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ON THE a-BROWDER AND a-WEYL SPECTRA OF TENSOR PRODUCTS

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ABSTRACT. Given Banach space operators $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, let $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ denote the tensor product of A and B. Let σ_a, σ_{aw} and σ_{ab} denote the approximate point spectrum, the Weyl approximate point spectrum and the Browder approximate point spectrum, respectively. Then $\sigma_{aw}(A \otimes B) \subseteq \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B) \subseteq \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B) = \sigma_{ab}(A \otimes B)$, and a sufficient condition for the (a-Weyl spectrum) identity $\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B)$ to hold is that $\sigma_{aw}(A \otimes B) = \sigma_{ab}(A \otimes B)$. Equivalent conditions are proved in Theorem 1, and the problem of the transference of a-Weyl's theorem for a-isoloid operators A and B to their tensor product $A \otimes B$ is considered in Theorem 2. Necessary and sufficient conditions for the (plain) Weyl spectrum identity are revisited in Theorem 3.

1. Introduction

Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X} \otimes \mathcal{Y}$ denote the completion (in some reasonable uniform cross norm) of the tensor product of $\mathcal{X}$ and $\mathcal{Y}$. For Banach space operators $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, let $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ denote the tensor product of A and B. Recall that for an operator S, the Browder spectrum $\sigma_b(S)$ and the Weyl spectrum $\sigma_w(S)$ of S are the sets

$$\sigma_b(S) = \{\lambda \in \sigma(S) \colon S - \lambda \text{ is not Fredholm or } \operatorname{asc}(S-\lambda) \neq \operatorname{dsc}(S-\lambda)\},$$
$$\sigma_w(S) = \{\lambda \in \sigma(S) \colon S - \lambda \text{ is not Fredholm or } \operatorname{ind}(S-\lambda) \neq 0\}.$$

(All our notation is explained in the following section). In the case in which $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, two of the authors proved in [8] that

$$\text{if } \sigma_b(A) = \sigma_w(A) \text{ and } \sigma_b(B) = \sigma_w(B), \text{ then } \sigma_b(A \otimes B) = \sigma_w(A \otimes B)$$
$$\text{if and only if } \sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B).$$

In other words, if A and B satisfy Browder's Theorem, then their tensor product satisfies Browder's theorem if and only if the Weyl spectrum identity holds true. The same proof still holds in a Banach space setting, and a new equivalent condition is added in Theorem 3 below.

The current paper considers the Browder approximate point spectrum σ_{ab},

$$\sigma_{ab}(S) = \{\lambda \in \sigma_a(S) \colon \lambda \notin \Phi_+(S) \text{ or } \operatorname{asc}(S-\lambda) = \infty\},$$

and the Weyl approximate point spectrum σ_{aw},

$$\sigma_{aw}(S) = \{\lambda \in \sigma_a(S) \colon \lambda \notin \Phi_+(S) \text{ or } \operatorname{ind}(S-\lambda) > 0\}.$$

Here σ_a denotes the approximate point spectrum and

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$$\Phi_+(S) = \{\lambda \in \sigma(S) \colon S - \lambda \text{ is upper semi-Fredholm}\}.$$

It is proved that

$$\begin{aligned}\sigma_{aw}(A \otimes B) &\subseteq \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B)\\ &\subseteq \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B) = \sigma_{ab}(A \otimes B),\end{aligned}$$

and that,

$$\begin{aligned}&\text{if } \sigma_{ab}(A) = \sigma_{aw}(A) \text{ and } \sigma_{ab}(B) = \sigma_{aw}(B), \text{ then } \sigma_{ab}(A \otimes B) = \sigma_{aw}(A \otimes B)\\ &\qquad\text{if and only if } \sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B),\end{aligned}$$

which extends the above displayed result from [8] to Browder and Weyl approximate point spectrum.

Let $\Pi_0^a(S) = \{\lambda \in \operatorname{iso}\sigma_a(S) : 0 < \dim(S-\lambda)^{-1}(0) < \infty\}$. We prove that if $\sigma_a(A) \setminus \sigma_{aw}(A) = \Pi_0^a(A)$ and $\sigma_a(B) \setminus \sigma_{aw}(B) = \Pi_0^a(B)$, the isolated points of $\sigma_a(A)$ (also, of $\sigma_a(B)$) are eigenvalues of A (resp. B), and $\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B)$, then $\sigma_a(A \otimes B) \setminus \sigma_{aw}(A \otimes B) = \Pi_0^a(A \otimes B)$.

