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A NOTE ON BROWDER SPECTRUM

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Abstract. In this paper we supply a new proof that the Browder spectrum is the largest part of the spectrum that remains unchanged under compact perturbations from the commutant.

1. Introduction

The Weyl spectrum $\sigma_w(T)$ of an operator T is the largest part of the spectrum $\sigma(T)$ that is invariant under compact perturbations. Thus the Browder spectrum $\sigma_b(T)$ is not invariant under compact perturbation (reason: $\sigma_w(T) \subseteq \sigma_b(T)$ and the inclusion may be proper). However, the Browder spectrum is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations from the commutant of T. This result was announced in [11], whose proof, based on ascent-descent techniques, appeared in [7]. In the realm of pure Banach algebra techniques, the result was considered in [3]. The main purpose of this paper is to offer a new proof, along the lines of [3], but using single operator theory techniques instead, thus supplying a simpler complete proof. This is done in Section 3. The necessary notational preliminaries are presented in Section 2, and the paper closes in Section 4 with a remark on the answer of a long awaited open question on the Weyl spectrum of tensor products. We work in a Hilbert space setting, although the results in this paper hold in a Banach space setting with essentially the same proofs.

2. Preliminaries

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, and $\mathcal{B}[\mathcal{H}]$ the unital Banach algebra of all (bounded linear) operators on $\mathcal{H}$, where I denotes the identity in $\mathcal{B}[\mathcal{H}]$. Let $\mathcal{B}_\infty[\mathcal{H}]$ be the ideal of all compact operators from $\mathcal{B}[\mathcal{H}]$, and let

$$\begin{aligned}\mathcal{F} &= \big\{T \in \mathcal{B}[\mathcal{H}]\colon\ I - AT \text{ and } I - TA \text{ are compact for some } A \in \mathcal{B}[\mathcal{H}]\big\}\\ &= \big\{T \in \mathcal{B}[\mathcal{H}]\colon\ \mathcal{R}(T) \text{ is closed, } \dim\mathcal{N}(T) < \infty \text{ and } \dim\mathcal{N}(T^*) < \infty\big\}\end{aligned}$$

stand for the class of all Fredholm operators from $\mathcal{B}[\mathcal{H}]$ (see e.g., [1, Remark 3.33] and [4, Corollary XI.2.4]), where $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote range and kernel, and $T^* \in \mathcal{B}[\mathcal{H}]$ the adjoint, of an arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$. The Fredholm index of an operator T in $\mathcal{F}$ is the integer $\operatorname{ind}(T) = \dim\mathcal{N}(T) - \dim\mathcal{N}(T^*)$. A Weyl operator is a Fredholm operator of index zero. Set

$$\mathcal{W} = \big\{T \in \mathcal{F}\colon\ \operatorname{ind}(T) = 0\big\},$$

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the class of all Weyl operators from $\mathcal{B}[\mathcal{H}]$. Consider the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, and let $\pi\colon \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ be the natural quotient map. The essential spectrum $\sigma_e(T)$ of T is the spectrum of $\pi(T)$ in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. By the Atkinson Theorem,

$$\sigma_e(T) \,=\, \sigma(\pi(T)) \,=\, \big\{\lambda\in\mathbb{C}\colon\; \lambda I - T \in \mathcal{B}[\mathcal{H}]\backslash\mathcal{F}\big\} \,\subseteq\, \sigma(T)$$

(see e.g., [9, Corollary 4.2]), where $\sigma(T)$ is the spectrum of T in $\mathcal{B}[\mathcal{H}]$. Set

$$\sigma_0(T) \,=\, \big\{\lambda\in\sigma(T)\colon\; \lambda I - T \in \mathcal{W}\big\} \,\subseteq\, \sigma_{PF}(T) \,\subseteq\, \sigma_P(T),$$

