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A NOTE ON k-PARANORMAL OPERATORS

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Abstract. It is still unknown whether the inverse of an invertible k-paranormal operator is normaloid, and so whether a k-paranormal operator is totally hereditarily normaloid. We provide sufficient conditions for the inverse of an invertible k-paranormal operator to be k-paranormal.

1. Preliminaries

Let $\mathcal{B}[\mathcal{H}]$ stand for the Banach algebra of all bounded linear transformations of a nonzero complex Hilbert space $\mathcal{H}$ into itself. By an operator we mean an element from $\mathcal{B}[\mathcal{H}]$. If T lies in $\mathcal{B}[\mathcal{H}]$, then T^* in $\mathcal{B}[\mathcal{H}]$ denotes the adjoint of X. The range and kernel of $T \in \mathcal{B}[\mathcal{H}]$ will be denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. By a contraction we mean an operator $T \in \mathcal{B}[\mathcal{H}]$ such that $\|T\| \leq 1$. An isometry is a contraction T such that $\|Tx\| = \|x\|$ for every $x \in \mathcal{H}$. If both T and T^* are isometries, then T is a unitary operator. A contraction is said to be completely nonunitary if it has no unitary direct summand. For any contraction T the sequence of positive numbers $\{\|T^n x\|\}$ is decreasing (thus convergent) for every $x \in \mathcal{H}$. A contraction T is of class $\mathcal{C}_{0\cdot}$ if it is strongly stable; that is, if $\{\|T^n x\|\}$ converges to zero for every $x \in \mathcal{H}$, and of class $\mathcal{C}_{1\cdot}$ if $\{\|T^n x\|\}$ does not converge to zero for every nonzero $x \in \mathcal{H}$. It is of class $\mathcal{C}_{\cdot 0}$ or of class $\mathcal{C}_{\cdot 1}$ if its adjoint T^* is of class $\mathcal{C}_{0\cdot}$ or $\mathcal{C}_{1\cdot}$, respectively, leading to the Nagy–Foiaş classes of contractions $\mathcal{C}_{00}$, $\mathcal{C}_{01}$, $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$ [23, p. 72].

The classes of subnormal and hyponormal operators were introduced more than half a century ago by Paul Halmos in [12]. Since then, these have been considered in current literature along with a myriad of classes of close to normal operators. We shall be concerned with just a few of these well-known classes of operators that properly include the hyponormals. An operator T is *dominant* if, for each $\lambda \in \mathbb{C}$, there exists a real number M_λ such that $\|(\lambda I - T)^* x\| \leq M_\lambda \|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$ or, equivalently, if $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$; and it is called M*-hyponormal* if there exists a real number $M \geq 1$ such that, for all $\lambda \in \mathbb{C}$, $\|(\lambda I - T)^* x\| \leq M\|(\lambda I - T)x\|$ for every $x \in \mathcal{H}$. A *hyponormal* is precisely a 1-hyponormal operator (i.e., an operator T such that $TT^* \leq T^*T$ or, equivalently, $\|(\lambda I - T)^* x\| \leq \|(\lambda I - T)x\|$ for every $\lambda \in \mathbb{C}$ and every $x \in \mathcal{H}$). As usual, put $|T| = (T^*T)^{\frac{1}{2}}$, the absolute value of T. A p*-hyponormal* is an operator T such that $|T^*|^{2p} \leq |T|^{2p}$ for some real number $0 < p \leq 1$. Again, a hyponormal is precisely a 1-hyponormal. An operator T is k*-quasihyponormal* if $T^{*k}(T^*T - TT^*)T^k \geq O$ for some integer $k \geq 1$, and *quasi-*p*-hyponormal* (also called p*-quasihyponormal*) if $T^*(|T|^{2p} - |T^*|^{2p})T \geq O$ for some

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real $0 < p \leq 1$. A *quasihyponormal* is a 1-quasihyponormal or a quasi-1-hyponormal operator or, equivalently, an operator T such that $|T|^4 \leq |T^2|^2$; and so a *semi-quasi-hyponormal* is an operator T such that $|T|^2 \leq |T^2|$ (also called *class* $\mathcal{A}$ or *class* $\mathcal{U}$). An operator T is k*-paranormal* if $\|Tx\|^{k+1} \leq \|T^{k+1}x\|\,\|x\|^k$ for some integer $k \geq 1$ and every $x \in \mathcal{H}$. Equivalently, T is k-paranormal if $\|Tx\|^{k+1} \leq \|T^{k+1}x\|$ for some integer $k \geq 1$ and every unit vector $x \in \mathcal{H}$ (i.e., for every $x \in \mathcal{H}$ such that $\|x\| = 1$). A *paranormal* is simply a 1-paranormal operator.

