the following definitions, and also the next result, from [6]. Consider a finite collection $\{\mathcal{H}_i\}_{i=1}^m$ of Hilbert spaces. A subspace of the orthogonal direct sum $\bigoplus_{i=1}^m \mathcal{H}_i$ is *intrinsic* if it is of the form $\bigoplus_{i=1}^m \mathcal{M}_i$, where each $\mathcal{M}_i$ is a subspace of $\mathcal{H}_i$. Otherwise it is said to be *extrinsic*. This notion can be brought to tensor product spaces yielding the concept of *regular* subspaces. A subspace of a tensor product space $\bigotimes_{i=1}^m \mathcal{H}_i$ is *regular* if it is of the form $\bigotimes_{i=1}^m \mathcal{M}_i$, where each $\mathcal{M}_i$ is a subspace of $\mathcal{H}_i$. Otherwise it is *irregular*. Regular invariant and reducing subspaces are characterized as follows.

Lemma 1. *Take any integer $m \geq 2$. For each integer $i \in [1, m]$ let A_i be an operator on a Hilbert space $\mathcal{H}_i$ and let $\mathcal{M}_i$ be a subspace of $\mathcal{H}_i$. Consider the tensor product $\bigotimes_{i=1}^m A_i$ of $\{A_i\}_{i=1}^m$ on the tensor product space $\bigotimes_{i=1}^m \mathcal{H}_i$.*

(a$_1$) *If each $\mathcal{M}_i$ is invariant (reducing) for A_i, then $\bigotimes_{i=1}^m \mathcal{M}_i$ is an invariant (reducing) subspace for $\bigotimes_{i=1}^m A_i$.*

(a$_2$) *If $\bigotimes_{i=1}^m \mathcal{M}_i$ is invariant for $\bigotimes_{i=1}^m A_i$, then one of the subspaces $\mathcal{M}_i$ is invariant for A_i.*

(a$_3$) *If $\bigotimes_{i=1}^m \mathcal{M}_i$ reduces $\bigotimes_{i=1}^m A_i$, then one of the subspaces $\mathcal{M}_i$ reduces A_i, or one of the subspaces $\mathcal{M}_i$ is invariant for A_i and the orthogonal complement $\mathcal{M}_j^\perp$ of another subspace $\mathcal{M}_j$ with $j \neq i$ is invariant for A_j.*

(a$_4$) *If $\bigotimes_{i=1}^m \mathcal{M}_i$ is invariant (reducing) for $\bigotimes_{i=1}^m A_i$ and if each $\mathcal{M}_i \not\subseteq \mathcal{N}(A_i)$, then each $\mathcal{M}_i$ is invariant (reducing) for A_i. Particular case:*

  (a$_4'$) *If $\bigotimes_{i=1}^m \mathcal{M}_i$ is nonzero and invariant (reducing) for $\bigotimes_{i=1}^m A_i$ and if every A_i is injective, then each $\mathcal{M}_i$ is invariant (reducing) for A_i.*

(b) *One of the subspaces $\mathcal{M}_i$ is nontrivial and the others $\{\mathcal{M}_j\}_{i \neq j=1}^m$ are nonzero if and only if $\bigotimes_{i=1}^m \mathcal{M}_i$ is nontrivial.*

(c$_1$) *If each $\mathcal{M}_i$ is A_i-invariant, then*

$$\big(\textstyle\bigotimes_{i=1}^m A_i\big)\big|_{\bigotimes_{i=1}^m \mathcal{M}_i} = \textstyle\bigotimes_{i=1}^m A_i|_{\mathcal{M}_i}.$$

(c$_2$) *If $\bigotimes_{i=1}^m \mathcal{M}_i$ is nonzero and $\bigotimes_{i=1}^m A_i$-invariant, and if each A_i is injective, then*

$$\big(\textstyle\bigotimes_{i=1}^m A_i\big)\big|_{\bigotimes_{i=1}^m \mathcal{M}_i} = \textstyle\bigotimes_{i=1}^m A_i|_{\mathcal{M}_i}.$$

