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ON EXPONENTIAL STABILITY OF CONTRACTION SEMIGROUPS

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Abstract. A semigroup $[T(t)]$ on a Hilbert space is exponentially stable if there exist real constants $M \geq 1$ and $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for every $t \geq 0$. If $[T(t)]$ is a strongly continuous *contraction* semigroup, then it is proved that we can set $M = 1$ in the definition of exponential stability *if and only if* the generator A of $[T(t)]$ is *boundedly strict* dissipative (just a strict dissipative A is not enough).

1. Introduction

If a *contraction* (strongly continuous) semigroup is exponentially stable, then it is not true that there exists an $\alpha > 0$ such that $\|T(t)\| \leq e^{-\alpha t}$ for every $t \geq 0$, even though its generator is dissipative. It might be assumed that this would be the case if the generator was strictly dissipative. We show here that this is not true either: just strict dissipativity is not enough. In fact, it is proved in Theorem 2 that an exponentially stable *contraction* C_0-semigroup is such that $\|T(t)\| \leq e^{-\alpha t}$ for every $t \geq 0$, for some $\alpha > 0$, *if and only if* its generator is *boundedly strict* dissipative. A refinement of such an equivalence is given in Corollary 2.

Throughout this paper $\mathcal{H}$ will stand for a complex Hilbert space, and $\mathcal{B}[\mathcal{H}]$ for the Banach algebra of all bounded linear transformations of $\mathcal{H}$ into itself. By an operator we mean an element from $\mathcal{B}[\mathcal{H}]$. Moreover, $[T(t)] = \{T(t)\,;\ t \geq 0\}$ will stand for a semigroup of operators $T(t)$ in $\mathcal{B}[\mathcal{H}]$, and the linear transformation $A\colon \mathcal{D} \to \mathcal{H}$ will stand for the generator of $[T(t)]$, where $\mathcal{D}$ is a linear manifold of $\mathcal{H}$.

A semigroup $[T(t)]$ is *exponentially stable* if there exist real constants $M \geq 1$ and $\alpha > 0$ such that
$$\|T(t)\| \leq Me^{-\alpha t} \quad \text{for every} \quad t \geq 0$$
or, equivalently, if $\|T(t)x\| \leq Me^{-\alpha t}\,\|x\|$ for every $t \geq 0$ and every $x \in \mathcal{H}$. It is *uniformly stable* if
$$\|T(t)\| \to 0 \quad \text{as} \quad t \to \infty$$
(i.e., $T(t) \to O$ as $t \to \infty$ in $\mathcal{B}[\mathcal{H}]$). Recall that $[T(t)]$ is uniformly stable if and only if it is exponentially stable [2, Lemma 1]. A semigroup $[T(t)]$ is strongly stable if
$$\|T(t)x\| \to 0 \quad \text{as} \quad t \to \infty \quad \text{for every } x \in \mathcal{H}.$$
It is clear that uniform stability (which is equivalent to exponential stability) implies strong stability. We assume that $\mathcal{H}$ is infinite-dimensional; otherwise all the above stability concepts coincide.

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A semigroup $[T(t)]$ is *contractive*, or it is a *contraction* semigroup, if

$$\|T(t)\| \leq 1 \quad \text{for all} \quad t \geq 0$$

or, equivalently, if $\|T(t)x\| \leq \|x\|$ every $x \in \mathcal{H}$ and all $t \geq 0$. It is a *proper contraction* semigroup if

$$\|T(t)x\| < \|x\| \quad \text{for every} \quad 0 \neq x \in \mathcal{H} \quad \text{and all} \quad t > 0.$$

A semigroup $[T(t)]$ is a *strict contraction* semigroup if

$$\|T(t)\| < 1 \quad \text{for every} \quad t > 0$$

or, equivalently, if $\sup_{\|x\|=1} \|T(t)x\| < 1$ for all $t > 0$. In general, a proper contraction semigroup is not a strict contraction semigroup, but the two concepts coincide for a semigroup of compact operators.

