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# QUASI-SIMILAR $k$-PARANORMAL OPERATORS

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Abstract. It is proved in this paper that k-paranormal operators satisfy (Bishop's) property ($\beta$); and also that if $S$ and $T$ are $k$-paranormal contractions such that the completely non-unitary part $S_c$ of $S$ has finite multiplicity, then $S$ is quasi-similar to $T$ if and only if their unitary parts are unitarily equivalent and their completely non-unitary parts are quasi-similar. This generalizes a result of W.W. Hastings [4] on subnormal operators and P.Y. Wu [11] on hyponormal operators.

## 1. Introduction

Let $B(\mathcal{H})$ denote the algebra of operators on an infinite dimensional complex Hilbert space. An operator $T \in B(\mathcal{H})$ is $k$-paranormal for some integer $k \geq 1$ if

$$\|Tx\|^{k+1} \leq \|T^{k+1}x\|$$

for every unit vector $x \in \mathcal{H}$. Let $P(k)$ denote the class of all $k$-paranormal operators. $T \in B(\mathcal{H})$ is a quasi-affinity if it is injective and has a dense range; $S, T \in B(\mathcal{H})$ are quasi-similar, $S \sim T$, if there exist quasi-similarities $X, Y \in B(\mathcal{H})$ such that

$$SX = XT \quad \text{and} \quad TY = YS.$$

Let $T_u$ denote the unitary part and $T_c$ denote the cnu (completely non-unitary) part of a contraction $T \in B(\mathcal{H})$. Nagy–Foiaş classes of contractions, $C_{00}$, $C_{01}$, $C_{10}$ and $C_{11}$ [8, p. 72] are defined as usual. Let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, and $\sigma_e(T)$ stand for spectrum, point spectrum, approximate point spectrum, and essential spectrum (or Fredholm spectrum) of $T \in B(\mathcal{H})$, respectively. Let $\mathbb{N}$ denote the set of non–negative integers. The ascent asc$(T)$ and descent dsc$(T)$ of $T \in B(\mathcal{H})$ are given by

$$\operatorname{asc}(T) = \inf\{n \in \mathbb{N} : T^{-n}(0) = T^{-(n+1)}(0)\}$$

and

$$\operatorname{dsc}(T) = \inf\{n \in \mathbb{N} : T^{n}(\mathcal{H}) = T^{n+1}(\mathcal{H})\}$$

(if no such integer $n$ exists, then $\operatorname{asc}(T) = \infty$, respectively $\operatorname{dsc}(T) = \infty$). We say that $T$ has the single valued extension property, or SVEP, at $\lambda \in \mathbb{C}$ if for every open neighborhood $U$ of $\lambda$, the only analytic solution $f$ to the equation

$$(T - \mu)f(\mu) = 0$$

for all $\mu \in U$ is the constant function $f \equiv 0$; we say that $T$ has SVEP if $T$ has a SVEP at every $\lambda \in \mathbb{C}$. It is well known that finite ascent implies SVEP; also, an operator has SVEP at every isolated point of its spectrum (as well as at every isolated point of its approximate point spectrum). An operator $T \in B(\mathcal{H})$ satisfies

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(Bishop's) property $(\beta)$ if, for every open subset $U$ of the complex plane $\mathbb{C}$ and every sequence of analytic functions $f_n : U \to \mathcal{H}$ with the property that

$$(T-\lambda)f_n(\lambda) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly on all compact subsets of $U$, $f_n(\lambda)\to 0$ as $n\to\infty$ locally uniformly on $U$.

## 2. Bishop's property $(\beta)$ for $P(k)$ operators

Recall that operators $S \in P(k)$ are normaloid, i.e., $\|S\| = r(S)$.

**Lemma 2.1.** ([12, Lemma 2.3 and Corollary 2.6]). *If $T \in P(k)$ and $(0 \neq)\lambda \in \sigma_p(T)$, then*

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} (T-\lambda)^{-1}(0) \\ \{(T-\lambda)^{-1}(0)\}^{\perp} \end{pmatrix},$$

*where $T_{22} \in P(k) \cap B(\{(T-\lambda)^{-1}(0)\}^{\perp})$ is such that $\lambda \notin \sigma_p(T_{22})$.*

**Lemma 2.2.** ([3, Corollary 1]). *If $T \in P(k)$ is a contraction, then it has a decomposition $T = T_u \oplus T_c$, where $T_c \in C_{.0}$.*

