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EXPONENTIAL STABILITY AND DISSIPATIVITY

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Abstract. Exponential stability and plain exponential stability of strongly continuous semigroups, and dissipativity and boundedly strict dissipativity of its generator, are investigated. It is proved that dissipativity implies boundedly strict dissipativity in an equivalent topology, which ensures that dissipativity and exponential stability together imply plain-e-stability in that topology.

1. Introduction

Throughout this paper $\mathcal{H}$ will stand for a complex Hilbert apace. Let $\mathcal{B}[\mathcal{H}]$ denote the Banach algebra of all (bounded linear) operators on $\mathcal{H}$. We will be dealing with strongly continuous semigroups $[T(t)] = \{T(t)\,;\ t \ge 0\}$ (i.e., semigroups of class C_0) of operators in $\mathcal{B}[\mathcal{H}]$. The inner product and norm on $\mathcal{H}$ are denoted by $\langle \cdot\,;\cdot \rangle$ and by $\|\cdot\|$. Besides the inner product $\langle \cdot\,;\cdot \rangle$ we will also be dealing with a second inner product $\langle \cdot\,;\cdot \rangle_P$ associated with an arbitrary positive operator P in $\mathcal{B}[\mathcal{H}]$, namely,

$$\langle x\,; y\rangle_P = \langle Px\,; y\rangle \quad \text{for every} \quad x, y \in \mathcal{H},$$

so that the norm induced by it is given by

$$\|x\|_P = \|P^{\frac{1}{2}}x\| \quad \text{for every} \quad x \in \mathcal{H},$$

where $P^{\frac{1}{2}}$ in $\mathcal{B}[\mathcal{H}]$ denotes the unique positive square root of P.

A semigroup $[T(t)]$ is *exponentially stable* (or *e-stable*) if there exist real constants $M \ge 1$ and $\alpha > 0$ such that

$$\|T(t)\| \le Me^{-\alpha t} \quad \text{for every} \quad t \ge 0$$

or, equivalently, if

$$\|T(t)x\| \le Me^{-\alpha t}\,\|x\| \quad \text{for every} \quad x \in \mathcal{H} \quad \text{and every} \quad t > 0.$$

We shall refer to any constant α that satisfies the above inequality for an e-stable semigroup as a *decay exponent*.

A semigroup $[T(t)]$ is *contractive*, or it is a *contraction* semigroup, if

$$\|T(t)\| \le 1 \quad \text{for all} \quad t \ge 0$$

or, equivalently, if

$$\|T(t)x\| \le \|x\| \quad \text{for every} \quad x \in \mathcal{H} \quad \text{and all} \quad t > 0.$$

Let $A\colon \mathcal{D} \to \mathcal{H}$ be the generator of the semigroup $[T(t)]$, which is a linear transformation. If $[T(t)]$ is a C_0-semigroup, then A is closed and densely defined. If A is densely defined (i.e., if $\mathcal{D}$ is a dense linear manifold of $\mathcal{H}$), then it is *dissipative* if

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$$\operatorname{Re}\langle Ax\,;x\rangle \leq 0 \quad \text{for every} \quad x \in \mathcal{D}.$$

If $[T(t)]$ is a strongly continuous *contractive* semigroup (i.e., a *contractive* C_0-semigroup), then its generator A is maximal dissipative in the sense that it is dissipative and there is no dissipative extension of it on $\mathcal{H}$ [3, 4, 9]. A dissipative generator A is called *strictly dissipative* if

$$\operatorname{Re}\langle Ax\,;x\rangle < 0 \quad \text{for every} \quad 0 \neq x \in \mathcal{D},$$

We say that a strictly dissipative generator A is *boundedly strict dissipative* (or *uniformly dissipative* — see [2, p.34]) if there exists a constant $\gamma > 0$ such that

$$\operatorname{Re}\langle Ax\,;x\rangle \leq -\gamma\,\|x\|^2 \quad \text{for every} \quad x \in \mathcal{D}.$$

Even if an e-stable semigroup $[T(t)]$ is contractive, it may happen that the inequality $\|T(t)\| \leq Me^{-\alpha t}$ for every $x \in \mathcal{H}$ and every $t > 0$ only holds for constants $M > 1$ [6]. If a semigroup is e-stable with $M = 1$, then it is said to be *plain-e-stable*. In other words, $[T(t)]$ is *plain-e-stable* if there exists a constant $\alpha > 0$ such that

$$\|T(t)\| \leq e^{-\alpha t} \quad \text{for every} \quad t \geq 0$$

or, equivalently, if

$$\|T(t)x\| \leq e^{-\alpha t}\,\|x\| \quad \text{for every} \quad x \in \mathcal{H} \quad \text{and every} \quad t > 0.$$

