Advances in Differential Equations and Control Processes **11** (2013) 135–144

s-MATRIX INVERSION LEMMA

NHAN LEVAN AND CARLOS S. KUBRUSLY

Abstract. It is shown that a state-feedback time-invariant linear system has its built-in s-Matrix Inversion Lemma which results directly in the system transfer matrix — without using the standard Matrix Inversion Lemma itself.

1. Introduction

This note deals with the well-known Matrix Inversion Lemma (see e.g., [5, Section 5.1], [7, Appendix A], [4], [3], and the references therein). This is a classical and extremely useful result, with an elementary proof, which, in its simplest form, reads as follows. Let L and T be linear transformations of a linear space into itself. If $I - LT$ is invertible, then so is $I - TL$. Moreover, in this case,

$$(I - TL)^{-1} = I + T(I - LT)^{-1}L.$$

In fact, this holds in any ring with identity (see e.g., [6, Problem 2.32]). The above relation can be easily extended as follows. Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces (over the same field), and let $S\colon \mathcal{X} \to \mathcal{X}$, $T\colon \mathcal{Y} \to \mathcal{X}$, and $L\colon \mathcal{X} \to \mathcal{Y}$ be linear transformations (in particular, let S be a square matrix and T and L be matrices with compatible dimensions). If S and $S - TL$ are invertible (so that $I - LS^{-1}T$ is invertible), then

$$(S - TL)^{-1} = S^{-1} + S^{-1}T(I - LS^{-1}T)^{-1}LS^{-1}.$$

The proof is essentially the same as that of the simplest form, consisting of mere algebraic manipulations. Indeed and tautologically, by setting $S = I$ above, we get the former simplest form. Conversely, by setting $L = L'S^{-1}$ in the former simplest form and multiplying it from the left by S^{-1} we get the latter form — and therefore, as the invertibility of $I - LT$ implies the invertibility $I - TL$, then so the invertibility of S and $S - TL$ together imply the invertibility of $I - LS^{-1}T$.

The Matrix Inversion Lemma (MIL) has been used in many problems including calculation of the transfer matrix of a state-feedback system (see e.g., [2]). It is not the purpose of this note to offer manipulations on how to compute, for a finite-dimensional linear system, the closed loop transfer matrix after static state feedback and static output feedback, nor to offer another proof of the well-known Matrix inversion Lemma, whose all proofs are certainly very elementary. The aim of this note is to show that a state-feedback system has its own built-in Matrix Inversion Lemma (called s-MIL), which readily results in an expression for its transfer matrix without using the original MIL. This is shown in Theorem 1 (Section 2): a state-feedback time-invariant linear system generates a version of the Matrix Inversion Lemma (the s-MIL). Consequences of Theorem 1 on the transfer matrix of state and output feedback systems are considered in Corollaries 1 and 2 (Section 3).

In other words, the outcome of the forthcoming Theorem 1 is a version of the Matrix Inversion Lemma whose proof was carried out, of course, without using the original Matrix Inversion Lemma. Actually, what Theorem 1 offers is quite

Date: March 25, 2013.
2010 *Mathematics Subject Classification.* Primary 15A09; Secondary 93C05.
Keywords. Matrix Inversion Lemma, state and output feedback, time-invariant linear system.

near the original Matrix Inversion Lemma (with respect to the matrices under consideration). The forthcoming Remark 1 exhibits the original Matrix Inversion Lemma (again with respect to the matrices under consideration), which trivially leads to the result of Theorem 1 (by multiplying the expression of Remark 1 on the right by B). The converse holds in the sense that if B is, for instance, invertible, then the expression of Theorem 1 leads to the expression in Remark 1. Section 3 presents a couple of results that are obtained via Theorem 1. Therefore, and unlike the usual practice, these results in Corollaries 1 and 2 are now verified without requiring the original Matrix Inversion Lemma.

Summing up: In Section 2 we derive an alternate form of the Matrix Inversion Lemma for time-invariant linear systems (whose proof does not use the original Matrix Inversion Lemma), and in Section 3 this is applied to recover a couple of results by presenting new (and still elementary) verifications that use Theorem 1 instead of using the original Matrix Inversion Lemma.