Proof. [6, Theorem 1]. □

Consider the direct sum $\bigoplus_{i=1}^m A_i$ and the tensor product $\bigotimes_{i=1}^m A_i$ of operators acting in $\mathcal{B}[\bigoplus_{i=1}^m \mathcal{H}_i]$ and $\mathcal{B}[\bigotimes_{i=1}^m \mathcal{H}_i]$, respectively. The existence of intrinsic invariant subspaces of direct sums is trivial; the existence of regular invariant subspaces of tensor products is discussed in Lemma 1. The existence of extrinsic and irregular invariant subspaces is exhibited in Remark 1 and Corollary 1 from [6] (see also [7]). We restrict our attention to direct sums or tensor products of operators. Thus operators in $\mathcal{B}[\bigoplus_{i=1}^m \mathcal{H}_i]$ or in $\mathcal{B}[\bigotimes_{i=1}^m \mathcal{H}_i]$ that are not direct sums or tensor products of operators (i.e., operators in $\mathcal{B}[\bigoplus_{i=1}^m \mathcal{H}_i]$ or in $\mathcal{B}[\bigotimes_{i=1}^m \mathcal{H}_i]$ that are not of the form $\bigoplus_{i=1}^m A_i$ or $\bigotimes_{i=1}^m A_i$) will not be discussed here.

3. Regular Lattices

Let $\{\mathcal{H}_i\}_{i=1}^m$ be a finite collection of Hilbert spaces. Take the Hilbert space $\bigoplus_{i=1}^m \mathcal{H}_i$ of their orthogonal direct sum, and also the Hilbert space $\bigotimes_{i=1}^m \mathcal{H}_i$ of their tensor product. Let $\mathrm{Lat}(A)$ denote lattice of all invariant subspaces for a given operator A. Consider a collection of operators $\{A_i\}_{i=1}^m$ with each A_i in $\mathcal{B}[\mathcal{H}_i]$.

Intrinsic Lattices: Let $\bigoplus_{i=1}^m \operatorname{Lat}(A_i)$ denote the collection of all intrinsic subspaces $\bigoplus_{i=1}^m \mathcal{M}_i \subseteq \bigoplus_{i=1}^m \mathcal{H}_i$ made up of A_i-invariant subspaces $\mathcal{M}_i$,

$$\textstyle\bigoplus_{i=1}^m \operatorname{Lat}(A_i) = \big\{\bigoplus_{i=1}^m \mathcal{M}_i \subseteq \bigoplus_{i=1}^m \mathcal{H}_i \colon\ \mathcal{M}_i \in \operatorname{Lat}(A_i)\big\},$$

and let $\operatorname{ILat}\big(\bigoplus_{i=1}^m A_i\big)$ denote the collection of all intrinsic invariant subspaces for the direct sum $\bigoplus_{i=1}^m A_i$ on $\bigoplus_{i=1}^m \mathcal{H}_i$,

$$\textstyle\operatorname{ILat}\big(\bigoplus_{i=1}^m A_i\big) = \big\{\bigoplus_{i=1}^m \mathcal{M}_i \subseteq \bigoplus_{i=1}^m \mathcal{H}_i \colon\ \bigoplus_{i=1}^m \mathcal{M}_i \in \operatorname{Lat}\big(\bigoplus_{i=1}^m A_i\big)\big\},$$

which is a lattice itself. In fact, take the lattice $\operatorname{Lat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ of all subspaces of the direct sum $\bigoplus_{i=1}^m \mathcal{H}_i$. Let $\operatorname{ILat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ denote the subcollection of $\operatorname{Lat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ of all intrinsic subspaces of $\bigoplus_{i=1}^m \mathcal{H}_i$. It is readily verified that $\operatorname{ILat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ is sublattice of $\operatorname{Lat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$. Observing that

$$\textstyle\operatorname{ILat}\big(\bigoplus_{i=1}^m A_i\big) = \operatorname{ILat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big) \cap \operatorname{Lat}\big(\bigoplus_{i=1}^m A_i\big),$$

we infer that $\operatorname{ILat}\big(\bigoplus_{i=1}^m A_i\big)$ is a sublattice of $\operatorname{Lat}\big(\bigoplus_{i=1}^m A_i\big)$. Indeed, as we had noticed before, an intrinsic subspace $\mathcal{I} = \bigoplus_{i=1}^m \mathcal{M}_i$ lies in $\operatorname{ILat}\big(\bigoplus_{i=1}^m A_i\big)$ if and only if $\mathcal{I} \in \bigoplus_{i=1}^m \operatorname{Lat}(A_i)$. Thus, since there exist extrinsic subspaces in $\operatorname{Lat}\big(\bigoplus_{i=1}^m A_i\big)$ [6, Remark 1], it follows that

$$\textstyle\operatorname{ILat}\big(\bigoplus_{i=1}^m A_i\big) = \bigoplus_{i=1}^m \operatorname{Lat}(A_i) \subseteq \operatorname{Lat}\big(\bigoplus_{i=1}^m A_i\big),$$

where the above inclusion may be proper.