In this case, the semigroup $[T(t)]$ is obviously contractive (and hence it has a dissipative generator A, as we saw above). Recall from [6] that *an e-stable contraction C_0-semigroup is plain-e-stable if and only if its generator is boundedly strict dissipative* [6, Theorem 2]. Actually, *a contraction C_0-semigroup is plain-e-stable if and only if its generator is boundedly strict dissipative* [6, Corollary 2] or, equivalently, a contraction C_0-semigroup is not plain-e-stable if and only if its generator is not boundedly strict dissipative, which means that for every $\alpha > 0$ there exist a vector $x_\alpha \in \mathcal{H}$ and a positive number $t_\alpha > 0$ such that

$$e^{-\alpha t_\alpha}\|x_\alpha\| < \|T(t_\alpha)x_\alpha\|$$

if and only if for every $\gamma > 0$ there exists a vector $x_\gamma \in \mathcal{D}$ such that

$$-\gamma\|x_\gamma\| < \operatorname{Re}\langle Ax_\gamma\,;x_\gamma\rangle \leq 0.$$

For a characterization of contraction C_0-semigroups that are strongly stable but not plain-e-stable see [7, Corollary 2]. The special case of plain-e-stability with $\|x\| = \operatorname{Re}\langle Ax\,;x\rangle$ for every $x \in \mathcal{D}$ was investigated in [5, Corollary 3]. Subspaces of plain-e-stability for contraction C_0-semigroups with strictly dissipative generators were considered in [8, Theorem 4].

In the following we will be dealing with dissipativity (in particular, with boundedly strict dissipativity) of the generator A with respect to the inner product$\langle\cdot\,;\cdot\rangle$ in $\mathcal{H}$, as well as with P-dissipativity of A; that is, dissipativity of A with respect to the inner product $\langle\cdot\,;\cdot\rangle_P$ in $\mathcal{H}$ for a positive operator P in $\mathcal{B}[\mathcal{H}]$. We investigate the role of dissipativity of the generator A of a C_0-semigroup under the assumptions of e-stability and plain-e-stability of the semigroup. We begin in Section 2 by showing that the necessary and sufficient conditions for e-stability of C_0-semigroups due to Datko (Theorem 1), can be expressed in terms of P-boundedly strict dissipativity of the generator A (Corollary 1). Since the results of Theorem 1 do not assume that the semigroup is contractive, our approach to e-stability of contraction semigroups is to impose dissipativity constraints on the generator A of Theorem 1. The central results appear in Section 3, where we investigate P_α-dissipativity of A for a positive operator associated both with the e-stability criterion of Theorem 1 and also with the decaying exponent of e-stability. Let P be the positive operator of the Lyapunov equation of Theorem 1, and let α denote any decaying exponent for an arbitrary e-stable semigroup. We show in Lemma 1 that *for an e-stable C_0-semigroup, the norm associated with the positive operator $P_\alpha = P + (2\alpha)^{-1}I$ is equivalent to the Hilbert space norm*, and also that *if the generator of an e-stable C_0-semigroup is dissipative, then it is P_α-boundedly strict dissipative*. This leads to the main result in Theorem 2, where we prove that *if a C_0-semigroup has a dissipative generator* (*in particular, if the semigroup is contractive*) *and is e-stable, then it is plain-e-stable with respect to the equivalent norm associated with the positive operator $P_\alpha = P + (2\alpha)^{-1}I$.*

2. e-Stability and P-Boundedly Strict Dissipativity

Take any positive operator P in $\mathcal{B}[\mathcal{H}]$ and consider the inner product $\langle\cdot\,\cdot\rangle_P$ and the norm $\|\cdot\|_P$ on $\mathcal{H}$ associated with P. The generator A of a semigroup $[T(t)]$ is

(i) *P-dissipative* if $\operatorname{Re}\langle Ax\,;x\rangle_P \le 0$ for every $x \in \mathcal{D}$,

(ii) *P-strictly dissipative* if $\operatorname{Re}\langle Ax\,;x\rangle_P < 0$ for every $0 \ne x \in \mathcal{D}$,

(iii) *P-boundedly strict dissipative* if $\operatorname{Re}\langle Ax\,;x\rangle_P \le -\gamma\,\|x\|_P^2$ for every $x \in \mathcal{D}$, for some constant $\gamma > 0$.