2. Notation and Complementary Results

For a bounded linear operator $S \in B(\mathcal{X})$, let $\sigma(S)$, $\sigma_p(S)$, $\sigma_a(S)$ and $\operatorname{iso}\sigma(S)$ denote, respectively, the spectrum, the point spectrum, the approximate point spectrum of S and the isolated points of $\sigma(S)$. Let $\alpha(S)$ and $\beta(S)$ denote the nullity and the deficiency of S, defined by

$$\alpha(S) = \dim S^{-1}(0) \quad \text{and} \quad \beta(S) = \operatorname{codim} S(\mathcal{X}).$$

If the range $S(\mathcal{X})$ of S is closed and $\alpha(S) < \infty$ (resp. $\beta(S) < \infty$), then S is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. If $S \in B(\mathcal{X})$ is either upper or lower semi-Fredholm, then S is called a *semi-Fredholm* operator, and $\operatorname{ind}(S)$, the *index* of S, is then defined by $\operatorname{ind}(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then S is a *Fredholm* operator. The *ascent*, denoted $\operatorname{asc}(S)$, and the *descent*, denoted $\operatorname{dsc}(S)$, of S are given by

$$\operatorname{asc}(S) = \inf\{n \colon S^{-n}(0) = S^{-(n+1)}(0)\}, \quad \operatorname{dsc}(S) = \inf\{n \colon S^n(\mathcal{X}) = S^{n+1}(\mathcal{X})\}$$

(where the infimum is taken over the set of non-negative integers); if no such integer n exists, then $\operatorname{asc}(S) = \infty$, respectively $\operatorname{dsc}(S) = \infty$. Let

$$\begin{aligned}\Phi_+(S) &= \{\lambda \in \mathbb{C} \colon S - \lambda \text{ is upper semi-Fredholm}\},\\ \Phi_e(S) &= \{\lambda \in \mathbb{C} \colon S - \lambda \text{ is Fredholm}\},\\ \sigma_{SF_+}(S) &= \{\lambda \in \sigma_a(S) \colon \lambda \notin \Phi_+(S)\},\\ \sigma_{aw}(S) &= \{\lambda \in \sigma_a(S) \colon \lambda \in \sigma_{SF_+}(S) \text{ or } \operatorname{ind}(S-\lambda) > 0\},\\ \sigma_{ab}(S) &= \{\lambda \in \sigma_a(S) \colon \lambda \in \sigma_{SF_+}(S) \text{ or } \operatorname{asc}(S-\lambda) = \infty\},\\ \Pi_0^a(S) &= \{\lambda \in \operatorname{iso}\sigma_a(S) \colon 0 < \dim(S-\lambda)^{-1}(0) < \infty\},\\ p_0^a(S) &= \{\lambda \in \operatorname{iso}\sigma_a(S) \colon \lambda \in \Phi_+(S), \operatorname{asc}(S-\lambda) < \infty\},\\ H_0(S) &= \{x \in \mathcal{X} : \lim_{n\to\infty} \|S^n x\|^{1/n} = 0\}.\end{aligned}$$

Recall that $\sigma_{aw}(S)$ is the Weyl approximate point spectrum of S, $\sigma_{ab}(S)$ is the Browder approximate point spectrum of S, and $H_0(S)$ is the quasi-nilpotent part of S [1].

We say that S has the *single valued extension property*, or SVEP, at $\lambda \in \mathbb{C}$ if for every open neighborhood U of λ, the only analytic solution f to the equation $(S-\mu)f(\mu)=0$ for all $\mu \in U$ is the constant function $f \equiv 0$; we say that S has SVEP if S has a SVEP at every $\lambda \in \mathbb{C}$. It is well known that finite ascent implies SVEP; also, an operator has SVEP at every isolated point of its spectrum (as well as at every isolated point of its approximate point spectrum).