A linear transformation $A\colon \mathcal{D} \to \mathcal{H}$ is *dissipative* if its domain $\mathcal{D}$ is a dense linear manifold of $\mathcal{H}$ and

$$\operatorname{Re}\langle Ax\,;x\rangle \leq 0 \quad \text{for every} \quad x \in \mathcal{D}.$$

Recall that if $[T(t)]$ is a strongly continuous contraction semigroup (i.e., a contraction C_0-semigroup), then its generator A is a closed operator, whose domain $\mathcal{D}$ is a dense linear manifold of $\mathcal{H}$, and A is maximal dissipative in the sense that it is dissipative and there is no dissipative extension of it on $\mathcal{H}$ [4, 5, 7]. A dissipative generator A is called *strictly dissipative* if

$$\operatorname{Re}\langle Ax\,;x\rangle < 0 \quad \text{for every} \quad 0 \neq x \in \mathcal{D}.$$

We shall say that a strictly dissipative generator A is *boundedly strict dissipative* (or *uniformly dissipative* — see [3, p.34]) if there exists a constant $\gamma > 0$ such that

$$\operatorname{Re}\langle Ax\,;x\rangle \leq -\gamma\,\|x\|^2 \quad \text{for every} \quad x \in \mathcal{D}.$$

We show in Section 2 that if $[T(t)]$ is a uniformly continuous contraction semigroup with a strictly dissipative generator, then we can set $M = 1$ in the definition of exponential stability. This will be extended in Section 3 to contraction semigroups that are just strongly continuous: if $[T(t)]$ is a strongly continuous contraction semigroup, then we can set $M = 1$ in the definition of exponential stability *if and only if* the generator A is boundedly strict dissipative.

2. Injective Semigroups

The main result of this section says that, for an injective strongly continuous contraction semigroup with a strictly dissipative generator, we can set $M = 1$ in the definition of exponential stability. This ensures that, for a uniformly continuous contraction semigroup with a strictly dissipative generator, we can set $M = 1$ in the definition of exponential stability. These are consequences of Lemma 1 below.

Take a semigroup $[T(t)]$. The operator $T(t_0)$ is injective for some $t_0 > 0$ if and only if $T(t)$ is injective for all $t > 0$. Equivalently, 0 is not an eigenvalue of $T(t_0)$ for some $t_0 > 0$ if and only if 0 is not an eigenvalue of $T(t)$ for every $t > 0$. In this case we say that $[T(t)]$ is an injective semigroup. Moreover, every uniformly continuous contraction semigroup $[T(t)]$ is such that $T(t)$ has a bounded inverse for all $t > 0$, which means that 0 is in the resolvent set of $T(t)$ for every $t > 0$, and so it is injective for all $t > 0$. These results are considered in the following lemma.

Lemma 1. *Let $[T(t)]$ be a semigroup.*

(a) *$T(t)$ is injective for some $t > 0$ if and only if it is injective for every $t \geq 0$.*

Let $[T(t)]$ be a contraction semigroup. If $[T(t)]$ is uniformly continuous, then

(b) *$T(t)$ is has a bounded inverse for every $t \geq 0$.*

Proof. Let $[T(t)]$ be a semigroup (not necessarily a C_0-semigroup). Recall that $T(t)$ is injective if and only if $\mathcal{N}(T(t)) = \{0\}$, where $\mathcal{N}(T(t)) = \{x \in \mathcal{H}\colon\; T(t)x = 0\}$ is the kernel of $T(t)$.

(a) By the semigroup property (viz. $T(t+s) = T(s)T(t)$ for every $s, t \geq 0$), we get

$$\mathcal{N}(T(t)) \subseteq \mathcal{N}(T(t+s))$$

for every $s, t \geq 0$, and hence

$$\mathcal{N}(T(t)) = \{0\} \quad \Longrightarrow \quad \mathcal{N}(T(\tau)) = \{0\} \quad \text{for every } \; 0 \leq \tau \leq t.$$