**Lemma 2.3.** *Operators $T \in P(k)$ have finite ascent $\leq 1$.*

*Proof.* Since $\operatorname{asc}(T-\lambda) = 0$ for every $\lambda \in \sigma(T) \setminus \sigma_p(T)$, we consider points $\lambda \in \sigma_p(T)$. If $\lambda = 0$, then the definition of $k$-paranormality implies that $T^{-(k+1)}(0) \subseteq T^{-1}(0)$; since $T^{-1}(0) \subseteq T^{-2}(0) \subseteq ...$, $T^{-(k+1)}(0) = T^{-1}(0)$. Now let $\lambda \neq 0$. Then

$$T - \lambda = \begin{pmatrix} 0 & T_{12} \\ 0 & T_{22}-\lambda \end{pmatrix} \begin{pmatrix} (T-\lambda)^{-1}(0) \\ \{(T-\lambda)^{-1}(0)\}^{\perp} \end{pmatrix}.$$

Recall, [10, Exercise 7, p. 293], that $\operatorname{asc}(T-\lambda) \leq \operatorname{asc}(0) + \operatorname{asc}(T_{22}-\lambda)$. Since $\operatorname{asc}(T_{22}-\lambda) = 0$, we have that $\operatorname{asc}(T-\lambda) = 1$. $\square$

An immediate consequence of Lemma 2.3 is the following:

**Corollary 2.4.** *Operators $T \in P(k)$ have SVEP.*

Given an open subset $U$ of $\mathbb{C}$, let $H(U, \mathcal{H})$ denote the Fréchet space of analytic functions from $U$ to $\mathcal{H}$. Then $T \in B(\mathcal{H})$ satisfies property $(\beta)$ precisely when the operator $T_U : H(U, \mathcal{H}) \to H(U, \mathcal{H})$, $(T_U f)(\lambda) := (T-\lambda)f(\lambda)$, (is injective and) has closed range [7, Proposition 3.3.5].

Let $\ell^\infty(\mathcal{H})$ denote the space of all bounded sequences of elements of $\mathcal{H}$, and let $c_0(\mathcal{H})$ denote the space of all null sequences of $\mathcal{H}$. Endowed with the canonical norm, the quotient space $\mathcal{K} = \ell^\infty(\mathcal{H})/c_0(\mathcal{H})$ can be made into a Hilbert space [1], into which $\mathcal{H}$ may be isometrically embedded. The Berberian–Quigley extension theorem, [7, p. 255], says that given an operator $T \in B(\mathcal{H})$ there exists an isometric $^*$-isomorphism $T \to T^o \in B(\mathcal{K})$ preserving order such that $\sigma(T) = \sigma(T^o)$ and $\sigma_a(T) = \sigma_a(T^o) = \sigma_p(T^o)$. Let $[x_n] \in \mathcal{K}$ denote the equivalence class of the sequence $\{x_n\} \subset \mathcal{H}$. If $T \in P(k)$, then

$$\|T^o[x]\|^{k+1} = \|Tx\|^{k+1} \leq \|T^{k+1}x\|\|x\|^k = \|T^{ok+1}[x]\|\,\|[x]\|^k$$

for each $x \in \mathcal{H}$. Hence the Berberian–Quigley extension $T^o$ of an operator $T \in P(k)$ is again $k$-paranormal.

**Theorem 2.5.** *Operators $T \in P(k)$ satisfy property $(\beta)$.*

*Proof.* Let $U$ be an open subset of $\mathbb{C}$, and assume that

$$(T-\lambda)f_n(\lambda) \to 0 \text{ on } H(U,\mathcal{H})$$

for every $\lambda \in U$. Then

$$(T^o - \lambda I^o)[f_n(\lambda)] = 0 \text{ on } H(U,\mathcal{K})$$

for every $\lambda \in U$. Since the $k$-paranormal operator $T^o$ has SVEP, $[f_n(\lambda)] = 0$ (i.e., $\{f_n\} \in c_0(\mathcal{H})$). We claim that $f_n(\lambda) \to 0$ on $H(U,\mathcal{H})$. Start by observing that if $D(\lambda;r) = \{\mu \in \mathbb{C} : |\lambda - \mu| < r\}$ is such that $\overline{D(\lambda;r)} \subset U$, then the analytic sequence $\{f_n(\lambda)\}$ is uniformly bounded on $\overline{D(\lambda;r)}$; furthermore, for every $\epsilon > 0$, there exists a natural number $N$ and $0 < \rho < r$ such that