We say that $S \in B(\mathcal{X})$ satisfies a-Browder's theorem (shortened to S satisfies a-Bt) if $\sigma_{aw}(S)=\sigma_{ab}(S)$ (if and only if $\sigma_a(S)\setminus\sigma_{aw}(S)=p_0^a(S)$, see [1, p. 156]); S satisfies a-Weyl's theorem (shortened to S satisfies a-Wt) if $\sigma_a(S)\setminus\sigma_{aw}(S)=\Pi_0^a(S)$ (if and only if S satisfies a-Bt and $p_0^a(S)=\Pi_0^a(S)$) [1, p. 177]. The implications a-Wt $\Longrightarrow$ a-Bt and a-Wt $\Longrightarrow$ Weyl's theorem are well known. Let iso $\sigma_a(S)$ denote the isolated points of $\sigma_a(S)$.

Lemma 1. [1, Theorem 3.23]. *If $S \in B(\mathcal{X})$ has SVEP at $\lambda \in \sigma(S)\setminus\sigma_{SF_+}(S)$, then $\lambda \in iso\, \sigma_a(S)$ and $asc(S-\lambda)<\infty$.*

Let $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$.

Lemma 2. [2] and [6, Theorem 4.4 (a),(b)].

(i) $\sigma_x(A\otimes B)=\sigma_x(A)\sigma_x(B)$, *where* $\sigma_x=\sigma$ *or* σ_a.

(ii) $\sigma_{SF_+}(A\otimes B)=\sigma_{SF_+}(A)\sigma_a(B)\cup\sigma_a(A)\sigma_{SF_+}(B)$.

The inclusions below are readily verified.

Lemma 3. *$iso\,\sigma_a(A\otimes B)\subseteq iso\,\sigma_a(A)\; iso\,\sigma_a(B)\cup\{0\}$ and $\sigma_p(A)\sigma_p(B)\subseteq\sigma_p(A\otimes B)$.*

Lemma 4. $0\notin\sigma_a(A\otimes B)\setminus\sigma_{SF_+}(A\otimes B)$.

Proof. Suppose that $0\in\sigma_a(A\otimes B)\setminus\sigma_{SF_+}(A\otimes B)$. Then $0\in\sigma_a(A\otimes B)\cap\Phi_+(A\otimes B)$, i.e., $A\otimes B$ has closed range and $0<\alpha(A\otimes B)<\infty$. Since $A\otimes B$ is injective if and only if A and B are injective, we have that $\alpha(A)>0$ or $\alpha(B)>0$. But then $\alpha(A\otimes B)=\infty$, and we have a contradiction. □

3. Results

We start with a lemma relating $\sigma_{aw}(A\otimes B)$ and $\sigma_{ab}(A\otimes B)$.

Lemma 5. *Let $A\in B(\mathcal{X})$ and $B\in B(\mathcal{Y})$.*

$$\begin{aligned}\sigma_{aw}(A\otimes B) &\subseteq \sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)\\ &\subseteq \sigma_a(A)\sigma_{ab}(B)\cup\sigma_{ab}(A)\sigma_a(B)=\sigma_{ab}(A\otimes B).\end{aligned}$$

Proof. Since $\sigma_{aw}(S)\subseteq\sigma_{ab}(S)$, for every operator S, the inclusion $\sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)\subseteq\sigma_a(A)\sigma_{ab}(B)\cup\sigma_{ab}(A)\sigma_a(B)$ is evident. To prove the inclusion $\sigma_{aw}(A\otimes B)\subseteq\sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)$, take $\lambda\notin\sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)$. Since $\sigma_{SF_+}(A\otimes B)\subseteq\sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)$, Lemma 4 implies that $\lambda\neq 0$. For every factorization $\lambda=\mu\nu$ of λ such that $\mu\in\sigma_a(A)$ and $\nu\in\sigma_a(B)$ we have that $\mu\in\sigma_a(A)\setminus\sigma_{aw}(A)$ and $\nu\in\sigma_a(B)\setminus\sigma_{aw}(B)$, i.e., $\lambda\in\Phi_+(A)$, $\nu\in\Phi_+(B)$, $\mathrm{ind}(A-\mu)\leq 0$ and $\mathrm{ind}(B-\nu)\leq 0$. In particular, $\lambda\notin\sigma_{SF_+}(A\otimes B)$.