On the other hand, the semigroup property ensures that, by induction,

$$T(nt) = T(t)^n$$

for every $t \geq 0$ and every nonnegative integer n. Take an arbitrary $t > 0$. Suppose $\mathcal{N}(T(t)) = \{0\}$. This means that $T(t)x \neq 0$ for every $0 \neq x \in \mathcal{H}$, which implies that $T(t)^n x \neq 0$ by induction, and so $T(nt)x \neq 0$ for every $0 \neq x \in \mathcal{H}$ and every positive integer n by the preceding identity. But this means that $\mathcal{N}(T(nt)) = \{0\}$ for every $n \geq 1$. Hence (by using the previous implication),

$$\mathcal{N}(T(t)) = \{0\} \quad \Longrightarrow \quad \mathcal{N}(T(\tau)) = \{0\} \quad \text{for every } \; 0 < t \leq \tau,$$

Therefore,

$$\mathcal{N}(T(t)) = \{0\} \quad \text{for some } \; t > 0 \quad \Longrightarrow \quad \mathcal{N}(T(t)) = \{0\} \quad \text{for every } \; t \geq 0.$$

That is, if $T(t)$ is injective for some $t > 0$, then $T(t)$ is injective for every $t \geq 0$. The converse is tautological. This completes the proof of (a).

(b) Now consider the nonnegative function $\|T(\cdot)x\|^2\colon [0, \infty) \to \mathbb{R}$. Suppose $[T(t)]$ is a contraction semigroup, so that $\|T(t+s)x\|^2 = \|T(s)T(t)x\|^2 \leq \|T(t)x\|^2 \leq \|x\|^2$ for every $s, t \geq 0$ and every $x \in \mathcal{H}$, and hence

$$\|T(\cdot)x\|^2 \;\text{ is nonincreasing with }\; \|T(0)x\|^2 = \|x\|^2$$

for every $x \in \mathcal{H}$. Moreover, if the contraction semigroup $[T(t)]$ is also strongly continuous (i.e., a contraction C_0-semigroup), then its generator A is dissipative and

$$\frac{d}{dt}\|T(t)x\|^2 = 2\,\mathrm{Re}\,\langle AT(t)x\,; T(t)x\rangle \leq 0$$

for every $t > 0$ and every $x \in \mathcal{D}$ (see e.g., [4, p.80,90]). If, in addition, $[T(t)]$ is uniformly continuous, then $A \in \mathcal{B}[\mathcal{H}]$ (i.e., $\mathcal{D} = \mathcal{H}$ and A is bounded — see e.g., [4, p.78]), and therefore (by the Schwartz inequality)

$$\left|\frac{d}{dt}\|T(t)x\|^2\right| \leq 2\|A\|\,\|T(t)x\|^2 \leq 2\|A\|\,\|x\|^2$$

for every $t > 0$ and every $x \in \mathcal{H}$, which implies that

$$-2\|A\| \leq \inf_{\|x\|=1} \frac{d}{dt}\|T(t)x\|^2 \quad \text{for all} \quad t > 0.$$

Since $\|T(\cdot)x\|^2 \colon [0,\infty) \to \mathbb{R}$ is nonincreasing and $\|T(0)x\|^2 = \|x\|^2 = 1$ for every $x \in \mathcal{H}$ with $\|x\| = 1$, it can be verified that, if $\varepsilon \in (0, \frac{1}{2\|A\|})$, then

$$0 < 1 - 2\|A\|\varepsilon \leq \inf_{\|x\|=1} \|T(t)x\|^2 \leq 1 \quad \text{for all} \quad t \in [0,\varepsilon].$$

Take an arbitrary $\varepsilon \in (0, \frac{1}{2\|A\|})$. The above inequality implies that

$$T(t) \text{ has bounded inverse for every } t \in [0,\varepsilon].$$

Now let s be an arbitrary positive real number such that $\varepsilon < s$, and take a positive integer n large enough such that $\frac{s}{n} \leq \varepsilon$. Set $t = \frac{s}{n} \leq \varepsilon$ so that $T(t)$ has a bounded inverse, say $T(t)^{-1} \in \mathcal{B}[\mathcal{H}]$. Thus, by the semigroup property,

$$T(s) = T(nt) = T(t)^n,$$

and hence $T(s)$ has a bounded inverse, viz. $T(s)^{-1} = T(t)^{-n} \in \mathcal{B}[\mathcal{H}]$. Therefore,