Regular Lattices: It is convenient to exclude zero subspaces when defining the tensor product space counterpart. In fact, since $\{0\} \in \operatorname{Lat}(A_i)$ for every $i \in [1, m]$, it follows that if $\mathcal{M}_{i_0} = \{0\}$ for some $i_0 \in [1, m]$, then $\mathcal{M}_{i_0} \in \operatorname{Lat}(A_{i_0})$ and $\bigotimes_{i=1}^m \mathcal{M}_i = \{0\} = \bigotimes_{i=1}^m \{0\}$ is a regular subspace consisting of A_i-invariant subspaces even if $\mathcal{M}_i \notin \operatorname{Lat}(A_i)$ for every $i \neq i_0$. Thus let $\bigotimes_{i=1}^m \operatorname{Lat}(A_i)$ denote the collection of all nonzero regular subspaces $\bigotimes_{i=1}^m \mathcal{M}_i \subseteq \bigotimes_{i=1}^m \mathcal{H}_i$, where each $\mathcal{M}_i$ is A_i-invariant,

$$\textstyle\bigotimes_{i=1}^m \operatorname{Lat}(A_i) = \big\{\{0\} \neq \bigotimes_{i=1}^m \mathcal{M}_i \subseteq \bigotimes_{i=1}^m \mathcal{H}_i \colon\ \{0\} \neq \mathcal{M}_i \in \operatorname{Lat}(A_i)\big\},$$

and let $\operatorname{RLat}\big(\bigotimes_{i=1}^m A_i\big)$ denote the collection of all regular invariant subspaces for the tensor product operator $\bigotimes_{i=1}^m A_i$ on $\bigotimes_{i=1}^m \mathcal{H}_i$,

$$\textstyle\operatorname{RLat}\big(\bigotimes_{i=1}^m A_i\big) = \big\{\bigotimes_{i=1}^m \mathcal{M}_i \subseteq \bigotimes_{i=1}^m \mathcal{H}_i \colon\ \bigotimes_{i=1}^m \mathcal{M}_i \in \operatorname{Lat}\big(\bigotimes_{i=1}^m A_i\big)\big\}.$$

Note that, if $A_{i_0} = O$ for some $i_0 \in [1, m]$, then $\bigotimes_{i=1}^m A_i = O$, and so every regular subspace $\bigotimes_{i=1}^m \mathcal{M}_i$ of $\bigotimes_{i=1}^m \mathcal{H}_i$ lies in $\operatorname{Lat}\big(\bigotimes_{i=1}^m A_i\big)$ independently of the operators A_i for every $i \neq i_0$. If $\mathcal{M}_{i_0} \subseteq \mathcal{N}(A_{i_0})$ for some $i_0 \in [1, m]$ (and so $\mathcal{M}_{i_0} \in \operatorname{Lat}(A_{i_0})$), then $\bigotimes_{i=1}^m \mathcal{M}_i \subseteq \mathcal{N}\big(\bigotimes_{i=1}^m A_i\big)$ so that $\bigotimes_{i=1}^m \mathcal{M}_i$ lies in $\operatorname{RLat}\big(\bigotimes_{i=1}^m A_i\big)$ independently of the subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ for every $i \neq i_0$.

Theorem 1. $\operatorname{RLat}\big(\bigotimes_{i=1}^m A_i\big)$ *is a lattice. If each A_i is injective, then*

$$\textstyle\operatorname{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\} = \bigotimes_{i=1}^m \operatorname{Lat}(A_i) \subseteq \operatorname{Lat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\},$$

and the inclusion may be proper even though every A_i is injective.