We prove next that $\mathrm{ind}(A\otimes B-\lambda)\leq 0$. Suppose that $\mathrm{ind}(A\otimes B-\lambda)>0$. Then $\alpha(A\otimes B-\lambda)<\infty$ implies that $\beta(A\otimes B-\lambda)<\infty$, so that $\lambda\in\Phi_e(A\otimes B)$. Let

$$E = \{(\mu_i, \nu_i)_{i=1}^{p} \in \sigma(A)\sigma(B) \colon \ \mu_i \nu_i = \lambda\}.$$

Then E is a finite set. Furthermore (see [5, Theorem 3.1] and [4]):

(i) if $n > 1$, then $\mu_i \in \operatorname{iso}\sigma(A)$, for $1 \leq i \leq n$;

(ii) if $p > n$, then $\nu_i \in \operatorname{iso}\sigma(B)$, for $n+1 \leq i \leq p$;

(iii) $\operatorname{ind}(A \otimes B - \lambda) = \sum_{j=n+1}^{p} \operatorname{ind}(A - \mu_j) \dim H_0(B - \nu_j)$
$\qquad + \sum_{j=1}^{n} \operatorname{ind}(B - \nu_j) \dim H_0(A - \mu_j)$.

Since $\operatorname{ind}(A - \mu_i)$ and $\operatorname{ind}(B - \nu_i)$ are non-positive, we have a contradiction. Hence, $\operatorname{ind}(A \otimes B - \lambda) \leq 0$, and consequently $\lambda \notin \sigma_{aw}(A \otimes B)$. This leaves us to prove the equality $\sigma_{ab}(A \otimes B) = \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B)$.

Suppose that $\lambda \notin \sigma_{ab}(A \otimes B)$. Then $\lambda \neq 0$, $\lambda \in \Phi_+(A \otimes B)$ and $\operatorname{asc}(A \otimes B - \lambda) < \infty$. Observe that $\lambda \in \operatorname{iso}\sigma_a(A \otimes B)$. Let $\lambda = \mu\nu$ be any factorization of λ such that $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B)$; then $\mu \in \Phi_+(A)$ and $\nu \in \Phi_+(B)$. Furthermore, since $\sigma_a(A \otimes B) \subseteq \operatorname{iso}\sigma_a(A) \operatorname{iso}\sigma_a(B) \cup \{0\}$, A has SVEP at μ and B has SVEP at ν. Consequently, $\mu \in \Phi_+(A)$, $\operatorname{asc}(A - \mu) < \infty$, $\nu \in \Phi_+(B)$ and $\operatorname{asc}(B - \nu) < \infty$, i.e., $\mu \notin \sigma_{ab}(A)$ and $\nu \notin \sigma_{ab}(B)$. But then $\lambda \notin \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B)$. Hence $\sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B) \subseteq \sigma_{ab}(A \otimes B)$.

To prove the reverse inclusion we start by recalling the (easily proved) fact that if $\mu \in \operatorname{iso}\sigma_a(A)$ and $\nu \in \operatorname{iso}\sigma_a(B)$ for every factorization $\lambda = \mu\nu$ of $\lambda \neq 0$, then $\lambda = \mu\nu \in \operatorname{iso}\sigma_a(A \otimes B)$. Let $\lambda \notin \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B)$. Then $\lambda \neq 0$. Furthermore, if $\lambda = \mu\nu$ is any factorization of λ such that $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B)$, then the following implications hold:

$$\begin{aligned}
&\mu \notin \sigma_{ab}(A) \text{ and } \nu \notin \sigma_{ab}(B) \quad \Longrightarrow \\
&\mu \in \Phi_+(A),\ \nu \in \Phi_+(B),\ \operatorname{asc}(A - \mu) < \infty \text{ and } \operatorname{asc}(B - \nu) < \infty \quad \Longrightarrow \\
&\lambda \in \Phi_+(A \otimes B),\ \mu \in \operatorname{iso}\sigma_a(A) \text{ and } \nu \in \operatorname{iso}\sigma_a(B) \quad \Longrightarrow \\
&\lambda \in \Phi_+(A \otimes B) \text{ and } \lambda \in \operatorname{iso}\sigma_a(A \otimes B) \quad \Longrightarrow \\
&\lambda \in \Phi_+(A \otimes B) \text{ and } \operatorname{asc}(A \otimes B - \lambda) < \infty \quad \Longrightarrow \\
&\lambda \notin \sigma_{ab}(A \otimes B).
\end{aligned}$$