See [3], [4], [8], [10], [14], [15], [22] and [25] for properties of operators belonging to the above classes. Recall that a paranormal operator is k-paranormal for every positive integer k (see e.g., [10, p. 271] or [14, Problem 9.17]), and so an operator is paranormal if and only if it is k-paranormal for every $k \geq 1$. The diagram below summarizes the relationship among these classes.

$$\begin{array}{ccccccccc}
 & & p\text{-hyponormal} & \longrightarrow & \text{quasi-}p\text{-hyponormal} & & & & \\
 & \nearrow & & & \downarrow_1 & \searrow_2 & & & \\
\text{hyponormal} & \longrightarrow & \text{quasihyponormal} & \longrightarrow & \text{semi-quasihyponormal} & \longrightarrow & \text{paranormal} & \longrightarrow & k\text{-paranormal} \\
 & \searrow & & \searrow & & & & & \\
 & & M\text{-hyponormal} & & k\text{-quasihyponormal} & & & & \\
 & & & \searrow & & & & & \\
 & & & & \text{dominant} & & & &
\end{array}$$

For the nontrivial implications in the central row (from hyponormal through k-paranormal) see e.g., [14, p. 94]. Those in 1 and 2 can be found in [9]–[11] and [1], respectively. The remaining implications are either readily verified or trivial.

2. Introduction

What all the above classes have in common besides including the hyponormal operators? Putnam [18] gave the first proof that completely nonunitary hyponormal contractions are of class $\mathcal{C}_{\cdot 0}$ (also see [16]). This was extended to paranormal contractions in [17] and to dominant contractions in [22] (also see [4], [24], and the references therein). This was further extended to both k-paranormal and k-quasihyponormal contractions in [7]. Therefore, every completely nonunitary contraction in any of those classes appearing in the diagram of Section 1 is of class $\mathcal{C}_{\cdot 0}$ — all of them are included in the union of dominant, k-quasihyponormal and k-paranormal contractions. We show that in this sense (that is, in the sense that completely nonunitary contractions are of class $\mathcal{C}_{\cdot 0}$) the diagram of Section 1 is tight enough. Posinormal operators (defined in Section 5) comprise a class that properly includes the dominant operators. Hereditarily normaloid operators (defined in Section 3) comprise a class that properly includes the k-paranormal operators. We exhibit in Section 5 a completely nonunitary posinormal contraction and a completely nonunitary hereditarily normaloid contraction that are not of class $\mathcal{C}_{\cdot 0}$,

It is known that every k-paranormal operator is hereditarily normaloid (every part of it is normaloid), and that a paranormal operator (i.e., a 1-paranormal operator) is totally hereditarily normaloid (it is hereditarily normaloid and every invertible part of it has a normaloid inverse). However it remains as an open question whether the inverse of an invertible k-paranormal operator for $k \geq 2$ is normaloid, and so whether a k-paranormal operator for $k \geq 2$ is totally hereditarily normaloid. Sufficient conditions for an invertible k-paranormal operator to have a k-paranormal inverse are given in Theorems 1 and 2 of Section 4, and hence for a k-paranormal operator to be totally hereditarily normaloid.

3. Intermediate Results: k-Paranormal

Recall that a part $T|_{\mathcal{M}}$ of an operator T is a restriction of it to an invariant subspace $\mathcal{M}$, and that an operator T is *normaloid* if its spectral radius coincides with its norm (i.e., if $r(T) = \|T\|$) or, equivalently, if $\|T^n\| = \|T\|^n$ for every nonnegative integer n. An operator is *hereditarily normaloid* if every part of it (including itself) is normaloid (also called *invariant normaloid* [10, p. 275]) and *totally hereditarily normaloid* if it is hereditarily normaloid and the inverse of every invertible part of it (including its own inverse if it is invertible) is normaloid [5]. Paranormal operators are totally hereditarily normaloid (which are trivially hereditarily normaloid, and tautologically normaloid), and all these inclusions are proper (cf. [6]). We start with a new, short and simple proof of a proposition that extends the right end of the above diagram, asserting that k-paranormal operators are hereditarily normaloid, as follows.

$$\begin{array}{ccccccc} & & \text{totally hereditarily normaloid} & & & & \\ & \nearrow & & \searrow & & & \\ \text{paranormal} & \longrightarrow & k\text{-paranormal} & \longrightarrow & \text{hereditarily normaloid} & \longrightarrow & \text{normaloid} \end{array}$$

For a different proof see [10, p. 267–273]).