$$\|f_n(\mu)\| < \frac{\epsilon}{2} \quad \text{and} \quad \|f_n(\lambda) - f_n(\mu)\| < \frac{\epsilon}{2}$$

for all $n > N$ and $\mu \in D(\lambda;\rho)$. Indeed, considering $\frac{f_n}{1+\|f_n\|}$ instead of $f_n$ if need be, we may assume that $\sup|f_n| = M < \infty$ on $\overline{D(\lambda;r)}$. The function $f_n$ being analytic, $f_n(\mu) - f_n(\lambda) = \sum_{m=1}^{\infty} a_{nm}(\mu-\lambda)^m$, and then $\|f_n(\lambda) - f_n(\mu)\| \leq \frac{M\rho}{r-\rho}$ for all $\mu \in \overline{D(\lambda;\rho)}$ such that $0 < \rho < r$. Now choose $N$ and $\rho$ such that $|f_n(\lambda)| < \frac{\epsilon}{4}$ (recall that $f_n(\lambda) \in c_0$) and $\frac{M\rho}{r-\rho} < \frac{\epsilon}{4}$. Then

$$\|f_n(\mu)\| \leq \|f_n(\lambda)\| + \|f_n(\lambda) - f_n(\mu)\| < \frac{\epsilon}{2}$$

for all $n > N$ and $\mu \in D(\lambda;\rho)$. Consequently, $f_n(\lambda) \to 0$ in $H(U,\mathcal{H})$, i.e., $T$ satisfies property $(\beta)$. □

The conclusion that $P(k)$ operators satisfy property $(\beta)$ generalizes an observation by Uchiyama and Takahashi [9] that paranormal operators (i.e., $P(1)$ operators) satisfy property $(\beta)$. Property $(\beta)$ has a number of consequences: we list below but a couple of these. Let $\mathbf{D}$ denote the closed unit disc in $\mathbb{C}$.

**Corollary 2.6.** *If $S \in P(k)$ is quasi-similar to an operator $T \in B(\mathcal{H})$ satisfying property $(\beta)$, then $\sigma_x(S) = \sigma_x(T)$, where $\sigma_x = \sigma$ or $\sigma_e$. In particular, if $S \in P(k)$ is quasi-similar to an isometry $V \in B(\mathcal{H})$, then $S$ is a contraction such that $\sigma_x(S) = \sigma_x(V) = \mathbf{D}$.*

*Proof.* That $\sigma_x(S) = \sigma_x(T)$ follows from an application of [7, Theorem 3.7.15]. In the particular case in which $T = V$, it follows that $\sigma_x(S) = \sigma_x(V) = \mathbf{D}$. Hence, since $S$ is normaloid, $r(T) = \|T\| = 1$, i.e., $S$ is a contraction. □

A number of the commonly considered classes of operators in $B(\mathcal{H})$ (for example, hyponormal, $M$-hyponormal, $p$-hyponormal for $0 < p \leq 1$, w-hyponormal, $(p,k)$-quasihyponormal operators for $0 < p \leq 1$ and integers $k \geq 1$) are known to satisfy property $(\beta)$; Corollary 2.6 applies to operators $T$ belonging to one of these classes. An operator $T$ on a separable Hilbert space $\mathcal{H}$ is said to be supercyclic if the homogeneous orbit $\{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbf{N} \cup \mathbf{0}\}$ is dense in $\mathcal{H}$ for some $x \in \mathcal{H}$. It is known that paranormal operators (i.e., operators in $P(1)$) are not supercyclic [2]. Does this extend to operators in $P(k)$ for $k \geq 2$?

We have a partial result for invertible $k$-paranormal operators. Recall that the inverse of an invertible paranormal is again paranormal. It is, however, an open question whether the inverse of an invertible $k$-paranormal operator for $k \geq 2$ is $k$-paranormal [6].