Proof. (a) Consider the lattice $\operatorname{Lat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big)$ of all subspaces of the tensor product $\bigotimes_{i=1}^m \mathcal{H}_i$. Recall that, if $\mathcal{E}$ and $\mathcal{E}'$ are subspaces of $\bigotimes_{i=1}^m \mathcal{H}_i$, then their inf and sup in $\operatorname{Lat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big)$ are given by $\mathcal{E} \wedge \mathcal{E}' = \mathcal{E} \cap \mathcal{E}'$ and $\mathcal{E} \vee \mathcal{E}' = (\mathcal{E} + \mathcal{E}')^-$. Now let $\mathrm{RLat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big)$ be the subcollection of $\mathrm{Lat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big)$ consisting of all regular subspaces of $\bigotimes_{i=1}^m \mathcal{H}_i$. If $\mathcal{R} = \bigotimes_{i=1}^m \mathcal{M}_i$ and $\mathcal{R}' = \bigotimes_{i=1}^m \mathcal{M}'_i$ are regular subspaces of $\bigotimes_{i=1}^m \mathcal{H}_i$, then their inf and sup are given by $\mathcal{R} \wedge \mathcal{R}' = \bigotimes_{i=1}^m \mathcal{M}_i \cap \mathcal{M}'_i$ and $\mathcal{R} \vee \mathcal{R}' = \bigotimes_{i=1}^m (\mathcal{M}_i + \mathcal{M}'_i)^-$, which lie in $\mathrm{RLat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big)$. Thus, in this sense, $\mathrm{RLat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big)$ is sublattice of $\mathrm{Lat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big)$. Therefore, by its very definition,

$$\mathrm{RLat}\big(\textstyle\bigotimes_{i=1}^m A_i\big) = \mathrm{RLat}\big(\bigotimes_{i=1}^m \mathcal{H}_i\big) \cap \mathrm{Lat}\big(\bigotimes_{i=1}^m A_i\big),$$

so that, being an intersection of latices, $\mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)$ is a lattice itself (actually, a sublattice of $\mathrm{Lat}\big(\bigotimes_{i=1}^m A_i\big)$).

(b) Next observe by Lemma 1(a'_4) that, if each A_i is injective, then

$$\mathrm{RLat}\big(\textstyle\bigotimes_{i=1}^m A_i\big)\backslash\{0\} \subseteq \bigotimes_{i=1}^m \mathrm{Lat}(A_i)$$

(i.e., for every $\{0\} \neq \mathcal{R} = \bigotimes_{i=1}^m \mathcal{M}_i$ in $\mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)$ it follows that $\mathcal{R}$ lies in $\bigotimes_{i=1}^m \mathrm{Lat}(A_i)$). Conversely, by Lemma 1(a_1), we get that

$$\textstyle\bigotimes_{i=1}^m \mathrm{Lat}(A_i) \subseteq \mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\} \subseteq \mathrm{Lat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$$

(i.e., if $\{0\} \neq \mathcal{R} = \bigotimes_{i=1}^m \mathcal{M}_i$ lies in $\bigotimes_{i=1}^m \mathrm{Lat}(A_i)$, then $\mathcal{R}$ lies in $\mathrm{Lat}\big(\bigotimes_{i=1}^m A_i\big)$). Therefore, if each A_i is injective, then

$$\mathrm{RLat}\big(\textstyle\bigotimes_{i=1}^m A_i\big)\backslash\{0\} = \bigotimes_{i=1}^m \mathrm{Lat}(A_i) \subseteq \mathrm{Lat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}.$$

(c) Finally set $\mathcal{H}_i = \ell_+^2$ and $A_i = S_+$ for all i, where S_+ is the canonical unilateral shift of multiplicity 1 acting on ℓ_+^2, which is injective and irreducible. Recall that $\bigotimes_{i=1}^m S_+$ is again a unilateral shift, now of higher multiplicity acting on $\bigotimes_{i=1}^m \ell_+^2$, the tensor product of m copies of ℓ_+^2 [6, Theorem 2]. Since the multiplicity of the unilateral shift $\bigotimes_{i=1}^m S_+$ is greater than 1, it is reducible (in fact, the tensor product of m copies of a injective operator always is reducible [6, Corollary 1(b)]. Thus, since S_+ is injective and irreducible, it follows by Lemma 1(a) that all nontrivial reducing subspaces of $\bigotimes_{i=1}^m S_+$ are irregular. Therefore,