where $\sigma_P(T)$ is the point spectrum, and $\sigma_{PF}(T)$ is the set of eigenvalues of finite multiplicity, of T. Let $\sigma_{\rm iso}(T)$ be the set of isolated points of $\sigma(T)$, and consider the Riesz idempotent associated with $\lambda\in\sigma_{\rm iso}(T)$, $E_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda}(\gamma I - T)^{-1}d\gamma$, where Γ_λ is any positively oriented circle enclosing λ but no other point of $\sigma(T)$. Let

$$\pi_0(T) \,=\, \sigma_{\rm iso}(T)\cap\sigma_0(T) \,=\, \big\{\lambda\in\sigma_{\rm iso}(T)\colon\; \dim\mathcal{R}(E_\lambda)<\infty\big\}$$

be the set of Riesz points of T (see e.g., [2, Proposition 2]), and set

$$\pi_{00}(T) \,=\, \sigma_{\rm iso}(T)\cap\sigma_{PF}(T),$$

the set of all isolated eigenvalues of T of finite multiplicity. Clearly, $\pi_0(T)\subseteq\pi_{00}(T)$. The Weyl spectrum $\sigma_w(T)$ of T is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations, which, according to Schechter's Theorem, coincides with the complement of $\sigma_0(T)$ in $\sigma(T)$ (see e.g., [9, Propositions 7.2 and 7.4]),

$$\sigma_w(T) = \bigcap\nolimits_{K\in\mathcal{B}_\infty[\mathcal{H}]}\sigma(T+K) \,=\, \sigma(T)\backslash\sigma_0(T) \,=\, \big\{\lambda\in\mathbb{C}\colon\; \lambda I - T\in\mathcal{B}[\mathcal{H}]\backslash\mathcal{W}\big\}.$$

A Browder operator is often defined as a Fredholm operator with finite ascent and finite descent. Equivalently, a Browder operator is a Fredholm operator such that, if zero is in the spectrum, then it is an isolated point (see e.g., [5, Theorem 7.9.3]) or, still equivalently, if zero is in the spectrum, then it is a Riesz point (since the set $\sigma_w(T)\backslash\sigma_e(T)$ is open [4, Proposition XI.6.11]). Thus the class $\mathcal{B}$ of all Browder operators from $\mathcal{B}[\mathcal{H}]$ is given by

$$\mathcal{B} = \big\{T\in\mathcal{F}\colon\; 0\in\rho(T)\cup\sigma_{\rm iso}(T)\big\} = \big\{T\in\mathcal{F}\colon\; 0\in\rho(T)\cup\pi_0(T)\big\},$$

where $\rho(T)=\mathbb{C}\backslash\sigma(T)$ is the resolvent set of T. Define the Browder spectrum of T as the set $\sigma_b(T)$ of all complex numbers λ for which $\lambda I - T$ is not Browder, which coincides with the complement of $\pi_0(T)$ in $\sigma(T)$ (see e.g., [9, Corollary 9.4]),

$$\sigma_b(T) \,=\, \big\{\lambda\in\mathbb{C}\colon\; \lambda I - T\in\mathcal{B}[\mathcal{H}]\backslash\mathcal{B}\big\} \,=\, \sigma(T)\backslash\pi_0(T).$$

Since $\mathcal{G}[\mathcal{H}]\subseteq\mathcal{B}\subseteq\mathcal{W}\subseteq\mathcal{F}$, where $\mathcal{G}[\mathcal{H}]$ is the group of all invertible operators from $\mathcal{B}[\mathcal{H}]$, it follows that $\sigma_e(T)\subseteq\sigma_w(T)\subseteq\sigma_b(T)\subseteq\sigma(T)$.

3. Browder Spectrum

In this section we supply a new proof that the Browder spectrum of T is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations from the commutant of T (proof of Theorem 1). What makes this proof different is that it does not rely on ascent-descent calculations, and uses single operator theory techniques, instead of using only pure Banach algebra techniques, which makes it simpler and more elementary. The complete proof is based on Lemmas 1 and 2.