$$T(s) \text{ has bounded inverse for every } s \in (\varepsilon,\infty).$$

which completes the proof of (b). $\square$

Theorem 1. *Let $[T(t)]$ be a strongly continuous contraction semigroup with a strictly dissipative generator such that $T(t)$ is injective for some $t > 0$. If $[T(t)]$ is exponentially stable, then there exists a constant $\alpha > 0$ such that*

$$\|T(t)\| \leq e^{-\alpha t} \quad \textit{for every} \quad t \geq 0.$$

Proof. Suppose $[T(t)]$ is exponentially stable, so that there exist real constants $M \geq 1$ and $\alpha > 0$ such that $\|T(t)x\|^2 \leq Me^{-\alpha t}\,\|x\|^2$ for every $t \geq 0$ and every $x \in \mathcal{H}$. Moreover, suppose $[T(t)]$ is contractive, so that the nonnegative function $\|T(\cdot)x\|^2 \colon [0,\infty) \to \mathbb{R}$ is nonincreasing with $\|T(0)x\|^2 = \|x\|^2$ for every $x \in \mathcal{H}$. Thus, if $M > 1$ necessarily, then, for each $x \in \mathcal{H}$ there is an $\varepsilon = \varepsilon(x) > 0$ such that

$$\|T(t)x\|^2 = \|x\|^2 \quad \text{for every} \quad t \in [0,\varepsilon].$$

Suppose $T(t)$ is injective for some $t > 0$. Lemma 1(a) says that $T(t)$ is injective for every $t \geq 0$. If the contraction semigroup $[T(t)]$ is strongly continuous, then it has a dissipative generator A. If A is strictly dissipative, then

$$\frac{d}{dt}\|T(t)x\|^2 = 2\,\mathrm{Re}\langle AT(t)x\,; T(t)x\rangle < 0$$

for every $t > 0$ and every $0 \neq x \in \mathcal{D}$ — which is dense in $\mathcal{H}$ — because $T(t)$ is injective for every $t \geq 0$. The above two expressions lead to a contradiction, namely, $\|T(\cdot)x\|^2 \colon [0,\infty) \to \mathbb{R}$ cannot be constant over a nondegenerate interval for any $0 \neq x \in \mathcal{D}$. Therefore, $M \leq 1$. $\square$

In other words, for every pair of constants (α, M) ensuring exponential stability for an *injective* strongly continuous contraction semigroup with a strictly dissipative generator, we can always set $M = 1$ without changing the α.

Corollary 1. *Let $[T(t)]$ be a uniformly continuous contraction semigroup with a strictly dissipative generator. If $[T(t)]$ is exponentially stable, then there exists a constant $\alpha > 0$ such that*

$$\|T(t)\| \le e^{-\alpha t} \quad \textit{for every} \quad t \ge 0.$$

Proof. If $[T(t)]$ is uniformly continuous, then $[T(t)]$ is strongly continuous and, if it is contractive, then each $T(t)$ has a bounded inverse by Lemma 1(b), and so each $T(t)$ is injective. If, in addition, $[T(t)]$ has a strictly dissipative generator. Thus the claimed result follows by Theorem 1. □

Remark 1. It is worth noticing how the preceding results behave for a discrete semigroup $[T^n] = \{T^n\,;\ n \ge 0\}$, where T is an arbitrary operator in $\mathcal{B}[\mathcal{H}]$. The result in Lemma 1(a) for discrete semigroups, namely, T^n is injective for some integer $n \ge 1$ if and only if T^n is injective for every $n \ge 0$, has essentially the same proof. The discrete counterpart of Lemma 1(b) reads as follows: T^n has an inverse in $\mathcal{B}[\mathcal{H}]$ for every $n \ge 1$ wherever T has (reason: $(T^n)^{-1} = (T^{-1})^n$). The definition of exponential stability for discrete semigroups is: $[T^n]$ is exponentially stable if there exist real constants $M \ge 1$ and $\alpha \in (0,1)$ such that $\|T^n\| \le M\alpha^n$ for every $n \ge 0$. It may happen that exponential stability for a particular exponentially stable contraction discrete semigroup holds only with a constant M greater than 1. For instance, this is the case for every exponentially stable contraction discrete semigroup $[T^n]$ with $\|T\| = 1$. Sample: Take the nilpotent contraction $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$ so that for every $M > 1$ there exists an $\alpha \in (0,1)$ such that $\|T^n\| \le M\alpha^n$, while there is no $\alpha \in (0,1)$ for which $\|T^n\| \le \alpha^n$. The counterpart of Corollary 1 for a discrete semigroup requires that T be a strict contraction (reason: $\|T^n\| \le \|T\|^n$).