$$\textstyle\bigotimes_{i=1}^m \mathrm{Lat}(S_+) = \mathrm{RLat}\big(\bigotimes_{i=1}^m S_+\big)\backslash\{0\} \subset \mathrm{Lat}\big(\bigotimes_{i=1}^m S_+\big)\backslash\{0\},$$

where the inclusion is proper. □

Example 1. A tensor product $A \otimes B$ is an isometry if and only if both γA and $\gamma^{-1} B$ are isometries for some nonzero scalar γ [13, Theorem 2.4]; a tensor product $A \otimes B$ is a unilateral shift if and only if $A \otimes B = V \otimes S_+$ or $A \otimes B = S_+ \otimes V$ where S_+ is a unilateral shift and V is an isometry [8, Lemma 1]. Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional Hilbert spaces, and let V be an isometry on a $\mathcal{H}$ and S_+ a unilateral shift of multiplicity 1 on $\mathcal{K}$, so that $V \otimes S_+$ is a unilateral shift of multiplicity grater than 1 on $\mathcal{H} \otimes \mathcal{K}$, thus reducible. Since isometries are injective,

$$\mathrm{RLat}(V \otimes S_+)\backslash\{0\} = \mathrm{Lat}(V) \otimes \mathrm{Lat}(S_+)$$

by Theorem 1. Therefore, if $\mathcal{M}$ and $\mathcal{N}$ are nonzero subspaces of $\mathcal{H}$ and $\mathcal{K}$, then $\mathcal{M} \otimes \mathcal{N}$ is $(V \otimes S_+)$-invariant if and only if $\mathcal{M}$ is V-invariant and $\mathcal{N}$ is S_+-invariant. Since $V \otimes S_+$ is reducible, V and S_+ are injective, and S_+ is irreducible, Lemma 1(a) ensures that the tensor product $V \otimes S_+$ is reducible whose all nontrivial reducing subspaces are irregular, so that $\mathcal{M} \otimes \mathcal{N}$ never reduces $V \otimes S_+$. Then *every nonzero reducing subspace $\mathcal{R}$ for $V \otimes S_+$ is irregular*; that is, $\mathcal{R} \notin \mathrm{RLat}(V \otimes S_+)$. Therefore,

$$\mathcal{R} \in \operatorname{Lat}(V \otimes S_+)\backslash \operatorname{Lat}(V) \otimes \operatorname{Lat}(S_+).$$

4. A Further Characterization

Let $\{\mathcal{H}_i\}_{i=1}^m$ and $\{\mathcal{K}_i\}_{i=1}^m$ be finite collections of Hilbert spaces. Take the Hilbert spaces made up of their orthogonal direct sum and of their tensor product, $\bigotimes_{i=1}^m \mathcal{H}_i$ and $\bigotimes_{i=1}^m \mathcal{K}_i$. Take $A_i \in \mathcal{B}[\mathcal{H}_i]$ and $\mathcal{B}_i \in \mathcal{B}[\mathcal{K}_i]$. Consider the definition of ILat and RLat, and define IRLat and RILat as follows.

$$\operatorname{IRLat}\big(\textstyle\bigoplus_{i=1}^m A_i \otimes B_i\big) = \bigoplus_{i=1}^m \big(\operatorname{RLat}(A_i \otimes B_i)\backslash\{0\}\big)$$

and

$$\operatorname{RILat}\big(\textstyle\bigotimes_{i=1}^m A_i \oplus B_i\big) = \bigotimes_{i=1}^m \operatorname{ILat}(A_i \oplus B_i),$$

on $\bigoplus_{i=1}^m \mathcal{H}_i \otimes \mathcal{K}_i$ and on $\bigotimes_{i=1}^m \mathcal{H}_i \oplus \mathcal{K}_i$, respectively.

Proposition 1. *Consider the previous setup.*

(a) $\operatorname{ILat}\big(\bigoplus_{i=1}^m A_i \otimes B_i\big) = \bigoplus_{i=1}^m \operatorname{Lat}(A_i \otimes B_i)$.

(b) $\operatorname{RILat}\big(\bigotimes_{i=1}^m A_i \oplus B_i\big) = \bigotimes_{i=1}^m \big(\operatorname{Lat}(A_i) \oplus \operatorname{Lat}(B_i)\big)$.