Recall the following results due to Datko [1].

Theorem 1. *Let $[T(t)]$ be a C_0-semigroup on $\mathcal{H}$ with generator A. The following conditions are equivalent.*

(a) *$[T(t)]$ is exponentially stable.*

(b) $\int_0^\infty \|T(t)x\|^2\,dt < \infty$ *for every $x \in \mathcal{H}$.*

(c) *There exists a unique positive operator P* (i.e., $P > O$) *on $\mathcal{H}$ such that*

$$\langle Px\,;x\rangle = \int_0^\infty \|T(t)x\|^2\,dt \quad \textit{for every} \quad x \in \mathcal{H},$$

and it satisfies the Lyapunov equation:

$$2\operatorname{Re}\langle PAx\,;x\rangle = -\|x\|^2 \quad \textit{for every} \quad x \in \mathcal{D}.$$

The next result is a straightforward consequence of Theorem 1 that will be needed in the sequel.

Corollary 1. *Let $[T(t)]$ be a C_0-semigroup on $\mathcal{H}$ with generator A. Suppose there exist constants $M \ge 1$ and $\alpha > 0$ such that*

$$\|T(t)\| \le Me^{-\alpha t} \quad \textit{for every} \quad t \ge 0$$

(i.e., *suppose $[T(t)]$ is exponentially stable*).

(a) *There exists a positive operator P in $\mathcal{B}[\mathcal{H}]$ such that*

$$\langle Px\,;x\rangle \le \tfrac{M^2}{2\alpha}\|x\|^2 \quad \textit{for every} \quad x \in \mathcal{H}.$$

Equivalently, such that

$$P \leq \tfrac{1}{2\alpha} M^2 I$$

(*where I denotes the identity operator in $\mathcal{B}[\mathcal{H}]$*).

(b) *Moreover,*

$$\mathrm{Re}\langle Ax\,;x\rangle_P \leq -\tfrac{\alpha}{M^2}\|x\|_P^2 \quad \textit{for every} \quad x \in \mathcal{D}$$

(*so that the generator A is P-boundedly strict dissipative*).

(c) *Furthermore, if*

$$P = \tfrac{1}{2\alpha} M^2 I,$$

then A is boundedly strict dissipative.

Proof. (a) Since $\int_0^\infty M^2 e^{-2\alpha t}\,dt = \frac{M^2}{2\alpha}$ for every nonzero constants M and α, it follows that, if $\|T(t)x\| \leq Me^{-\alpha t}\|x\|$ for every $x \in \mathcal{H}$ and every $t > 0$, then the result in (a) is a consequence of Theorem 1(c).

(b) Moreover, since $2\mathrm{Re}\langle Ax\,;x\rangle_P = 2\,\mathrm{Re}\langle PAx\,;x\rangle = -\|x\|^2$ for every $x \in \mathcal{D}$ by the Lyapunov equation in Theorem 1(c), it follows by item (a) that, for every $x \in \mathcal{D}$,

$$2\,\mathrm{Re}\langle Ax\,;x\rangle_P = -\|x\|^2 \leq -\tfrac{2\alpha}{M^2}\langle Px\,;x\rangle = -\tfrac{2\alpha}{M^2}\|x\|_P^2.$$

(c) Finally, recall from (a) that $P \leq \frac{1}{2\alpha}M^2$. If $P = \frac{1}{2\alpha}M^2$, then, by the Lyapunov equation in Theorem 1(a), $\mathrm{Re}\langle Ax\,;x\rangle = -\frac{\alpha}{M^2}\|x\|^2$ for every $x \in \mathcal{D}$, and so A is boundedly strict dissipative. □

Observe from Corollary 1(a,c) that if the generator of the e-stable C_0-semigroup is not boundedly strict dissipative, then $P \leq \frac{1}{2\alpha}M^2 I$ and $P \neq \frac{1}{2\alpha}M^2 I$ (which does not imply that $P < \frac{1}{2\alpha}M^2 I$).

Conditions for e-stability of *uniformly continuous* semigroups, in terms of boundedly strict dissipative generators were discussed in [2]. For contraction semigroups it follows that *if a uniformly continuous contraction semigroup with a strictly dissipative generator is e-stable, then it is plain-e-stable* [6, Corollary 1].