2. s-Matrix Inversion Lemma

From now on we proceed formally. Consider the state-space description of a time-invariant linear system $\mathcal{S}$,

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{2.1}$$

for $t \geq 0$, where $x(\cdot)$, $u(\cdot)$, and $y(\cdot)$ are state, input and output, vector-valued functions, respectively (in an appropriate space of finite-dimensional-valued functions on $[0, \infty)$), while A is a square matrix, B and C are possibly rectangular with compatible dimensions. The transfer matrix $G(\cdot)$ of $\mathcal{S}$ is given by (see e.g., [2])

$$G(s) = C(sI - A)^{-1}B$$

for every $s \notin \sigma(A)$ — equivalently for every $s \in \rho(A)$ — where $\sigma(A)$ stands for the set of all eigenvalues of A (i.e., for the spectrum of the finite-dimensional operator A), and $\rho(A) = \mathbb{C}\backslash\sigma(A)$ is the resolvent set of A, so that $sI - A$ is invertible if and only if $s \in \rho(A)$. (Extensions to systems acting on infinite-dimensional normed spaces with *bounded* linear transformations are straightforward.)

A closed-loop (full) state-feedback system $\mathcal{S}_K$ is derived from $\mathcal{S}$ by setting its input $u(\cdot)$ to be of the feedback form,

$$u(t) = r(t) - Kx(t),$$

for $t \geq 0$, where $r(\cdot)$ is an arbitrary reference input to $\mathcal{S}_K$, and K is a given state-feedback gain matrix. Thus, plugging this feedback input $u(\cdot)$ into (2.1), we get the following state-space description for $\mathcal{S}_K$.

$$\dot{x}(t) \;=\; Ax(t) + Br(t) - BKx(t) \;=\; (A - BK)x(t) + Br(t), \qquad y(t) = Cx(t), \tag{2.2}$$

for every $t \geq 0$. The transfer matrix $G_K(\cdot)$ of $\mathcal{S}_K$ is then [2]

$$G_K(s) = C[sI - (A - BK)]^{-1}B,$$

for every $s \notin \sigma(A-B)$ (i.e., for every $s \in \rho(A-BK)$). This is the place where, in general, the original Matrix Inversion Lemma can be used to compute the matrix $[sI-(A-BK)]^{-1}$. However, as we now show, this is not necessary.

Let the Laplace transform of a function $f(\cdot)$ be denoted by $F(\cdot)$, so that $X(\cdot)$ and $R(\cdot)$ stand for the Laplace transforms of $x(\cdot)$ and $r(\cdot)$. Set, for every $s \in \mathbb{C}$,

$$\mathcal{A}(s) = sI - A,$$

and take the Laplace transforms of (2.2), with $x(0)=0$, so that

$$\mathcal{A}(s)X(s) \;=\; (sI-A)X(s) \;=\; BR(s) - BKX(s)$$

or, equivalently,

$$[\mathcal{A}(s)+BK]X(s) \;=\; [sI-(A-BK)]X(s) \;=\; BR(s).$$

Thus, if $s \in \rho(A)$, then

$$X(s) = \mathcal{A}(s)^{-1}BR(s) - \mathcal{A}(s)^{-1}BKX(s), \tag{2.3}$$

and, if $s \in \rho(A-BK)$, then

$$X(s) = [\mathcal{A}(s)+BK]^{-1}BR(s).$$

Therefore, if $s \in \rho(A) \cap \rho(A-BK)$ (i.e., if $s \notin \sigma(A) \cup \sigma(A-BK)$), then

$$[\mathcal{A}(s)+BK]^{-1}BR(s) = \mathcal{A}(s)^{-1}BR(s) - \mathcal{A}(s)^{-1}BKX(s). \tag{2.4}$$

The above equation will yield an identity for $[\mathcal{A}(s)+BK]^{-1}B$, as soon as the vector $KX(s)$ on the right can be expressed in terms of the vector $R(s)$.

Theorem 1. *The state-feedback system $\mathcal{S}_K$ with state-space description*

$$\dot{x}(t) = (A-BK)x(t) + Br(t), \qquad y(t) = Cx(t),$$

for every $t \geq 0$, has its own built-in s-Matrix Inversion Lemma, viz.,

$$\begin{aligned}&[sI-A+BK]^{-1}B\\ &= \; (sI-A)^{-1}B - (sI-A)^{-1}B[I+K(sI-A)^{-1}B]^{-1}K(sI-A)^{-1}B;\end{aligned}$$

equivalently, by setting $\mathcal{A}(s) = sI-A$ for every $s \in \mathbb{C}$,

$$[\mathcal{A}(s)+BK]^{-1}B = \mathcal{A}(s)^{-1}B - \mathcal{A}(s)^{-1}B[I+K\mathcal{A}(s)^{-1}B]^{-1}K\mathcal{A}(s)^{-1}B;$$

for every $s \notin \sigma(A)$ and $s \notin \sigma(A-BK)$ (i.e., *for every $s \in \rho(A) \cap \rho(A-BK)$*).