If each A_i and B_i is injective, then

(c) $\operatorname{RLat}\big(\bigotimes_{i=1}^m A_i \oplus B_i\big)\backslash\{0\} = \bigotimes_{i=1}^m \operatorname{Lat}(A_i \oplus B_i)$

and

(d) $\operatorname{IRLat}\big(\bigoplus_{i=1}^m A_i \otimes B_i\big) = \bigoplus_{i=1}^m \operatorname{Lat}(A_i) \otimes \operatorname{Lat}(B_i)$.

Proof. Identities in (a) and (b) are straightforward by the very definition of ILat:

$$\operatorname{ILat}\big(\textstyle\bigoplus_{i=1}^m A_i \otimes B_i\big) = \bigoplus_{i=1}^m \operatorname{Lat}(A_i \otimes B_i)$$

and

$$\operatorname{RILat}\big(\textstyle\bigotimes_{i=1}^m A_i \oplus B_i\big) = \bigotimes_{i=1}^m \operatorname{Lat}(A_i) \oplus \operatorname{Lat}(B_i).$$

If each A_i and each B_i are injective, then we get (c) and (d),

$$\operatorname{RLat}\big(\textstyle\bigotimes_{i=1}^m A_i \oplus B_i\big)\backslash\{0\} = \bigotimes_{i=1}^m \operatorname{Lat}(A_i \oplus B_i)$$

and

$$\operatorname{IRLat}\big(\textstyle\bigoplus_{i=1}^m A_i \otimes B_i\big) = \bigoplus_{i=1}^m \operatorname{Lat}(A_i) \otimes \operatorname{Lat}(B_i),$$

according to Theorem 1. □

Let $\bigotimes_{i=1}^m C_i$ be the tensor product operator on the tensor product space $\bigotimes_{i=1}^m \mathcal{X}_i$, consisting of m operators taken from the collection $\{A_i, B_i\}_{i=1}^m$ such that C is either A or B and, accordingly, $\mathcal{X}$ is either $\mathcal{H}$ or $\mathcal{K}$. Order these by lexicographic ordering (so that there are 2^m tensor products where $\bigotimes_{i=1}^m A_i$ on $\bigotimes_{i=1}^m \mathcal{H}_i$ is the first and $\bigotimes_{i=1}^m B_i$ on $\bigotimes_{i=1}^m \mathcal{K}_i$ is the last). Let $\mathbf{X}$ and $\mathbf{C}$ denote the classes of all tensor products $\bigotimes_{i=1}^m \mathcal{X}_i$ and $\bigotimes_{i=1}^m C_i$, respectively, equipped with the lexicographic ordering. Take the operators

$$\textstyle\bigotimes_{i=1}^m (A_i \oplus B_i) \quad \text{on} \quad \bigotimes_{i=1}^m (\mathcal{H}_i \oplus \mathcal{K}_i)$$

and

$$\textstyle\bigoplus_{j=1}^{2^m} \big(\bigotimes_{i=1}^m C_i\big) \quad \text{on} \quad \bigoplus_{j=1}^{2^m} \big(\bigotimes_{i=1}^m \mathcal{X}_i\big),$$

where the direct suns $\bigoplus_{j=1}^{2^m}$ are taken over all elements of $\mathbf{C}$ and of $\mathbf{X}$, respectively. For instance, if $m=2$, then take the operators

$$(A_1\oplus B_1)\otimes(A_2\oplus B_2)\quad\text{on}\quad(\mathcal{H}_1\oplus\mathcal{K}_1)\otimes(\mathcal{H}_2\oplus\mathcal{K}_2)$$

and

$$(A_1\otimes A_2)\oplus(A_1\otimes B_2)\oplus(A_2\otimes B_1)\oplus(B_1\otimes B_2)$$

on

$$(\mathcal{H}_1\otimes\mathcal{H}_2)\oplus(\mathcal{H}_1\otimes\mathcal{K}_2)\oplus(\mathcal{H}_2\otimes\mathcal{K}_1)\oplus(\mathcal{K}_1\otimes\mathcal{K}_2).$$

Corollary 1. *Consider again the previous setup.*

(a) *The operators* $\bigotimes_{i=1}^m(A_i\oplus B_i)$ *and* $\bigoplus_{j=1}^{2^m}\big(\bigotimes_{i=1}^m C_i\big)$ *are unitarily equivalent.*

(b) *Thus the latices* $\operatorname{Lat}\big(\bigotimes_{i=1}^m(A_i\oplus B_i)\big)$ *and* $\operatorname{Lat}\big(\bigoplus_{j=1}^{2^m}\big(\bigotimes_{i=1}^m C_i\big)\big)$ *are unitarily equivalent as well.*

(c) *Moreover, if each A_i and B_i is injective, then*

$$\textstyle\bigotimes_{m=1}^m\big(\operatorname{Lat}(A_i)\oplus\operatorname{Lat}(B_i)\big)\quad\textit{and}\quad\bigoplus_{j=1}^{2^m}\big(\bigotimes_{i=1}^m\operatorname{Lat}(C_i)\big)$$

are unitarily equivalent.