Proposition 1. *Every k-paranormal operator is hereditarily normaloid.*

Proof. The proof is split into two parts.

(a) Every k-paranormal operator is normaloid.

(b) Every part of a k-paranormal operator is again k-paranormal.

Proof of (a). Let $T \neq O$ in $\mathcal{B}[\mathcal{H}]$ be k-paranormal so that, for some integer $k \geq 1$,

$$\|Tx\|^{k+1} \leq \|T^{k+1}x\| \, \|x\|^k \quad \text{for every} \quad x \in \mathcal{H}.$$

Take any integer $j \geq 1$. Observe that

$$\|T^j x\|^{k+1} \leq \|T^{k+j}\| \, \|T^{j-1}\|^k \|x\|^{k+1}$$

for every $x \in \mathcal{H}$, which implies $\|T^j\|^{k+1} \leq \|T^{k+j}\| \, \|T^{j-1}\|^k$. Suppose $\|T^j\| = \|T\|^j$ for some $j \geq 1$ (which holds tautologically for $j = 1$). Then, by the above inequality,

$$\|T\|^{(k+1)j} = (\|T\|^j)^{k+1} = \|T^j\|^{k+1} \leq \|T^{k+j}\| \, \|T^{j-1}\|^k \leq \|T^{k+j}\| \, \|T\|^{(j-1)k},$$

and therefore

$$\|T^{k+j}\| = \|T\|^{k+j}.$$

Thus, by induction, $\|T^{1+jk}\| = \|T\|^{1+jk}$ for every $j \geq 1$. This yields a subsequence $\{T^{n_j}\}$ of $\{T^n\}$, say $T^{n_j} = T^{1+jk}$, such that $\lim_j \|T^{n_j}\|^{\frac{1}{n_j}} = \lim_j (\|T\|^{n_j})^{\frac{1}{n_j}} = \|T\|$. Since $\{\|T^n\|^{\frac{1}{n}}\}$ is a convergent sequence that converges to the spectral radius of T (Beurling–Gelfand formula for the spectral radius), and since it has a subsequence that converges to $\|T\|$, it follows that $r(T) = \|T\|$, which means that T is normaloid.

Proof of (b). If $\mathcal{M}$ is a T-invariant subspace, then, for every u in $\mathcal{M}$,

$$\|T|_{\mathcal{M}} u\|^{k+1} = \|Tu\|^{k+1} \leq \|T^{k+1}u\| \, \|u\|^k = \|(T|_{\mathcal{M}})^{k+1}u\| \, \|u\|^k,$$

and so $T|_{\mathcal{M}}$ is k-paranormal whenever $T \in \mathcal{B}[\mathcal{H}]$ is k-paranormal for some $k \geq 1$. □

Observe that k-paranormality and normaloidness are closed under nonzero scaling (i.e., for every $\alpha \neq 0$, αT is k-paranormal or normaloid if and only if T is), and so is hereditarily and totally hereditarily normaloidness (since the lattice of invariant subspaces and inversion are closed under nonzero scaling). Moreover, since any power of a paranormal operator is paranormal, it follows that if the power T^m for some $m \geq 1$ is paranormal, then T^{mn} is paranormal for every $n \geq 1$, but T itself may not be paranormal.

> However *if T^{k+1} is a multiple of an isometry for some $k \geq 1$* (i.e.,if $\|T^{k+1}x\| = \|T\|^{k+1}\|x\|$ for every $x \in \mathcal{H}$) *then T is k-paranormal.*

Indeed, in this case, $\|Tx\|^{k+1} \leq \|T\|^{k+1}\|x\|^{k+1} = \|T^{k+1}x\|\,\|x\|^k$ for each $x \in \mathcal{H}$. Note that if T^{k+1} is a multiple of an isometry then T^{k+1} is paranormal, since isometries are hyponormal — quasinormal, actually — and so T^{k+1} is j-paranormal for every $j \geq 1$. Further conditions for k-paranormality are given in the next lemmas.

Lemma 1. *Take any $T \in \mathcal{B}[\mathcal{H}]$ and an arbitrary integer $k \geq 1$. Suppose either*

$$\|T^k x\|^{k+1} \leq \|T^{k+1}x\|^k \tag{1}$$

or

$$\|T^k x\|\,\|Tx\| \leq \|T^{k+1}x\| \tag{2}$$

for every unit vector $x \in \mathcal{H}$. If T is $(k$-$1)$-paranormal, then T is k-paranormal. Conversely, suppose either

$$\|T^{k+1}x\|^k \leq \|T^k x\|^{k+1} \tag{1'}$$

or

$$\|T^{k+1}x\| \leq \|T^k x\|\,\|Tx\| \tag{2'}$$

for every unit vector $x \in \mathcal{H}$. If T is k-paranormal, then T is $(k$-$1)$-paranormal.