3. Boundedly Strict Dissipative Generator

Throughout this section we assume that $[T(t)]$ is a C_0-semigroup (i.e., a strongly continuous semigroup) with a generator A, and we consider the following setup. Take the nonnegative function $\|T(\cdot)x\|^2\colon [0,\infty) \to \mathbb{R}$. If $[T(t)]$ is a contraction semigroup, then

$$\|T(\cdot)x\|^2 \ \text{ is nonincreasing with } \|T(0)x\|^2 = \|x\|^2$$

for every $x \in \mathcal{H}$. Moreover, recall that

$$\frac{d}{dt}\|T(t)x\|^2 = 2\,\mathrm{Re}\langle AT(t)x\,;T(t)x\rangle$$

for every $t > 0$ and every $x \in \mathcal{D}$. Since $T(\cdot)x\colon [0,\infty) \to \mathcal{H}$ is continuous for every $x \in \mathcal{H}$ (and so is the inner product) we get, for every $x \in \mathcal{D}$,

$$\frac{d}{dt}\|T(t)x\|^2\Big|_{t=0^+} = \lim_{t\to 0}\frac{d}{dt}\|T(t)x\|^2 = \lim_{t\to 0} 2\,\mathrm{Re}\langle AT(t)x\,;T(t)x\rangle = 2\,\mathrm{Re}\langle Ax\,;x\rangle.$$

Also set $\partial\mathbb{D}|_{\mathcal{D}} = \{x \in \mathcal{D}\colon \|x\| = 1\}$ and $\partial\mathbb{D}|_{\mathcal{H}} = \{x \in \mathcal{H}\colon \|x\| = 1\}$, which are the unit spheres in $\mathcal{D}$ and in $\mathcal{H}$, respectively.

We show in this section that if $[T(t)]$ is an exponentially stable strongly continuous contraction semigroup, then we can set $M = 1$ in the definition of exponential stability *if and only if* $[T(t)]$ has a boundedly strict dissipative generator.

Theorem 2. *Let $[T(t)]$ be an exponentially stable contraction C_0-semigroup. There exists a constant $\alpha > 0$ such that*

$$\|T(t)\| \le e^{-\alpha t} \quad \textit{for every} \quad t \ge 0$$

if and only if the generator of $[T(t)]$ is boundedly strict dissipative.

Proof. If $[T(t)]$ is contractive, and if its generator A is strictly dissipative, then

(i) $2\,\mathrm{Re}\langle Ax\,; x\rangle = d\,\|T(t)x\|^2/dt\,|_{t=0^+} < 0$ for every nonzero $x \in \mathcal{D}$, and
(ii) $\|T(\cdot)x\|^2$ is nonincreasing with $\|T(0)x\|^2 = \|x\|^2$ for every x in $\mathcal{H}$.

If, in addition to (i) and (ii), $[T(t)]$ is exponentially stable, then there exist constants $M \ge 1$ and $\alpha > 0$ such that

$$(*) \qquad \|T(t)x\|^2 \le Me^{-\alpha t}\,\|x\|^2 \quad \text{for every} \quad t \ge 0 \quad \text{and every} \quad x \in \mathcal{H}.$$

According to properties (i), (ii), and $(*)$, for each $0 \ne x \in \mathcal{D}$ there is a number $\beta(x)$ in $(0, |d\|T(t)x\|^2/dt|_{t=0^+}|)$ such that

$$(**) \qquad \|T(t)x\|^2 \le e^{-\beta(x)t}\|x\|^2 \quad \text{for every } t \ge 0.$$