Hence, $\sigma_{ab}(A \otimes B) \subseteq \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B)$, and the proof is complete. □

The equality $\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B)$ fails to hold in general, as follows from Remark 2(a) below. The following lemma gives a sufficient condition for the equality to hold.

Lemma 6. *If $A \otimes B$ satisfies a-Bt, then*

$$\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B).$$

Proof. $A \otimes B$ satisfies a-Bt if and only if $\sigma_{aw}(A \otimes B) = \sigma_{ab}(A \otimes B)$. Thus the stated result is an immediate consequence of Lemma 5. □

Remark 1. The hypothesis $\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A) \cup \sigma_a(B)$ is not sufficient to guarantee that $A \otimes B$ satisfies a-Bt. Indeed, the conclusion $A \otimes B$ satisfies a-Bt is liable to fail if one of A and B does not satisfy a-Bt, as the following example shows. Let $U \in B(\ell^2_+)$ denote the forward unilateral shift $U(x_1, x_2, x_3, \ldots) =$

$(0, x_1, x_2, x_3, ...)$. Let $A = U^*$ (or, $A = U \oplus U^*$) and let $B \in B(\ell_+^2)$ be the operator $B = I \oplus \frac{1}{2}I$, $I(x_1, x_2, ...) = (x_1, x_2, ...)$ for all $(x_1, x_2, x_3, ...) \in \ell_+^2$. Then, letting $\partial\mathbb{D}$ denote the boundary of the closed unit disc $\mathbb{D}$, $\sigma_a(A) = \mathbb{D}$, $\sigma_{aw}(A) = \partial\mathbb{D}$, $\sigma(B) = \sigma_{aw}(B) = \sigma_w(B) = \{\frac{1}{2}, 1\}$, $\sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A) \cup \sigma_a(B) = \partial\mathbb{D} \cup \frac{1}{2}\mathbb{D}$ and $\sigma_a(A \otimes B) = \mathbb{D} = \sigma(A \otimes B)$. Since $\mathrm{asc}(A - \mu) = \infty$ for all $\mu \in \sigma(A)$ (in both choices of A), $\sigma_{ab}(A \otimes B) = \mathbb{D}$. Evidently, A and $A \otimes B$ do not satisfy a-Bt. We remark here that if either $\sigma_a(C) = \sigma_{aw}(C)$ or $\sigma_a(D) = \sigma_{aw}(D)$ for some operators $C \in B(\mathcal{X})$ and $D \in B(\mathcal{Y})$, then $\sigma(C \otimes D) = \sigma_a(C)\sigma_{aw}(D) \cup \sigma_{aw}(C) \cup \sigma_a(D)$; consequently, $\sigma_w(C \otimes D) = \sigma_a(C)\sigma_{aw}(D) \cup \sigma_{aw}(C) \cup \sigma_a(D)$ implies $C \otimes D$ satisfies a-Bt.

The next theorem, our main result, proves that A and B satisfy a-Bt implies $A \otimes B$ satisfies a-Bt if and only if $\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B)$.

Theorem 1. *If A and B satisfy a-Bt, then the following conditions are equivalent:*

(i) *$A \otimes B$ satisfies a-Bt.*

(ii) *$\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B)$.*

(iii) *A has SVEP at every $\mu \in \Phi_+(A)$ and B has SVEP at every $\nu \in \Phi_+(B)$ such that $(0 \neq)\lambda = \mu\nu \in \sigma_a(A \otimes B) \setminus \sigma_{aw}(A \otimes B)$.*

Proof. If A and B satisfy a-Bt, then $\sigma_{aw}(A) = \sigma_{ab}(A)$ and $\sigma_{aw}(B) = \sigma_{ab}(B)$.