Set $\mathcal{A} = \mathcal{B}[\mathcal{H}]$. Let $\mathcal{A}'$ be a unital closed subalgebra of the unital complex Banach algebra $\mathcal{A}$, thus a unital complex Banach algebra itself. Take an arbitrary operator T in $\mathcal{A}'$. Let $\mathcal{B}_\infty[\mathcal{H}]' = \mathcal{B}_\infty[\mathcal{H}] \cap \mathcal{A}'$ denote the collection of all compact operators from $\mathcal{A}'$, and let $\mathcal{F}'$ denote the class of all Fredholm operators in $\mathcal{A}'$. That is,

$$\mathcal{F}' = \big\{T \in \mathcal{A}' \colon\; I - AT \text{ and } I - TA \text{ are in } \mathcal{B}_\infty[\mathcal{H}]' \text{ for some } A \in \mathcal{A}'\big\}.$$

It is clear that $\mathcal{F}' \subseteq \mathcal{F} \cap \mathcal{A}'$, and the inclusion is proper in general. However, by the Atkinson Theorem, if $\mathcal{F}' = \mathcal{F} \cap \mathcal{A}'$, then $\sigma_e'(T) = \sigma_e(T)$, where

$$\sigma_e'(T) = \big\{\lambda \in \sigma'(T) \colon\; \lambda I - T \in \mathcal{A}' \backslash \mathcal{F}'\big\}$$

stands for the essential spectrum of $T \in \mathcal{A}'$ with respect to $\mathcal{A}'$, with $\sigma'(T)$ denoting the spectrum of $T \in \mathcal{A}'$ with respect to the unital complex Banach algebra $\mathcal{A}'$. Let $\mathcal{W}' = \big\{T \in \mathcal{F}' \colon\; \operatorname{ind}(T) = 0\big\}$ denote the class of Weyl operators in $\mathcal{A}'$, and let

$$\sigma_w'(T) = \big\{\lambda \in \sigma'(T) \colon\; \lambda I - T \in \mathcal{A}' \backslash \mathcal{W}'\big\}$$

stand for the Weyl spectrum of $T \in \mathcal{A}'$ with respect to $\mathcal{A}'$. If $T \in \mathcal{A}'$, then set

$$\sigma_0'(T) \;=\; \sigma'(T) \backslash \sigma_w'(T) \;=\; \big\{\lambda \in \sigma'(T) \colon\; \lambda I - T \in \mathcal{W}'\big\}.$$

Moreover, let $\mathcal{B}'$ stand for the class of Browder operators in $\mathcal{A}'$,

$$\mathcal{B}' = \big\{T \in \mathcal{F}' \colon\; 0 \in \rho'(T) \cup \sigma_{\rm iso}'(T)\big\}$$

where $\sigma_{\rm iso}'(T)$ denotes the set of all isolated points of $\sigma'(T)$, and $\rho'(T) = \mathbb{C} \backslash \sigma'(T)$ is the resolvent set of $T \in \mathcal{A}'$ with respect to $\mathcal{A}'$. Let

$$\sigma_b'(T) = \big\{\lambda \in \sigma'(T) \colon\; \lambda I - T \in \mathcal{A}' \backslash \mathcal{B}'\big\},$$

be the Browder spectrum of $T \in \mathcal{A}'$ with respect to $\mathcal{A}'$.

Lemma 1. *Take any $T \in \mathcal{A}'$. If $\sigma'(T) = \sigma(T)$, then $\sigma_b'(T) = \sigma_b(T)$.*

Proof. Take T in $\mathcal{A}'$. Suppose $\sigma'(T) = \sigma(T)$. If $\lambda \in \sigma_{\rm iso}'(T)$, then the Riesz idempotent associated with it, $E_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda}(\gamma I - T)^{-1} d\gamma$, lies in $\mathcal{A}'$ (reason: $\sigma_{\rm iso}'(T) = \sigma_{\rm iso}(T)$, and $(\nu I - T)^{-1} \in \mathcal{A}'$ whenever $(\nu I - T)^{-1}$ lies in $\mathcal{B}[\mathcal{H}]$ since $\rho'(T) = \rho(T)$). Thus, according to the expressions for $\pi_0(T)$, if $\lambda \in \sigma_{\rm iso}'(T) = \sigma_{\rm iso}(T)$, then

$$\lambda \in \sigma_0'(T) \quad\iff\quad \dim \mathcal{R}(E_\lambda) < \infty \quad\iff\quad \lambda \in \sigma_0(T).$$