**Corollary 2.7.** *Operators $T \in P(k)$ such that $T^{-1}$, whenever it exists, is also a $P(k)$ operator are not supercyclic.*

*Proof.* Suppose that $T \in P(k)$ is supercyclic. The class $P(k)$ being closed under multiplication by non-zero scalars, we may assume that $\|T\| = 1$. Since the supercyclic contraction $T$ satisfies property $(\beta)$, $\sigma(T)$ is contained in the boundary $\partial\mathbf{D}$ of $\mathbf{D}$ [7, Proposition 3.3.18]. Thus $T$ is invertible, and hence (by hypothesis) $T^{-1} \in P(k)$. But then $\|T^{-1}\| = 1$ $(= \|T\|)$. Consequently, $T$ is a unitary. Since no unitary on an infinite dimensional Hilbert space can be supercyclic, we have a contradiction. □

Next, we state a couple of corollaries to Corollary 2.7

**Corollary 2.8.** *Invertible operators in $P(k)$ such that their inverse lies in $P(k-1)$ are not supercyclic.*

*Proof.* [6, Theorem 1] implies that $T^{-1} \in P(k)$; apply Corollary 2.7. □

**Corollary 2.9.** *If $T \in P(k)$ is invertible, and if*

$$\|T^k x\|^{k+1} \leq \|Tx\|^{k+1} \|T^{k+1} x\|^{k-1}$$

*for every unit vector $x \in \mathcal{H}$, then $T$ is not supercyclic.*

*Proof.* [6, Theorem 2] implies that $T^{-1} \in P(k)$; apply Corollary 2.7. □

## 3. Quasi-similar $P(k)$ operators

The multiplicity $\mu_T$ of an operator $T \in B(\mathcal{H})$ is the minimum cardinality of a set $K \subseteq \mathcal{H}$ such that $\mathcal{H} = \bigvee_{n=0}^{\infty} T^n K$. Evidently, if $S, T \in B(\mathcal{H})$ and $SX = XT$ for some operator $X \in B(\mathcal{H})$ with dense range, then $\mu_S \leq \mu_T$; hence, if there exist operators $X, Y \in B(\mathcal{H})$ with dense range such that $SX = XT$ and $TY = YS$, then $\mu_S = \mu_T$. The following technical lemma will be required.

**Lemma 3.1.** ([11, Theorem 3.7]). *If $X \in B(\mathcal{H})$ has dense range and is in the commutant of a $C_{1\cdot}$-contraction $T \in B(\mathcal{H})$, then $X$ is injective.*

In the following we shall denote the normal part and the pure part (i.e., completely non-normal part) of an operator $S \in B(\mathcal{H})$ by $S_n$ and $S_p$, respectively; if $S$ is a contraction, then we shall denote its unitary and cnu parts by $S_u$ and $S_c$, respectively.

**Theorem 3.2.** *Let $S, T \in B(\mathcal{H})$ be $P(k)$ contractions such that $\mu_{S_c} < \infty$. Then $S \sim T$ if and only if $S_u, T_u$ are unitarily equivalent and $S_c \sim T_c$.*

*Proof.* The "if" part being obvious, we prove the "only if" part. Since $S$ and $T$ have $C_{.0}$ cnu parts by Lemma 2.2,

$$S = S_u \oplus S_c = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & * \\ 0 & 0 & S_{33} \end{pmatrix} \quad \text{and} \quad T = T_u \oplus T_c = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & * \\ 0 & 0 & T_{33} \end{pmatrix},$$

where $S_{11} = S_u$, $T_{11} = T_u$, $S_{22}$ and $T_{22} \in C_{00}$, and $S_{33}$ and $T_{33} \in C_{10}$ [8, Chapter II, Theorem 4.1]. Let $SX = XT$ and $TY = YS$, where $X, Y \in B(\mathcal{H})$ are quasi-affinities. Then $X$ and $Y$ have representations $X = [X_{ij}]_{i,j=1}^{3}$ and $Y = [Y_{ij}]_{i,j=1}^{3}$. Observe that $S_{11}X_{12} = X_{12}T_{22}$; since $S_{11}$ is unitary and $T_{22} \in C_{00}$,

$$\|X_{12}x\| = \|S_{11}^{n}X_{12}x\| \leq \|X_{12}\| \, \|T_{22}^{n}x\| \to 0$$

as $n \to \infty$ for all $x$. Hence $X_{12} = 0$. A similar argument shows that indeed $X_{21} = X_{31} = X_{32} = 0 = Y_{12} = Y_{21} = Y_{31} = Y_{32}$. Thus $X_{11}$ and $Y_{11}$ are injective, and