(i) $\Leftrightarrow$ (ii). By Lemma 6 we have, without any extra condition, that (i) $\Rightarrow$ (ii). If (ii) is satisfied, then

$$\begin{aligned}\sigma_{aw}(A \otimes B) &= \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B) \\ &= \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B) = \sigma_{ab}(A \otimes B)\end{aligned}$$

(by Lemma 5). Hence $A \otimes B$ satisfies a-Bt.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. Let $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{aw}(A \otimes B) = \sigma_a(A \otimes B) \setminus \sigma_{ab}(A \otimes B)$. Then ($\lambda \neq 0$ and) for every factorization $\lambda = \mu\nu$ of λ such that $\mu \in \sigma_a(A) \cap \Phi_+(A)$ and $\nu \in \sigma_a(B) \cap \Phi_+(B)$ we have that $\mathrm{asc}(A - \mu)$ and $\mathrm{asc}(B - \nu)$ are finite. Hence, A and B have SVEP at (all such) μ and ν, respectively.

(iii) $\Rightarrow$ (ii). In view of Lemma 5, we have to prove that $\sigma_{ab}(A \otimes B) \subseteq \sigma_{aw}(A \otimes B)$. Suppose that (iii) is satisfied. Take a $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{aw}(A \otimes B)$. Then $(0 \neq)\lambda \in \Phi_+(A \otimes B)$ and $\mathrm{ind}(A \otimes B) \leq 0$. The equality $\sigma_{SF_+}(A \otimes B) = \sigma_{SF_+}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{SF_+}(B)$ (see Lemma 2) implies that for every factorization $\lambda = \mu\nu$ of λ (such that $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B)$) we have that $\mu \in \Phi_+(A)$ and $\nu \in \Phi_+(B)$. The SVEP hypothesis on A and B implies that $\mathrm{asc}(A - \mu)$ and $\mathrm{asc}(B - \nu)$ are finite. Hence, $\mu \notin \sigma_{ab}(A)$ and $\nu \notin \sigma_{ab}(B)$. But then $\lambda \notin \sigma_{ab}(A \otimes B)$; hence $\sigma_{ab}(A \otimes B) \subseteq \sigma_{aw}(A \otimes B)$. □

It is worth noticing that the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 1 extends the result [8, Corollary 6] from plain Browder's theorem and plain Weyl spectrum identity to a-Browder's theorem and a-Weyl spectrum identity.

Remark 2. Recall that $S \in B(\mathcal{X})$ satisfies Browder's theorem if $\sigma_w(S) = \sigma_b(S)$. Let $U \in B(\ell^2)$ denote the forward unilateral shift, and define the operators A, B by $A = (1 - UU^*) \oplus (\frac{1}{2}U - 1)$, $B = -(1 - UU^*) \oplus (\frac{1}{2}U^* + 1)$. Then A and B satisfy

Browder's theorem, but $A\otimes B$ does not satisfy Browder's theorem [7, Section 3]. More is true.

(a) Since A and B^* have SVEP, they satisfy a-Browder's theorem. Furthermore, since the operator $A\otimes B$ fails to have SVEP on the complement of $\sigma_w(A\otimes B)$, it fails to have SVEP on $\sigma_a(A\otimes B)\setminus\sigma_{aw}(A\otimes B)\supset\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)$, hence does not satisfy a-Browder's theorem [3, Lemma 2.18]. Theorem 1 implies that $A\otimes B$ does not satisfy the equality $\sigma_{aw}(A\otimes B)=\sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)$.

(b) A well-known open question, implicitly posed in [5, Theorem 4.2], asks whether the following inclusion, which holds for all operators A and B,

$$\sigma_w(A\otimes B)\subseteq\sigma_w(A)\cdot\sigma(B)\,\cup\,\sigma(A)\cdot\sigma_w(B),$$

may become a proper inclusion for some pair of operators. Equivalently, whether the above inclusion is an identity for every pair of operators. However, it was proved in [8, Proposition 7(a)] that *if A and B are such that*

$$\sigma_w(A\otimes B)=\sigma_w(A)\cdot\sigma(B)\,\cup\,\sigma(A)\cdot\sigma_w(B)$$

(i.e., if the Weyl spectrum identity holds) *and if both A and B satisfy Browder's theorem, then the tensor product $A\otimes B$ satisfies Browder's theorem.* Therefore, if there exist A and B that satisfy Browder's theorem, but $A\otimes B$ does not satisfy Browder's theorem, then the Weyl spectrum identity does not hold for them.