Proof. First, acting K on both sides of (2.3) we obtain, for $s \notin \sigma(A)$,

$$KX(s) = K\mathcal{A}(s)^{-1}BR(s) - K\mathcal{A}(s)^{-1}BKX(s),$$

Therefore, if $s \notin \sigma(A)$ and $s \notin \sigma(A-BK)$, then

$$KX(s) = [I + K\mathcal{A}(s)^{-1}B]^{-1}K\mathcal{A}(s)^{-1}BR(s),$$

where the invertibility of $I + K\mathcal{A}(s)^{-1}B$ is ensured by the invertibility of $\mathcal{A}(s)$ and $\mathcal{A}(s) + BK$, as we saw in Section 1. The preceding expression and (2.4) lead to

$$\begin{aligned}&[\mathcal{A}(s) + BK]^{-1}BR(s)\\ &= \mathcal{A}(s)^{-1}BR(s) - \mathcal{A}(s)^{-1}B[I + K\mathcal{A}(s)^{-1}B]^{-1}K\mathcal{A}(s)^{-1}BR(s).\end{aligned}$$

Since the above expression holds for every admissible reference input signal $r(\cdot)$, and since we may choose it such that $R(s) = I$, we get the claimed result. □

Remark 1. If $\mathcal{A}(s)$ and $\mathcal{A}(s) + BK$ are invertible (i.e., if $s \in \rho(A) \cap \rho(A - BK)$), then the original Matrix Inversion Lemma of Section 1 ensures that

$$[\mathcal{A}(s) + BK]^{-1} = \mathcal{A}(s)^{-1} - \mathcal{A}(s)^{-1}B[I + K\mathcal{A}(s)^{-1}B]^{-1}K\mathcal{A}(s)^{-1},$$

which leads to the identity stated in Theorem 1 (by multiplying the above identity on the right by B). Conversely, if B is well-behaved (e.g., if B is invertible), then the identity that was deduced in Theorem 1 without using the original Matrix Inversion Lemma leads to the above identity, which is precisely the original Matrix Inversion Lemma (for the matrices under consideration). It is in this sense that we say: *a state-feedback time-invariant linear system generates a version of the Matrix Inversion Lemma* (the so-called s-Matrix Inversion Lemma of Theorem 1).

3. Concluding Remarks

Straightforward consequences of Theorem 1 are stated in Corollaries 1 and 2 below. Thus, as applications of Theorem 1, we can get new (and still elementary) verifications of them, following directly from Theorem 1, and so not requiring the use of the original Matrix Inversion Lemma.

Corollary 1. *The transfer matrix of the state-feedback system S_K is*

$$\begin{aligned}G_K(s) &= C[sI - (A - BK)]^{-1}B,\\ &= G(s) - G(s)\big[I + K(sI - A)^{-1}B\big]^{-1}K(sI - A)^{-1}B.\end{aligned}$$

The above suggests the existence of a feedback-like set-up behind the MIL. Indeed, consider the following set-up.

$$Ax = Bu, \qquad u = r - Kx.$$

Thus

$$Ax = Br - BKx, \tag{3.1}$$

and so,

$$(A + BK)x = Br. \tag{3.2}$$

If A and $(A + BK)$ are invertible, then (3.1) leads to,

$$x = A^{-1}Br - A^{-1}BKx \tag{3.3}$$

and, from (3.2),

$$x = (A + BK)^{-1}Br. \tag{3.4}$$

Hence, as in the proof of Theorem 1, equating (3.3) and (3.4) we get

$$(A + BK)^{-1}Br = A^{-1}Br - A^{-1}BKx. \tag{3.5}$$

The final step is, as before, to substitute for Kx on the right by that obtained from acting K on both sides of (3.3). That is,

