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ON WEYL'S THEOREM FOR TENSOR PRODUCTS

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Abstract. Let A and B operators acting on infinite-dimensional spaces. In this note we prove that *if A and B are isoloid, satisfy Weyl's theorem, and the Weyl spectrum identity holds, then $A \otimes B$ satisfies Weyl's theorem.*

1. Notation and Terminology

By an *operator* we mean a *bounded* linear transformation of a Hilbert space into itself. We work in a Hilbert space setting, although the results in this paper hold in a Banach space setting with essentially the same proofs. Let T be an operator, and let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote kernel and range of T, respectively. Consider the classical partition $\{\sigma_P(T), \sigma_R(T), \sigma_C(T)\}$ of the spectrum $\sigma(T)$, where $\sigma_P(T) = \big\{\lambda \in \mathbb{C} \colon \mathcal{N}(\lambda I - T) \neq \{0\}\big\}$ is the point spectrum (i.e., the set of all eigenvalues of T), $\sigma_R(T) = \sigma_P(T^*)^* \backslash \sigma_P(T)$ is the residual spectrum (where T^* denotes the adjoint of T and $\Lambda^* = \{\overline{\lambda} \in \mathbb{C} \colon \lambda \in \Lambda\}$ denotes the set of all complex conjugates from a subset Λ of $\mathbb{C}$), and $\sigma_C(T) = \sigma(T) \backslash (\sigma_P(T) \cup \sigma_R(T))$ is the continuous spectrum. Let $\sigma_w(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not a Fredholm operator of index zero}\}$ be the Weyl spectrum of T, which is a subset of the whole spectrum $\sigma(T)$; that is,

$$\sigma_w(T) = \big\{\lambda \in \sigma(T) \colon \lambda I - T \text{ is not a Fredholm operator of index zero}\big\}.$$

The set

$$\sigma_0(T) = \big\{\lambda \in \sigma_P(T) \colon \mathcal{R}(\lambda I - T) \text{ is closed and } \dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty\big\}$$

is precisely the complement of the Weyl spectrum $\sigma_w(T)$ in the whole spectrum $\sigma(T)$ (see e.g., [5] or [6, Section 5.3]). Hence

$$\sigma_w(T) = \sigma(T) \backslash \sigma_0(T),$$

and so $\{\sigma_w(T), \sigma_0(T)\}$ forms another partition of the spectrum $\sigma(T)$. Set $\sigma_{PF}(T) = \big\{\lambda \in \sigma_P(T) \colon \dim \mathcal{N}(\lambda I - T) < \infty\big\}$; the set of all eigenvalues of T of finite multiplicity, so that $\sigma_0(T) \subseteq \sigma_{PF}(T)$ and $\sigma_R(T) \cup \sigma_C(T) \cup (\sigma_P(T) \backslash \sigma_{PF}(T)) \subseteq \sigma_w(T)$. Set

$$\pi_{00}(T) = \sigma_{\text{iso}}(T) \cap \sigma_{PF}(T),$$

where $\sigma_{\text{iso}}(T)$ denotes the set of all isolated points of the spectrum $\sigma(T)$. One says that an operator T **satisfies Weyl's theorem** if

$$\sigma_0(T) = \pi_{00}(T),$$

and it is said to **satisfy Browder's theorem**

$$\sigma_0(T) \subseteq \pi_{00}(T).$$

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An operator T **is isoloid** if $\sigma_{\rm iso}(T) \subseteq \sigma_P(T)$ (i.e., if every isolated point of the spectrum is an eigenvalue).

2. The Question

Consider the tensor product $A \otimes B$ of a pair of operators A and B. It is known from [1] that the spectrum of a tensor product coincides with the product of the spectra of the factors,

$$\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B).$$

For the Weyl spectrum it was proved in [3] that the inclusion

$$\sigma_w(A \otimes B) \subseteq \sigma_w(A) \cdot \sigma(B) \,\cup\, \sigma(A) \cdot \sigma_w(B)$$

holds. However, since then, it remained as an open question whether the preceding inclusion might be an identity. That is, it was not know if there existed a pair of operators A and B for which the above inclusion was proper. This question was solved quite recently by using the counterexample from [4, Section 3] (which exhibits a pair of operators that satisfy Weyl's theorem whose tensor product does not satisfy Browder's theorem) together with Corollary 6 from [7] (which says that Browder's theorem is transferred from a pair of operators to their tensor product if and only if the above inclusion is an identity). Thus, there exists a pair of operators for which the above inclusion is proper. If a pair of operators A, B is such that

$$\sigma_w(A \otimes B) = \sigma_w(A) \cdot \sigma(B) \,\cup\, \sigma(A) \cdot \sigma_w(B),$$

then we say that the **Weyl spectrum identity holds** for A and B. It was shown in [7, Proposition 5] that, if the Weyl spectrum identity holds for A and B, then

$$\sigma_0(A \otimes B) \subseteq \sigma_0(A) \cdot \sigma_0(B),$$

and it was also shown that the above inclusion may be proper even if the Weyl spectrum identity holds for A and B, with A, B and $A \otimes B$ being isoloid operators that satisfy Weyl's theorem (see [7, Remark 2]).