(a) If A is boundedly strict dissipative (i.e., if $\mathrm{Re}\langle Ax\,; x\rangle \le -\gamma\|x\|^2$ for every $x \in \mathcal{D}$ and some $\gamma > 0$), then property (i) says that $\sup_{x\in\partial\mathbb{D}|_{\mathcal{D}}} d\|T(t)x\|^2/dt|_{t=0^+} < 0$. So

$$\inf_{x\in\partial\mathbb{D}|_{\mathcal{D}}} |\,d\|T(t)x\|^2/dt\,|_{t=0^+}| \ne 0.$$

It can be verified by the above inequality and $(**)$ that there exists a function

$$\beta(\cdot)\colon \partial\mathbb{D}|_{\mathcal{D}} \to (0, \inf_{x\in\partial\mathbb{D}|_{\mathcal{D}}} |\,d\|T(t)x\|^2/dt\,|_{t=0^+}|),$$

bounded away from zero (i.e., such that $0 < \inf_{x\in\partial\mathbb{D}|_{\mathcal{D}}} \beta(x)$), for which $(**)$ holds for every unit vector in $\mathcal{D}$. Since $\mathcal{D}$ is dense in $\mathcal{H}$, the induced uniform norm of an operator in $\mathcal{B}[\mathcal{H}]$ and of its restriction in $\mathcal{B}[\mathcal{D}]$ coincide, and so

$$\|T(t)\| = \sup_{x\in\partial\mathbb{D}|_{\mathcal{H}}} \|T(t)x\| = \sup_{x\in\partial\mathbb{D}|_{\mathcal{D}}} \|T(t)x\|.$$

Now suppose there is no constant $\alpha > 0$ such that

$$\|T(t)\| \le e^{-\alpha t} \quad \text{for every} \quad t \ge 0.$$

Then for every $\alpha > 0$ there exists a $t_\alpha > 0$ such that, form $(**)$,

$$e^{-\alpha t_\alpha} < \|T(t_\alpha)\|^2 = \sup_{x\in\partial\mathbb{D}|_{\mathcal{H}}} \|T(t_\alpha)x\|^2 = \sup_{x\in\partial\mathbb{D}|_{\mathcal{D}}} \|T(t_\alpha)x\|^2 \le e^{-\inf_{x\in\partial\mathbb{D}|_{\mathcal{D}}} \beta(x)\,t_\alpha}.$$

In particular, by setting $\alpha = \inf_{x\in\partial\mathbb{D}|_{\mathcal{D}}} \beta(x) > 0$, the above strict inequality leads to a contradiction. Therefore, there exists a constant $\alpha > 0$ such that

$$\|T(t)\| \le e^{-\alpha t} \quad \text{for every} \quad t \ge 0.$$

(b) Conversely, suppose $(*)$ holds with $M = 1$. Equivalently, suppose the above expression holds for some constant $\alpha > 0$. Then $\|e^{\alpha t}\,T(t)\| \le 1$ for all $t \ge 0$, which means that the semigroup $[e^{\alpha t}\,T(t)]$ is contractive. Since its generator is $A + \alpha I$ (see e.g., [4, p.90]), it follows that $A + \alpha I$ is dissipative on $\mathcal{D}$, and therefore

$$\mathrm{Re}\langle Ax\,;x\rangle + \alpha\,\|x\|^2 = \mathrm{Re}\langle (A+\alpha I)x\,;x\rangle \leq 0 \quad \text{for every} \quad x \in \mathcal{D}.$$

Thus there is an $\alpha > 0$ such that

$$\alpha\|x\|^2 \leq -\mathrm{Re}\langle Ax\,;x\rangle \quad \text{for every} \quad x \in \mathcal{D};$$

equivalently, A is boundedly strict dissipative. □

A different proof of part (b) has been considered in [6, Lemma 2].

Remark 2. Let $[T(t)]$ be a contraction C_0-semigroup.

(a) If the generator of $[T(t)]$ is strictly dissipative, then $[T(t)]$ is a proper contraction on $\mathcal{D}$.

(b) If the generator of $[T(t)]$ is boundedly strict dissipative, then $[T(t)]$ is a proper contraction on $\mathcal{H}$.

Indeed, let A be generator of a contraction C_0-semigroup $[T(t)]$.