The next theorem gives a sufficient condition for $A\otimes B$ to satisfy a-Wt, given that A and B satisfy a-Wt. But before that a couple of technical lemmas. Recall that an operator S is said to be a-isoloid if $\lambda\in\operatorname{iso}\sigma_a(S)$ implies $\lambda\in\sigma_p(S)$.

Lemma 7. *A and B are a-isoloid implies $A\otimes B$ is a-isoloid.*

Proof. If $\operatorname{iso}\sigma_a(A)=\operatorname{iso}\sigma_a(B)=\varnothing$, then $\operatorname{iso}\sigma_a(A\otimes B)=\varnothing$. Observe also that if either of $\operatorname{iso}\sigma_a(A)$ or $\operatorname{iso}\sigma_a(B)$ is the empty set, say $\operatorname{iso}\sigma_a(A)=\varnothing$, then $\operatorname{iso}\sigma_A(A\otimes B)\subseteq\{0\}$ and $0\in\operatorname{iso}\sigma_a(B)$. But then $0\in\sigma_p(B)$, which implies that $0\in\sigma_p(A\otimes B)$. Now let $\lambda\in\operatorname{iso}\sigma_a(A\otimes B)$ be such that $\lambda=\mu\nu$, $\mu\in\operatorname{iso}\sigma_a(A)$ and $\nu\in\operatorname{iso}\sigma_a(B)$. Then $\mu\in\sigma_p(A)$ and $\nu\in\sigma_p(B)$. Since $\sigma_p(A)\cdot\sigma_p(B)\subseteq\sigma_p(A\otimes B)$, we have $\lambda\in\sigma_p(A\otimes B)$. □

Lemma 8. *Suppose that A, B and $A\otimes B$ satisfy a-Bt. If $\mu\in p_0^a(A)$ and $\nu\in p_0^a(B)$, then $\lambda=\mu\nu\in p_0^a(A\otimes B)$.*

Proof. $\mu\in\sigma_a(A)\setminus\sigma_{aw}(A)$, $\nu\in\sigma_a(B)\setminus\sigma_{aw}(B)$ and $\sigma_{aw}(A\otimes B)=\sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)$. Hence, $\lambda=\mu\nu\in\sigma_a(A\otimes B)\setminus\sigma_{aw}(A\otimes B)=p_0^a(A\otimes B)$. □

Theorem 2. *Suppose that A and B are a-isoloid operators which satisfy a-Wt. If $\sigma_{aw}(A\otimes B)=\sigma_a(A)\sigma_{aw}(B)\cup\sigma_{aw}(A)\sigma_a(B)$, then $A\otimes B$ satisfies a-Wt.*

Proof. The hypotheses imply that $A\otimes B$ satisfies a-Bt, i.e., $\sigma_a(A\otimes B)\setminus\sigma_{aw}(A\otimes B)=p_0^a(A\otimes B)$. Since $p_0^a(A\otimes B)\subseteq\Pi_0^a(A\otimes B)$, we have to prove that $\Pi_0^a(A\otimes B)\subseteq p_0^a(A\otimes B)$. Let $\lambda\in\Pi_0^a(A\otimes B)$. Then $(0\neq)\lambda=\mu\nu$ for some $\mu\in\operatorname{iso}\sigma_a(A)$ and $\nu\in\operatorname{iso}\sigma_a(B)$. The operators A and B being a-isoloid, it follows (from $\lambda\in\Pi_0^a(A\otimes B)$) that $\mu\in\Pi_0^a(A)=p_0^a(A)$ and $\nu\in\Pi_0^a(B)=p_0^a(B)$. By Lemma 8, $\lambda\in p_0^a(A\otimes B)$. □