Proof. First consider the case of $m=2$. Thus let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be Hilbert spaces, and take arbitrary operators $A_1\in\mathcal{B}[\mathcal{H}_1]$, $A_2\in\mathcal{B}[\mathcal{H}_2]$, $B_1\in\mathcal{B}[\mathcal{K}_1]$ and $B_2\in\mathcal{B}[\mathcal{K}_2]$. The operators

$$(A_1\oplus B_1)\otimes(A_2\oplus B_2)\quad\text{and}\quad(A_1\otimes A_2)\oplus(A_1\otimes B_2)\oplus(B_1\otimes A_2)\oplus(B_1\otimes B_2),$$

which act on

$$(\mathcal{H}_1\oplus\mathcal{K}_1)\otimes(\mathcal{H}_2\oplus\mathcal{K}_2)\quad\text{and}\quad(\mathcal{H}_1\otimes\mathcal{H}_2)\oplus(\mathcal{H}_1\otimes\mathcal{K}_2)\oplus(\mathcal{K}_1\otimes\mathcal{K}_2)\oplus(\mathcal{K}_1\otimes\mathcal{K}_2),$$

are unitarily equivalent. Indeed, as is readily verified,

$$\begin{aligned}\begin{pmatrix}A_1 & O\\ O & B_1\end{pmatrix}\otimes\begin{pmatrix}A_2 & O\\ O & B_2\end{pmatrix}&\cong\begin{pmatrix}A_1\otimes\left(\begin{smallmatrix}A_2 & O\\ O & B_2\end{smallmatrix}\right) & O\\ O & B_1\otimes\left(\begin{smallmatrix}A_2 & O\\ O & B_2\end{smallmatrix}\right)\end{pmatrix}\\ &\cong\begin{pmatrix}A_1\otimes A_2 & O & O & O\\ O & A_1\otimes B_2 & O & O\\ O & O & B_1\otimes A_2 & O\\ O & O & O & B_1\otimes B_2\end{pmatrix},\end{aligned}$$

and recalling that for any operators S and T their tensor products $S\otimes T$ and $T\otimes S$ are unitarily equivalent (i.e., tensor product is unitarily equivalent commutative), it follows that

$$(A_1\oplus B_1)\otimes(A_2\oplus B_2)\;\cong\;(A_1\otimes A_2)\oplus(A_1\otimes B_2)\oplus(A_2\otimes B_1)\oplus(B_1\otimes B_2)$$

and

$$(\mathcal{H}_1\oplus\mathcal{K}_1)\otimes(\mathcal{H}_2\oplus\mathcal{K}_2)\;\cong\;(\mathcal{H}_1\otimes\mathcal{H}_2)\oplus(\mathcal{H}_1\otimes\mathcal{K}_2)\oplus(\mathcal{H}_2\otimes\mathcal{K}_1)\oplus(\mathcal{K}_1\otimes\mathcal{K}_2),$$

where $\cong$ denotes unitary equivalence. So the lattices $\operatorname{Lat}[(A_1\oplus B_1)\otimes(A_2\oplus B_2)]$ and $\operatorname{Lat}[(A_1\otimes A_2)\oplus(A_1\otimes B_2)\oplus(A_2\otimes B_1)\oplus(B_1\otimes B_2)]$ are unitarily equivalent (invariant subspaces are preserved by unitary equivalence):

$$\operatorname{Lat}[(A_1\oplus B_1)\otimes(A_2\oplus B_2)]\cong\operatorname{Lat}[(A_1\otimes A_2)\oplus(A_1\otimes B_2)\oplus(A_2\otimes B_1)\oplus(B_1\otimes B_2)].$$