Hence $\pi_0'(T) = \sigma_0'(T) \cap \sigma_{\rm iso}'(T) = \sigma_0(T) \cap \sigma_{\rm iso}(T) = \pi_0(T)$. Thus, taking the complement of the set of Riesz points, $\sigma_b(T) = \sigma(T) \backslash \pi_0(T) = \sigma'(T) \backslash \pi_0'(T) = \sigma_b'(T)$. $\square$

Let $\{T\}'$ be the commutant of $T \in \mathcal{B}[\mathcal{H}]$, which is a unital closed (in fact, weakly closed) subalgebra of the unital complex Banach algebra $\mathcal{B}[\mathcal{H}]$.

Lemma 2. *Take any $T \in \mathcal{B}[\mathcal{H}]$. If $\mathcal{A}' = \{T\}'$ and $0 \in \sigma_0'(T)$, then $0 \in \sigma_{\rm iso}'(T)$.*

Proof. Take $T \in \mathcal{B}[\mathcal{H}]$. Let $\mathcal{A}'$ be a unital closed subalgebra of $\mathcal{A} = \mathcal{B}[\mathcal{H}]$ including T. Suppose $0 \in \sigma_0'(T)$, which means that $0 \in \sigma'(T)$ and $T \in \mathcal{W}'$. Since $T \in \mathcal{W}'$, there exists a compact $K \in \mathcal{B}_\infty[\mathcal{H}]' = \mathcal{B}_\infty[\mathcal{H}] \cap \mathcal{A}'$, actually a finite-rank operator, such that $T + K$ is invertible (see e.g., [4, Exercise XI.3.7]. That is,

$$A(T + K) = (T + K)A = I$$

for some $A \in \mathcal{A}'$, and so A is itself invertible with inverse $A^{-1} = T + K \in \mathcal{A}'$. If $\mathcal{A}' = \{T\}'$, then $AK = KA$, and so $A^{-1}K = KA^{-1}$. Let $\mathcal{A}''$ be the unital closed commutative subalgebra of $\mathcal{A}'$ generated by A, A^{-1}, and K. Since $T = A^{-1} - K$, it follows that $\mathcal{A}''$ includes T. Let $\sigma'_{\rm iso}(T)$ and $\sigma''_{\rm iso}(T)$ stand for the sets of isolated points of the spectra $\sigma'(T)$ and $\sigma''(T)$ of T with respect to the Banach algebras $\mathcal{A}'$ and $\mathcal{A}''$, respectively. Since $\mathcal{A}'' \subseteq \mathcal{A}'$, it follows that $\rho(T)'' \subseteq \rho'(T)$, and so $\sigma'(T) \subseteq \sigma''(T)$. Hence $0 \in \sigma''(T)$.

Claim. $0 \in \sigma''_{\rm iso}(T)$.

Proof. Let $\widehat{\mathcal{A}}''$ denote the collection of all algebra homomorphisms of $\mathcal{A}''$ into $\mathbb{C}$. Recall that (see e.g., [12, Theorem 0.4])

$$\sigma''(A^{-1}) = \big\{\Phi(A^{-1}) \in \mathbb{C}\colon\; \Phi \in \widehat{\mathcal{A}}''\big\},$$

which is bounded away from zero (since $0 \in \rho''(A^{-1})$), and

$$\sigma''(K) = \big\{\Phi(K) \in \mathbb{C}\colon\; \Phi \in \widehat{\mathcal{A}}''\big\} = \{0\} \cup \big\{\Phi(K) \in \mathbb{C}\colon\; \Phi \in \widehat{\mathcal{A}}''_F\big\},$$