4. Browder's Theorem

A bounded linear operator S satisfies Browder's theorem, Bt for short, if $\sigma_w(S) = \sigma_b(S)$; S satisfies Weyl theorem, Wt for short, if $\sigma(S) \backslash \sigma_w(S) = \Pi_0(S)$ (equivalently, if S satisfies Bt and $p_0(S) = \Pi_0(S)$), where $p_0(S)$ ($\Pi_0(S)$) is the set of isolated point in $\sigma(S)$ with finite ascent and descent (resp. with finite dimensional kernel). It is known, [1], that

$$\begin{array}{ccccc} & & \text{Weyl's theorem} & & \\ & \nearrow & & \searrow & \\ \text{a-Weyl's theorem} & & & & \text{Browder's theorem.} \\ & \searrow & & \nearrow & \\ & & \text{a-Browder's theorem} & & \end{array}$$

Theorem 3. *If A and B satisfy Bt, then the following conditions are equivalent:*

(i) *$A \otimes B$ satisfies Bt.*

(ii) *$\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$.*

(iii) *A has SVEP at points $\mu \in \Phi(A)$ and B has SVEP at points $\nu \in \Phi(B)$ such that $(0 \neq)\lambda = \mu\nu \notin \sigma_w(A \otimes B)$.*

Proof. (i) $\Leftrightarrow$ (ii). [8, Corollary 6].

(ii) $\Leftrightarrow$ (iii). The proof is similar to that of Theorem 1: we include it here for completeness. Suppose that (ii) holds. Let $\lambda \in \sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_b(A \otimes B)$. Then ($\lambda \neq 0$ and) for every factorization $\lambda = \mu\nu$ of λ such that $\mu \in \sigma(A) \cap \Phi(A)$ and $\nu \in \sigma(B) \cap \Phi(B)$ we have that $\operatorname{asc}(A - \mu)$ and $\operatorname{asc}(B - \nu)$ are finite. Hence, A and B have SVEP at (all such) μ and ν, respectively. We suppose next that (iii) is satisfied and prove that $\sigma_b(A \otimes B) \subseteq \sigma_w(A \otimes B)$. Take a $\lambda \in \sigma(A \otimes B) \setminus \sigma_w(A \otimes B)$. Then $(0 \neq)\lambda \in \Phi(A \otimes B)$ and $\operatorname{ind}(A \otimes B - \lambda) = 0$. The equality $\sigma_e(A \otimes B) = \sigma_e(A)\sigma(B) \cup \sigma(A)\sigma_e(B)$ implies that for every factorization $\lambda = \mu\nu$ of λ (such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$) we have $\mu \in \Phi(A)$ and $\nu \in \Phi(B)$. The SVEP hypothesis on A and B implies that $\operatorname{asc}(A - \mu)$ and $\operatorname{asc}(B - \nu)$ are finite (which in turn implies that $\operatorname{ind}(A - \mu)$ and $\operatorname{ind}(B - \nu)$ are both ≤ 0). Thus, in view of the fact that $\operatorname{ind}(A \otimes B - \lambda) = 0$, it follows from the index formula (iii) of the proof of Lemma 5 that $\operatorname{ind}(A - \mu)$ and $\operatorname{ind}(B - \nu)$ are both 0. Consequently, both $A - \mu$ and $B - \nu$ have finite ascent and descent [1, Theorem 3.4 (iv)]. But then $\mu \notin \sigma_b(A)$ and $\nu \notin \sigma_b(B)$, which implies that $\lambda \notin \sigma_b(A \otimes B)$. Hence $\sigma_b(A \otimes B) \subseteq \sigma_w(A \otimes B)$. $\square$

Remark 3. A result similar to that in Lemma 6 shows that the hypothesis $A \otimes B$ satisfies Bt ensures that $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$ (see [8, Proposition 6]). Also, the example in Remark 1 shows that the opposite implication is not true in general. Indeed, let $A = U \oplus U^*$ and B be defined as in Remark 1. Then $\sigma(A) = \mathbb{D}$, $\sigma_w(A) = \partial\mathbb{D}$, $\sigma(B) = \sigma_w(B) = \{\frac{1}{2}, 1\}$, $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B) = \partial\mathbb{D} \cup \frac{1}{2}\mathbb{D}$, $\sigma_b(A \otimes B) = \mathbb{D}$ and $\sigma(A \otimes B) = \mathbb{D}$. Evidently, A and $A \otimes B$ do not satisfy Bt.

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