3. e-Stability, Dissipativity, and Plain-e-Stability

We now show that dissipativity and P-boundedly strict dissipativity of the generator of an e-stable C_0-semigroup can be connected by means of the inner product $\langle\cdot\,;\cdot\rangle_{P+(2\alpha)^{-1}I}$ on $\mathcal{H}$, which is defined, for every positive number α, by

$$\langle x\,;y\rangle_{P+(2\alpha)^{-1}I} = \langle (P + (2\alpha)^{-1}I)x\,;y\rangle \quad \text{for every} \quad x,\, y \in \mathcal{H},$$

where $P > 0$ is the unique solution to the Lyapunov equation of Theorem 1(c). Suppose α is any decay exponent for an e-stable semigroup, and consider the norm $\|\cdot\|_{P+(2\alpha)^{-1}I}$ on $\mathcal{H}$ generated by the inner product $\langle\cdot\,;\cdot\rangle_{P+(2\alpha)^{-1}I}$ on $\mathcal{H}$.

Lemma 1. *Suppose $[T(t)]$ is an e-stable C_0-semigroup on $\mathcal{H}$, so that there exist constants $M \geq 1$ and $\alpha > 0$ such that*

$$\|T(t)\| \leq Me^{-\alpha t} \quad \textit{for every} \quad t \geq 0.$$

Then the norms $\|\cdot\|$ and $\|\cdot\|_{P+(2\alpha)^{-1}I}$ satisfy the inequalities

(a) $\frac{2\alpha}{M^2+1}\|x\|^2_{P+(2\alpha)^{-1}I} \leq \|x\|^2 \leq 2\alpha\|x\|^2_{P+(2\alpha)^{-1}I}$ *for every* $x \in \mathcal{H}$,

and so they are equivalent. If the generator A *of* $[T(t)]$ *is dissipative (in particular, if* $[T(t)]$ *is contractive), then*

$$\operatorname{Re}\langle Ax\,;x\rangle_{P+(2\alpha)^{-1}I} \leq -\|x\|^2 \quad \textit{for every} \quad x \in \mathcal{D},$$

and therefore

(b) $\operatorname{Re}\langle Ax\,;x\rangle_{P+(2\alpha)^{-1}I} \leq -\frac{\alpha}{M^2+1}\|x\|^2_{P+(2\alpha)^{-1}I}$ *for every* $x \in \mathcal{D}$,

so that A *is* $(P+(2\alpha)^{-1}I)$*-boundedly strict dissipative.*

Proof. The very definition of the norm $\|\cdot\|_{P+(2\alpha)^{-1}I}$ leads to

$$\|x\|^2_{P+(2\alpha)^{-1}I} = \langle (P+(2\alpha)^{-1}I)x\,;x\rangle = \|x\|^2_P + \tfrac{1}{2\alpha}\|x\|^2$$

for every $x \in \mathcal{H}$. Thus, under the e-stability assumption, Corollary 1(a) says that $\|x\|^2_P \leq \frac{M^2}{2\alpha}\|x\|^2$, and so

$$\|x\|^2_{P+(2\alpha)^{-1}I} \leq \tfrac{M^2}{2\alpha}\|x\|^2 + \tfrac{1}{2\alpha}\|x\|^2 = \tfrac{M^2+1}{2\alpha}\|x\|^2$$

for every $x \in \mathcal{H}$. The above two expressions ensure the inequalities in (a). Take an arbitrary $x \in \mathcal{D}$. Since $\operatorname{Re}\langle Ax\,;x\rangle_{P+(2\alpha)^{-1}I} = \operatorname{Re}\langle Ax\,;x\rangle_P + \frac{1}{2\alpha}\operatorname{Re}\langle Ax\,;x\rangle$, and since $2\operatorname{Re}\langle Ax\,;x\rangle_P = -\|x\|^2$ by Theorem 1(c), it follows that

$$2\operatorname{Re}\langle Ax\,;x\rangle_{P+(2\alpha)^{-1}I} = -\|x\|^2 + \tfrac{1}{\alpha}\operatorname{Re}\langle Ax\,;x\rangle.$$

Therefore, if A is dissipative, then $\operatorname{Re}\langle Ax\,;x\rangle \leq 0$, and so

$$2\operatorname{Re}\langle Ax\,;x\rangle_{P+(2\alpha)^{-1}I} \leq -\|x\|^2.$$

Thus we get the inequality in (b) by using the first inequality in (a). □

If a C_0*-semigroup with a dissipative generator is e-stable, then it is plain-e-stable with respect to the equivalent norm associated with the operator* $P+(2\alpha)^{-1}I > O$.