The problem of transferring Weyl's theorem from isoloid operators A and B to their tensor product $A \otimes B$ was considered in this journal in [8], where it was stated that *if A and B are isoloid operators that satisfy Weyl's theorem, then $A \otimes B$ satisfies Weyl's theorem.* Their proof stands only if it is further assumed that the Weyl spectrum identity holds for A and B (which is not true in general), and also that $\sigma_0(A) \cdot \sigma_0(B) \subseteq \sigma_0(A \otimes B)$ (which may fail even if the Weyl spectrum identity holds and all operators are isoloid and satisfy Weyl's theorem). However, under these extra two assumptions, the problem of transferring Weyl's theorem from isoloid operators A and B to their tensor product $A \otimes B$ survives.

Proposition 1. *Let A and B be operators acting on infinite-dimensional spaces. If*

(a) *A and B are isoloid,*

(b) *A and B satisfy Weyl's theorem, and*

(c) *the Weyl spectrum identity holds for A and B,*

then the tensor product $A \otimes B$ satisfies Weyl's theorem if, in addition,

(d) $\sigma_0(A \otimes B) = \sigma_0(A) \cdot \sigma_0(B)$.

Proof. [7, Corollary 5]. □

Question: Is condition (d) necessary in Proposition 1? In other words, is the next conjecture true?

Conjecture 1. *If A and B are isoloid, satisfy Weyl's theorem, and the Weyl spectrum identity holds, then $A \otimes B$ satisfies Weyl's theorem.*

We shall show in the next section (Theorem 1) that the above conjecture is, in fact, a theorem: it holds true for operators acting on infinite-dimensional spaces. (If the spaces are finite-dimensional, where every subspace is closed and every operator is isoloid with spectrum consisting only of eigenvalues of finite multiplicity, then the conjecture holds trivially.) The proof is carried out by using results originally proved in a Hilbert space setting, which clearly still hold in a Banach space setting.

3. The Answer

Before proving Theorem 1 we need the following auxiliary results. Let A, B, and T be arbitrary operators.

Lemma 1. *If A and B satisfy Browder's theorem, and if the Weyl spectrum identity holds, then the tensor product $A \otimes B$ satisfies Browder's theorem.*

Proof. [7, Proposition 7(a)]. □

Lemma 2. *If T satisfies Browder's but not Weyl's theorem, then*

$$\sigma_w(T) \cap \pi_{00}(T) \neq \varnothing.$$

Proof. T satisfies Browder's but not Weyl's theorem if and only if the proper inclusion $\sigma_0(T) \subset \pi_{00}(T)$ holds true, which implies that there exists an isolated eigenvalue of finite multiplicity not in $\sigma_0(T)$ (i.e., in $\sigma_w(T) = \sigma(T)\backslash\sigma_0(T)$). □

Lemma 3. *If A and B are isoloid operators on infinite-dimensional spaces, then*

$$\pi_{00}(A \otimes B) \subseteq \pi_{00}(A) \cdot \pi_{00}(B).$$

Proof. [7, Proposition 4]. □

Lemma 4. *If A and B are operators on infinite-dimensional spaces, then*

$$0 \notin \sigma_{PF}(A \otimes B).$$

Proof. [7, Proposition 1]. □

Lemma 5. *Suppose $\sigma_{\rm iso}(A) \neq \varnothing$ and $\sigma_{\rm iso}(B) \neq \varnothing$. If $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$, and $\lambda\mu \in \sigma_{\rm iso}(A \otimes B)$, then*

$$\lambda \in \sigma_{\rm iso}(A) \quad \textit{and} \quad \mu \in \sigma_{\rm iso}(B).$$

Proof. [7, Proof of Proposition 3(a)]. □

Lemma 6. *If $\lambda \in \sigma_P(A)$, $\mu \in \sigma_P(B)$, and $\lambda\mu \in \sigma_{PF}(A \otimes B)$, then*

$$\lambda \in \sigma_{PF}(A) \quad \textit{and} \quad \mu \in \sigma_{PF}(B).$$

Proof. [7, Proof of Proposition 2(a)]. □

Suppose A and B are operators acting on infinite-dimensional spaces.