$$Kx = KA^{-1}Br - KA^{-1}BKx,$$

so that

$$(I + KA^{-1}B)Kx = Kx + KA^{-1}BKx = KA^{-1}Br,$$

and hence

$$Kx = (I + KA^{-1}B)^{-1}KA^{-1}Br. \tag{3.6}$$

We finally get the MIL from (3.5) and (3.6), as expected:

$$(A + BK)^{-1}B = A^{-1}B - A^{-1}B(I + KA^{-1}B)^{-1}KA^{-1}B.$$

We now consider the output-feedback via state-feedback. Given an open-loop system $\mathcal{S}$ with the input-output description in the s-domain [2],

$$Y(s) = G(s)U(s)$$

for $s \in \mathbb{C}$, where, as before, $U(\cdot)$ and $Y(\cdot)$ denote the Laplace transform of the input $u(\cdot)$ and output $y(\cdot)$, respectively, while $G(\cdot)$ is the system transfer matrix. The associated closed-loop output-feedback system $\mathcal{S}_E$ is characterized by

$$Y(s) = G_E(s)R(s), \qquad R(s) - EY(s) = U(s), \tag{3.7}$$

for $s \in \mathbb{C}$, where $R(\cdot)$ the Laplace transform of a reference input $r(\cdot)$ to $\mathcal{S}_E$, and E is a feedback gain matrix, while $G_E(\cdot)$ is the transfer matrix of $\mathcal{S}_E$. Now since the open-loop system $\mathcal{S}$ also has a state-space description characterized by (2.1), we have, in addition to (3.7),

$$Y(s) = CX(s).$$

for $s \in \mathbb{C}$. This and (3.7) result in a state-feedback characterization in the s-domain,

$$U(s) = R(s) - ECX(s)$$

for $s \in \mathbb{C}$ or, equivalently, in the time-domain,

$$u(t) = r(t) - ECx(t)$$

for $t \geq 0$. Comparing this with the input in the feedback form of Section 2, namely,

$$u(t) = r(t) - Kx(t)$$

for $t \geq 0$, we conclude that an output-feedback system — with the feedback gain matrix E — is also a state-feedback system — with the feedback gain matrix $K = EC$. The next result then follows easily from Theorem 1.

Corollary 2. (i) *The s-MIL resulting from an output-feedback system $\mathcal{S}_E$ — with the gain matrix E — is*

$$[sI - (A - BEC)]^{-1}B = (sI - A)^{-1}B - (sI - A)^{-1}B \cdot$$
$$= [I + EC(sI - A)^{-1}B]^{-1}EC(sI - A)^{-1}B.$$

(ii) *The transfer matrix $G_E(s)$ of $\mathcal{S}_E$ is therefore*

$$G_E(s) = C\big[sI - (A - BEC)\big]^{-1}B = G(s) - G(s)[I + EG(s)]^{-1}EG(s),$$

where

$$G(s) = C(sI - A)^{-1}B$$

is the transfer matrix of the open-loop system $\mathcal{S}$.

We have shown that a state-feedback system has an s-Matrix Inversion Lemma which results directly in the transfer matrix of the system — without using the original Matrix Inversion Lemma. There are various proofs of the Matrix Inversion Lemma, all of them rather elementary. However, only Boyd and Vandenberghe [1] consider the following set-up for deriving it for the matrix $A + BC$.

$$(A + BC)x = b, \qquad y = Cx.$$

This can be identified with the following output-feedback setup.

$$Ax = u, \qquad y = Cx, \qquad u = b - By.$$

Acknowledgment

The authors thank O.L.V. Costa who did a careful reading of the manuscript, raising many sensible remarks, and suggesting important modifications.

References

1. S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.
2. J.B. Burl, *Linear Optimal Control*, Addison Wesley, Menlo Park, 1999.
3. W.W. Hager, *Updating the inverse of a matrix*, SIAM Review **31** (1989), 221–239.
4. H.V. Henderson and S.R. Searle, *On deriving the inverse of a sum of matrices*, SIAM Review **23** (1981), 53–60.
5. A.S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964.
6. C.S. Kubrusly, *The Elements of Operator Theory*, Birkhäuser/Springer, New York, 2011.
7. A.P. Sage and J.L. Melsa, *Estimation Theory with Applications to Communications and Control*, MaGraw-Hill, New York, 1971.

Department of Electrical Engineering, University of California in Los Angeles, Los Angeles, CA 90024-1594, USA
E-mail address: levan@ee.ucla.edu

Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: carlos@ele.puc-rio.br