Theorem 2. *Suppose* $[T(t)]$ *is an e-stable* C_0*-semigroup on* $\mathcal{H}$*, so that there exist constants* $M \geq 1$ *and* $\alpha > 0$ *such that*

$$\|T(t)x\| \leq Me^{-\alpha t}\,\|x\| \quad \textit{for every} \quad t \geq 0 \quad \textit{for every} \quad x \in \mathcal{H}.$$

If its generator A *is dissipative, then* $[T(t)]$ *is plain-e-stable in the* $(P+(2\alpha)^{-1}I)$*-norm, which means that there exists a constant* $\beta > 0$ *such that*

$$\|T(t)x\|_{P+(2\alpha)^{-1}I} \leq e^{-\beta t}\|x\|_{P+(2\alpha)^{-1}I} \quad \textit{for every} \quad x \in \mathcal{H}.$$

Proof. Take an arbitrary $x \in \mathcal{D}$. Set $\beta = \frac{\alpha}{M^2+1} > 0$ and $P_\alpha = P + (2\alpha)^{-1}I > O$ in $\mathcal{B}[\mathcal{H}]$. Lemma 1(b) can be rewritten as $\operatorname{Re}\langle Ax\,;x\rangle_{P_\alpha} \leq -\beta\|x\|^2_{P_\alpha}$, and therefore $\operatorname{Re}\langle (A+\beta I)x\,;x\rangle_{P_\alpha} \leq 0$. Then, since $\mathcal{D}$ is T-invariant, it follows that

$$\operatorname{Re}\big\langle (A+\beta I)e^{\beta t}T(t)x\,;e^{\beta t}T(t)x\big\rangle_{P_\alpha} \leq 0.$$

Thus, since the generator of the semigroup $[e^{\beta t}T(t)]$ is $A+\beta I$, it also follows that

$$\frac{d}{dt}\|e^{\beta t}T(t)x\|^2_{P_\alpha} = 2\operatorname{Re}\big\langle (A+\beta I)e^{\beta t}T(t)x\,;e^{\beta t}T(t)x\big\rangle_{P_\alpha} \leq 0$$

[3, p.80,90]. Integrating over $[0,t]$ we get $\|T(t)x\|^2_{P_\alpha} - e^{-\beta t}\|T(0)x\|^2_{P_\alpha} \leq 0$, and so

$$\|T(t)x\|^2_{P_\alpha} \leq e^{-\beta t}\|x\|^2_{P_\alpha} \quad \text{for every} \quad t \geq 0.$$

Extending by continuity from the dense domain $\mathcal{D}$ to the whole space $\mathcal{H}$,

$$\|T(t)x\|^2_{P_\alpha} \leq e^{-\beta t}\|x\|^2_{P_\alpha} \quad \text{for every} \quad t \geq 0 \quad \text{for every} \quad x \in \mathcal{H}. \qquad \square$$

Note that the assumption "A is dissipative" in the statement of Theorem 2 is implied by the stronger assumption "$[T(t)]$ is contractive". Thus we get the following straightforward consequence of Theorem 2. *If a contraction C_0-semigroup is e-stable, then it is plain-e-stable with respect to the equivalent norm associated with the positive operator* $P + (2\alpha)^{-1}I$. Also note that the decay exponent $\beta = \frac{\alpha}{M^2+1}$ is smaller than the original decay exponent α.

4. Conclusion

We have presented an approach to exponential stability of C_0-semigroups with dissipative generator (in particular, to contraction semigroups; but not necessarily to contraction semigroups). We began in Corollary 1 by expressing e-stability in terms of P-boundedly strict dissipativity. After verifying the equivalence between the original norm and the P_α-norm, we have shown in Lemma 1 that if the generator of an e-stable C_0-semigroup is dissipative, then it is P_α-boundedly strict dissipative. This lead to the main result in Theorem 2, which says that dissipativity of the generator and e-stability of the semigroup imply plain-e-stability of the semigroup with respect to the equivalent norm associated with the (invertible) positive operator $P_\alpha = P + (2\alpha)^{-1}I$.

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