Theorem 1. *If A and B are isoloid, satisfy Weyl's theorem, and the Weyl spectrum identity holds, then $A \otimes B$ satisfies Weyl's theorem.*

Proof. Suppose A and B satisfy Weyl's theorem. Thus, A and B satisfy Browder's theorem. Therefore, if the Weyl's spectrum identity holds, then $A \otimes B$ satisfies Browder's theorem according to Lemma 1. Suppose $A \otimes B$ does not satisfy Weyl's theorem. Then it follows by Lemma 2 that

$$\sigma_w(A \otimes B) \cap \pi_{00}(A \otimes B) \neq \varnothing.$$

Since the Weyl's spectrum identity holds; that is, since

$$\sigma_w(A \otimes B) = \sigma_w(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_w(B),$$

it follows that every product $\lambda\mu$ with $\lambda \in \sigma_w(A)$ and $\mu \in \sigma(B)$ or with $\lambda \in \sigma(A)$ and $\mu \in \sigma_w(B)$ lies in $\sigma_w(A \otimes B)$. Thus, $\nu \in \sigma_w(A \otimes B)$ if and only if $\nu = \lambda\mu$ for arbitrary λ, μ such that $(\lambda, \mu) \in \sigma_w(A) \times \sigma(B)$ or $(\lambda, \mu) \in \sigma(A) \times \sigma_w(B)$. First suppose $\nu \in \sigma_w(A \otimes B)$ is such that

$$\nu = \lambda\mu \quad \text{for an arbitrary pair} \quad (\lambda, \mu) \in \sigma_w(A) \times \sigma(B).$$

If $\nu \in \pi_{00}(A \otimes B)$, then

$$\nu \in \pi_{00}(A) \cdot \pi_{00}(B)$$

by Lemma 3 because A and B are isoloid, so that

$$\nu = \lambda'\mu' \quad \text{for some pair} \quad (\lambda', \mu') \in \pi_{00}(A) \times \pi_{00}(B).$$

Thus, if

$$\nu \in \sigma_w(A \otimes B) \cap \pi_{00}(A \otimes B),$$

then it follows by Lemma 4 that $0 \neq \nu = \lambda\mu = \lambda'\mu'$, with

$$\lambda = \tfrac{\nu}{\mu} \in \sigma_w(A), \qquad \lambda' = \tfrac{\nu}{\mu'} \in \pi_{00}(A) = \sigma_0(A),$$
$$\mu = \tfrac{\nu}{\lambda} \in \sigma(B), \qquad \mu' = \tfrac{\nu}{\lambda'} \in \pi_{00}(B) = \sigma_0(B),$$

because A and B satisfy Weyl's theorem. Thus, $\lambda \neq \lambda'$, since they live in complementary sets, and so $\mu \neq \mu'$. Since

$$\sigma_{\rm iso}(A) \neq \varnothing \quad \text{and} \quad \sigma_{\rm iso}(B) \neq \varnothing$$

(because $\lambda' \in \pi_{00}(A) \subseteq \sigma_{\rm iso}(A)$ and $\mu' \in \pi_{00}(B) \subseteq \sigma_{\rm iso}(B)$), and since

$$\nu = \lambda\mu \in \pi_{00}(A \otimes B) \subseteq \sigma_{\rm iso}(A \otimes B),$$

it follows by Lemma 5 that

$$\lambda \in \sigma_{\rm iso}(A) \quad \text{and} \quad \mu \in \sigma_{\rm iso}(B).$$

Recalling again that A an B are isoloid, we get

$$\lambda \in \sigma_P(A) \quad \text{and} \quad \mu \in \sigma_P(B).$$

Thus, since $\lambda\mu = \nu \in \pi_{00}(A \otimes B) \subseteq \sigma_{PF}(A \otimes B)$, it follows by Lemma 6 that

$$\lambda \in \sigma_{PF}(A) \quad \text{and} \quad \mu \in \sigma_{PF}(B).$$

Therefore, since A satisfies Weyl's theorem,

$$\lambda \in \sigma_w(A) \cap \sigma_{\rm iso}(A) \cap \sigma_{PF}(A) = \sigma_w(A) \cap \pi_{00}(A) = \sigma_w(A) \cap \sigma_0(A) = \varnothing,$$

which is a contradiction. On the other hand, if $\nu \in \sigma_w(A \otimes B)$ is such that

$$\nu = \lambda\mu \;\text{ for an arbitrary pair }\; (\lambda, \mu) \in \sigma(A) \times \sigma_w(B),$$

then, similarly and symmetrically,

$$\mu \in \sigma_w(B) \cap \sigma_{\rm iso}(B) \cap \sigma_{PF}(B) = \sigma_w(B) \cap \pi_{00}(B) = \sigma_w(B) \cap \sigma_0(B) = \varnothing,$$

which is again a contradiction because B satisfies Weyl's theorem as well. Therefore, if A and B are isoloid, both satisfy Weyl's theorem, and the Weyl spectrum identity holds, then $A \otimes B$ must satisfy Weyl's theorem. □

Remark. The a-version of Theorem 1 was considered in [2]: If A and B are a-isoloid, satisfy a-Weyl's theorem, and the a-Weyl spectrum identity holds, then $A \otimes B$ satisfies a-Weyl's theorem (where the prefix "a" means that spectrum is replaced with approximate point spectrum in every definition).

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