$$X_0 = \begin{pmatrix} X_{22} & X_{23} \\ 0 & X_{33} \end{pmatrix} \quad \text{and} \quad Y_0 = \begin{pmatrix} Y_{22} & Y_{23} \\ 0 & Y_{33} \end{pmatrix}$$

have dense range. The equalities $S_{11}X_{11} = X_{11}T_{11}$ and $T_{11}Y_{11} = Y_{11}S_{11}$ imply that $\overline{\mathrm{ran}X_{11}}$ reduces $S$, $\overline{\mathrm{ran}Y_{11}}$ reduces $T$, $T_{11}$ is unitarily equivalent to $S_{11}|_{\overline{\mathrm{ran}X_{11}}}$ and $S_{11}$ is unitarily equivalent to $T_{11}|_{\overline{\mathrm{ran}Y_{11}}}$. Thus, $S_{11}$ and $T_{11}$ are unitarily equivalent to direct summands of each other. Hence, [5], $S_{11}$ and $T_{11}$ are unitarily equivalent.

By hypothesis, $\mu_{S_c} < \infty$. Since $S_cX_0 = X_0T_c$ and $T_cY_0 = Y_0S_c$, and $X_0$ and $Y_0$ have dense range, $\mu_{S_c} = \mu_{T_c} < \infty$; this, since $\mu_{S_{33}} \leq \mu_{S_c}$ and $\mu_{T_{33}} \leq \mu_{T_c}$, implies that both $\mu_{S_{33}}$ and $\mu_{T_{33}}$ are finite. Evidently, $S_{33}X_{33}Y_{33} = X_{33}Y_{33}S_{33}$ and $T_{33}Y_{33}X_{33} = Y_{33}X_{33}T_{33}$, where $X_{33}Y_{33}$ and $Y_{33}X_{33}$ have dense range. Applying Lemma 3.1 it follows that $X_{33}Y_{33}$ and $Y_{33}X_{33}$ are quasi-affinities; hence $X_{33}$ and $Y_{33}$ are quasi-affinities. But then $X_0$ and $Y_0$ are quasi-affinities; hence $S_c \sim T_c$. □

Theorem 3.2 extends a result on hyponormal contractions of Wu [11, Corollary 3.10], see also [4], to $k$-paranormal contractions. The following corollary extends [11, Corollary 3.11]. Recall that every isometry $V \in B(\mathcal{H})$ has a decomposition $V = V_u \oplus V_c$, where $V_c \in C_{10}$ is a unilateral shift.

**Corollary 3.3.** *Let $S \in P(k)$ be such that (its pure part) $S_p$ has finite multiplicity. Then $S \sim V$ for some isometry $V \in B(\mathcal{H})$ if and only if $S_n$ is unitarily equivalent to $V_u$ and $S_p \sim V_c$.*

*Proof.* Since every isometry satisfies property $(\beta)$, $S \sim V$ implies that $\sigma(S) = \sigma(V) = \mathbf{D}$. Consequently, $S$ is a contraction. Decompose $S$ into its normal and pure parts by $S = S_n \oplus S_p$; then $S_p \in C_{.0}$. Let $V = V_u \oplus V_c$. If $SX = XV$ and $VY = YS$, $X = [X_{ij}]_{i,j=1}^{2}$ and $Y = [Y_{ij}]_{i,j=1}^{2}$, then $S_cX_{21} = X_{21}V_u$ and $V_cY_{21} = Y_{21}S_n$. Clearly, $X_{21} = 0$. Applying the Putnam–Fuglede theorem to $V_cY_{21} = Y_{21}S_n$ it is seen that $\overline{\mathrm{ran}Y_{21}}$ reduces $V_c$ and $V_c|_{\overline{\mathrm{ran}Y_{21}}}$ is unitary. Consequently, $Y_{21} = 0$, $Y_{11}$ is injective and $V_uY_{11} = Y_{11}S_n$. Another application of the Putnam–Fuglede theorem to $V_uY_{11} = Y_{11}S_u$ now shows that $\overline{\mathrm{ran}Y_{11}}$ reduces $V_u$ and $S_n$ is unitarily equivalent to $V_u|_{\overline{\mathrm{ran}Y_{11}}}$. Hence $S_n$ is unitary (and unitarily equivalent to $V_n$). Applying Theorem 3.2, $S_p \sim V_c$, and the proof is complete. □

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