where $\widehat{\mathcal{A}}''_F \subseteq \widehat{\mathcal{A}}''$ is a set of nonzero homomorphisms, which is finite because K is finite-rank (and so K has a finite spectrum). Note that $0 \in \sigma''(T) = \sigma''(A^{-1} - K)$ if and only if $0 = \Phi(A^{-1} - K) = \Phi(A^{-1}) - \Phi(K)$ for some $\Phi \in \mathcal{A}''$. If $\Phi \in \mathcal{A}''\backslash\mathcal{A}''_F$, then $\Phi(K) = 0$ so that $\Phi(A^{-1}) = 0$, which is a contradiction (because $0 \notin \sigma(A^{-1})$). Thus $\Phi \in \mathcal{A}''_F$, and hence $\Phi(A^{-1} - K) = \Phi(A^{-1}) - \Phi(K) = 0$, so that $\Phi(A^{-1}) = \Phi(K)$, for at most a finite number of homomorphisms Φ in $\mathcal{A}''_F$. Therefore, since $\{\Phi(A^{-1}) \in \mathbb{C}\colon\; \Phi \in \widehat{\mathcal{A}}''\} = \sigma''(A^{-1})$ is bounded away from zero, and since

$$\big\{\Phi \in \mathcal{A}''_F\colon\; \Phi(A^{-1}) = \Phi(K)\big\} = \big\{\Phi \in \mathcal{A}''_F\colon\; 0 \in \sigma''(A^{-1} - K)\big\}$$

is finite, it follows that 0 is an isolated point of $\sigma''(A^{-1} - K) = \sigma''(T)$, which concludes the proof of the claimed result.

Since $0 \in \sigma'(T) \subseteq \sigma''(T)$, it then follows that $0 \in \sigma'_{\rm iso}(T)$. $\square$

Theorem 1. *For every $T \in \mathcal{B}[\mathcal{H}]$,*

$$\sigma_b(T) = \bigcap\nolimits_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \{T\}'} \sigma(T + K).$$

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$, and let $\mathcal{A}'$ be a unital closed subalgebra of the unital complex Banach algebra $\mathcal{A} = \mathcal{B}[\mathcal{H}]$. Let $\sigma'(T)$, $\sigma'_b(T)$, and $\sigma'_w(T)$ be the spectrum, the Browder spectrum, and the Weyl spectrum of T with respect to $\mathcal{A}'$, respectively.

Claim 1. If $\mathcal{A}' = \{T\}'$, then $\sigma'(T) = \sigma(T)$.

Proof. Suppose $\mathcal{A}' = \{T\}'$. Trivially, $T \in \mathcal{A}'$. Let $\mathcal{P} = \mathcal{P}(T)$ be the collection of all polynomials $p(T)$ in T with complex coefficients, which is a unital commutative subalgebra of $\mathcal{B}[\mathcal{H}]$. Consider the collection $\mathcal{T}$ of all unital commutative subalgebras of $\mathcal{B}[\mathcal{H}]$ containing T. Note that every element of $\mathcal{T}$ is included in $\mathcal{A}'$, and also that $\mathcal{T}$ is partially ordered (in the inclusion ordering) and nonempty (e.g., $\mathcal{P} \in \mathcal{T}$). Every chain in $\mathcal{T}$ has an upper bound in $\mathcal{T}$ (the union of all subalgebras in a given chain of subalgebras in $\mathcal{T}$ is again a subalgebra in $\mathcal{T}$). Zorn's Lemma says that $\mathcal{T}$ has a maximal element, say $\mathcal{A}'' = \mathcal{A}''(T) \in \mathcal{T}$. Thus there is a maximal commutative subalgebra $\mathcal{A}''$ of $\mathcal{B}[\mathcal{H}]$ containing T (which is unital and closed). Hence $\mathcal{A}'' \subseteq \mathcal{A}' \subseteq \mathcal{A}$.