(a) If A is strictly dissipative, then

$$\lim_{\varepsilon\to 0^+} \frac{\|T(\varepsilon)x\|^2 - \|T(0)x\|^2}{\varepsilon} = \frac{d}{dt}\|T(t)x\|^2\big|_{t=0^+} = 2\,\mathrm{Re}\langle Ax\,;x\rangle < 0$$

for every $0 \neq x \in \mathcal{D}$; that is, $\|T(0^+)x\|^2 < \|T(0)x\|^2 = \|x\|^2$ for every $0 \neq x \in \mathcal{D}$. Therefore, since $\|T(\cdot)x\|^2$ is nonincreasing with $\|T(0)x\|^2 = \|x\|^2$, it follows that $\|T(t)x\|^2 < \|x\|^2$ for every $t > 0$, whenever $0 \neq x \in \mathcal{D}$.

(b) If A is boundedly strict dissipative, then there exists $\gamma > 0$ such that

$$\lim_{\varepsilon\to 0^+} \frac{\|T(\varepsilon)x\|^2 - \|T(0)x\|^2}{\varepsilon} = \frac{d}{dt}\|T(t)x\|^2\big|_{t=0^+} = 2\,\mathrm{Re}\langle Ax\,;x\rangle \leq -2\,\gamma\|x\|^2$$

for every $0 \neq x \in \mathcal{D}$. Hence, since $\mathcal{D}$ is dense in $\mathcal{H}$,

$$\lim_{\varepsilon\to 0^+} \frac{\|T(\varepsilon)x\|^2 - \|T(0)x\|^2}{\varepsilon} \leq -2\,\gamma\|x\|^2 < 0$$

for every $0 \neq x \in \mathcal{H}$. Therefore, by using the same argument, $\|T(t)x\|^2 < \|x\|^2$ for every $t > 0$, whenever $0 \neq x \in \mathcal{H}$.

Remark 3. The assumption that the generator A is strictly dissipative is enough to prove Theorem 1 and Corollary 1. However, Theorem 2 ensures that, in fact, that A must be boundedly strict dissipative.

Theorem 2 says that, if $[T(t)]$ is an exponentially stable contraction C_0-semigroup, then it is exponentially stable with $M = 1$ *if and only if* its generator is boundedly strict dissipative. We close the paper by showing that boundedly strict dissipativity actually implies exponentially stability. Therefore, *a strongly continuous contraction semigroup is exponentially stable with $M = 1$ if and only if its generator is boundedly strict dissipative.*

Corollary 2. *Let A be the generator of a contraction C_0-semigroup $[T(t)]$. There exists a constant $\alpha > 0$ such that*

$$\|T(t)\| \leq e^{-\alpha t} \quad \textit{for every} \quad t \geq 0$$

if and only if there exists a constant $\gamma > 0$ *such that*

$$\operatorname{Re}\langle Ax\,; x\rangle \leq -\,\gamma\,\|x\|^2 \quad \textit{for every} \quad x \in \mathcal{D}.$$

Proof. Suppose A is boundedly strict dissipative. Since $\mathcal{D}$ is $[T(t)]$-invariant, there exists a constant $\gamma > 0$ such that

$$\gamma\,\|T(t)x\|^2 \leq -\operatorname{Re}\langle AT(t)x\,; T(t)x\rangle = -\frac{d}{dt}\|T(t)x\|^2$$

for every $x \in \mathcal{D}$. Integrating, and extending by continuity (since integral is a continuous functional) form the dense domain $\mathcal{D}$ to the whole space $\mathcal{H}$, we get

$$\gamma \int_0^\infty \|T(t)x\|^2\,dt \leq \|x\|^2 < \infty$$

for every $x \in \mathcal{H}$. But, as it is well-known [1, Corollary 1], the preceding inequality implies that $[T(t)]$ is exponentially stable, which means that there exist constants $M \geq 1$ and $\alpha > 0$ such that

$$\|T(t)\| \leq Me^{-\alpha t} \quad \text{for every} \quad t \geq 0.$$

However, Theorem 2 ensures that an exponentially stable contraction C_0-semigroup with a boundedly strict dissipative generator is such that

$$\|T(t)\| \leq e^{-\alpha t} \quad \text{for every} \quad t \geq 0$$

for some $\alpha > 0$. The converse is an immediate consequence of Theorem 2 itself. □

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