Moreover, if A_1, A_2, B_1 and B_2 are injective, then by Proposition 1(b,d),

RILat$[(A_1\oplus B_1)\otimes(A_2\oplus B_2)] \cong$ IRLat$[(A_1\otimes A_2)\oplus(A_1\otimes B_2)\oplus(A_2\otimes B_1)\oplus(B_1\otimes B_2)]$,

Therefore (cf. Proposition 1(b,d) again),

$$\begin{aligned}[\mathrm{Lat}(A_1)\oplus\mathrm{Lat}(B_1)] \;\otimes\; &[\mathrm{Lat}(A_2)\oplus\mathrm{Lat}(B_2)]\\ &\cong\;[\mathrm{Lat}(A_1)\otimes\mathrm{Lat}(A_2)]\oplus[\mathrm{Lat}(A_1)\otimes\mathrm{Lat}(B_2)]\\ &\oplus\;[\mathrm{Lat}(A_2)\otimes\mathrm{Lat}(B_1)]\oplus[\mathrm{Lat}(B_1)\otimes\mathrm{Lat}(B_2)],\end{aligned}$$

which completes the proof for $m=2$. If $m\geq 3$ then, for any $j\in[2,m-1]$,

$$\textstyle\bigotimes_{i=1}^m C_i=\bigotimes_{i=1}^{j-1}C_i\otimes C_j\otimes\bigotimes_{i=j+1}^m C_i$$

and

$$\textstyle\bigotimes_{i=1}^m \mathcal{X}_i=\bigotimes_{i=1}^{j-1}\mathcal{X}_i\otimes \mathcal{X}_j\otimes\bigotimes_{i=j+1}^m \mathcal{X}_i,$$

so that the result for $m=2$ extends by induction to any integer $m\geq 3$. $\square$

Example 2. Set $m=2$, $\mathcal{H}=\mathcal{K}=\ell^2_+$, $A_1=B_1=I$, the identity operator, and $A_2=B_2=S_+$, the canonical unilateral shift of multiplicity 1 on ℓ^2_+. In this case,

$$\begin{aligned}I'\otimes S'_+ &= (I\oplus I)\otimes(S_+\oplus S_+)=(A_1\oplus B_1)\otimes(A_2\oplus B_2)\\ &\cong (A_1\otimes A_2)\oplus(A_1\otimes B_2)\oplus(B_1\otimes A_2)\oplus(B_1\otimes B_2)\\ &= (I\otimes S_+)\oplus(I\otimes S_+)\oplus(I\otimes S_+)\oplus(I\otimes S_+)=\textstyle\bigoplus_{i=1}^4(I\otimes S_+),\end{aligned}$$

where I' is the identity, and S'_+ is canonical unilateral shift of multiplicity 2, both acting on $\ell^2_+\oplus\ell^2_+$. Note that $\mathrm{Lat}(I')\otimes\mathrm{Lat}(S'_+)=(\ell^2_+\oplus\ell^2_+)\otimes\mathrm{Lat}(S_+\oplus S_+)$. Also note that both $\mathrm{Lat}(S'_+)$ and $\mathrm{Lat}\big(\bigoplus_{i=1}^4(I\otimes S_+)\big)$ have been fully characterized in [3] because S'_+ and $\bigoplus_{i=1}^4(I\otimes S_+)$ are unilateral shifts ($I\otimes S_+$, and so $\bigoplus_{i=1}^4(I\otimes S_+)$, are unilateral shifts of infinite multiplicity). Thus, since all operators involved in this example are injective, Corollary 1(c) shows that

$$\big(\mathrm{Lat}(I)\oplus\mathrm{Lat}(I)\big)\otimes\big(\mathrm{Lat}(S_+)\oplus\mathrm{Lat}(S_+)\big)=\big(\ell^2_+\oplus\ell^2_+\big)\otimes\big(\mathrm{Lat}(S_+)\oplus\mathrm{Lat}(S_+)\big)$$

is unitarily equivalent to

$$\textstyle\bigoplus_{i=1}^4\big(\mathrm{Lat}(I)\otimes\mathrm{Lat}(S_+)\big)=\bigoplus_{i=1}^4\big(\ell^2_+\otimes\mathrm{Lat}(S_+)\big),$$

where regular subspaces have been considered; irregular subspaces as in Example 1 are not characterized by the above equivalence, but require an independent analysis.

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