Let $\sigma''(T)$ be the spectrum of T with respect to $\mathcal{A}''$. Since $\mathcal{A}''$ is a maximal commutative subalgebra of $\mathcal{A}$, the preceding inclusions ensure that [12, Theorem 0.4]

$$\sigma(T) \subseteq \sigma'(T) \subseteq \sigma''(T) = \sigma(T).$$

Claim 2. If $\mathcal{A}' = \{T\}'$, then $\sigma_b(T) = \sigma_b'(T)$.

Proof. Claim 1 and Lemma 1.

Claim 3. If $\mathcal{A}' = \{T\}'$, then $\sigma_b'(T) = \sigma_w'(T)$.

Proof. $\lambda \in \sigma_0'(T)$ if and only if $\lambda \in \sigma'(T)$ and $\lambda I - T \in \mathcal{W}'$. But $\lambda \in \sigma'(T)$ if and only if $0 \in \sigma'(\lambda I - T)$ by the Spectral Mapping Theorem. Hence, $\lambda \in \sigma_0'(T)$ if and only if $0 \in \sigma_0'(\lambda I - T)$. Since $\mathcal{A}' = \{T\}' = \{\lambda I - T\}'$ it follows by Lemma 2 that $0 \in \sigma_0'(\lambda I - T)$ implies $0 \in \sigma_{\text{iso}}'(\lambda I - T)$. However, applying the Spectral Mapping Theorem again, $0 \in \sigma_{\text{iso}}'(\lambda I - T)$ if and only if $\lambda \in \sigma_{\text{iso}}'(T)$. Therefore, $\sigma_0'(T) \subseteq \sigma_{\text{iso}}'(T)$, which means that T satisfies Browder's theorem in $\mathcal{A}'$, which in turn is equivalent to saying that $\sigma_w'(T) = \sigma_b'(T)$ (see e.g., [9, Proposition 9.8]).

Claim 4. $\sigma_w'(T) = \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \mathcal{A}'} \sigma(T + K)$.

Proof. This is the very definition of the Weyl spectrum of T with respect to $\mathcal{A}'$.

By Claims 2, 3 and 4 we get $\sigma_b(T) = \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \{T\}'} \sigma(T + K)$. □

4. A Final Remark

Take any operator $T \in \mathcal{B}[\mathcal{H}]$. The following assertions are pairwise equivalent.

$$\sigma_0(T) \subseteq \pi_{00}(T), \qquad \sigma_0(T) = \pi_0(T), \qquad \sigma_0(T) \subseteq \sigma_{\text{iso}}(T), \qquad \sigma_w(T) = \sigma_b(T)$$

(see e.g., [9, Proposition 9.8]). An operator is said to satisfy Browder's theorem if any of the above assertions holds true. Consider the tensor product $T \otimes S$ of a pair of arbitrary Hilbert space operators T and S. It was proved in [6] that

$$\sigma_w(T \otimes S) \subseteq \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S).$$

However, since then, it remained as an open question whether the above inclusion might be, in fact, an identity. In other words, it was not know if there existed a pair of operators T and S for which the above inclusion was proper. This question was solved quite recently, and in this final section it is shown how it was. First note that Browder's theorem does not necessarily transfer from T and S to their tensor product $T \otimes S$. The following theorem gives a necessary and sufficient condition.

Theorem 2. *If both operators T and S satisfy Browder's theorem, then the tensor product $T \otimes S$ satisfies Browder's theorem if and only if*

$$\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S).$$

Proof. [10, Corollary 6]. □

Therefore, if there exist operators T and S that satisfy Browder's theorem, but $T \otimes S$ does not satisfy Browder's theorem, then the Weyl spectrum identity, viz., $\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S)$, does *not* hold for such a pair of operators. An example of a pair of operators that satisfy Browder's theorem but their tensor product does not satisfy Browder's theorem was recently supplied in [8].

Thus [8] and [10] together ensure that there exists a pair of operators T and S for which the inclusion

$$\sigma_w(T \otimes S) \subset \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S)$$

is proper; that is, for which the Weyl spectrum identity fails.

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