Proof. Take an operator $T \in \mathcal{B}[\mathcal{H}]$ and an integer $k \geq 1$. Suppose T is $(k$-$1)$-paranormal (i.e., $\|Tx\|^k \leq \|T^k x\|$ for every unit vector $x \in \mathcal{H}$). If (1) holds true, then

$$\|Tx\|^{k(k+1)} \leq \|T^k x\|^{k+1} \leq \|T^{k+1}x\|^k,$$

and, if (2) holds true, then

$$\|Tx\|^{k+1} = \|Tx\|^k\|Tx\| \leq \|T^k x\|\,\|Tx\| \leq \|T^{k+1}x\|,$$

and so, in both cases, $\|Tx\|^{k+1} \leq \|T^{k+1}x\|$ for every unit vector $x \in \mathcal{H}$, which means that T is k-paranormal. Conversely, suppose T is k-paranormal (i.e., $\|Tx\|^{k+1} \leq \|T^{k+1}x\|$ for every unit vector $x \in \mathcal{H}$). If $(1')$ holds true, then

$$\|Tx\|^{k(k+1)} \leq \|T^{k+1}x\|^k \leq \|T^k x\|^{k+1},$$

and, if $(2')$ holds true, then

$$\|Tx\|\,\|Tx\|^k = \|Tx\|^{k+1} \leq \|T^{k+1}x\| \leq \|T^k x\|\,\|Tx\|,$$

and so, in both cases, $\|Tx\|^k \leq \|T^k x\|$ for every unit vector $x \in \mathcal{H}$, which means that T is $(k$-$1)$-paranormal. □

We assume in (3) of Lemma 2 below that T^{k+1} is injective. If T is k-paranormal, then this means that T is injective itself because for a k-paranormal operator we have $\mathcal{N}(T^{k+1}) \subseteq \mathcal{N}(T)$. A similar observation holds for (2) in Lemma 3.

Lemma 2. *Take any $T \in \mathcal{B}[\mathcal{H}]$ and an arbitrary integer $k \geq 1$. If*

$$\|T^k x\|^{k+1} \leq \|T^{k+1}x\|^k \tag{1}$$

and

$$0 < \|T^{k+1}x\|^{k-1} \quad \text{and} \quad \|Tx\|^{k+1}\|T^{k+1}x\|^{k-1} \leq \|T^k x\|^{k+1} \tag{3}$$

for every unit vector $x \in \mathcal{H}$, then T is k-paranormal. Conversely, if T is k-paranormal and

$$\|T^k x\|^{k+1} \leq \|Tx\|^{k+1}\|T^{k+1}x\|^{k-1} \tag{3'}$$

for every unit vector $x \in \mathcal{H}$, then (1) holds for every unit vector $x \in \mathcal{H}$.

Proof. If (1) and (3) hold true, then $0 \neq \|T^{k+1}x\|^{k-1}$ and

$$\|Tx\|^{k+1}\|T^{k+1}x\|^{k-1} \leq \|T^k x\|^{k+1} \leq \|T^{k+1}x\|^k = \|T^{k+1}x\|^{k-1}\|T^{k+1}x\|,$$

and so

$$\|Tx\|^{k+1} \leq \|T^{k+1}x\|$$

for every unit vector $x \in \mathcal{H}$. Conversely if (3′) and the above inequality hold true for every unit vector $x \in \mathcal{H}$, then

$$\|T^k x\|^{k+1} \leq \|Tx\|^{k+1}\|T^{k+1}x\|^{k-1} \leq \|T^{k+1}x\|\,\|T^{k+1}x\|^{k-1} = \|T^{k+1}x\|^k,$$

and so (1) holds true for every unit vector $x \in \mathcal{H}$. □

Lemma 3. *Take any $T \in \mathcal{B}[\mathcal{H}]$ and an arbitrary integer $k \geq 1$. If*

$$\|T^{k+1}x\|^k \leq \|T^k x\|^{k+1} \tag{1'}$$

and

$$0 < \|T^k x\| \quad \text{and} \quad \|T^k x\|\,\|Tx\| \leq \|T^{k+1}x\| \tag{2}$$

for every unit vector $x \in \mathcal{H}$, then T is both $(k$-1$)$-paranormal and k-paranormal. Conversely, if T is either $(k$-1$)$-paranormal or k-paranormal and

$$\|T^{k+1}x\| \leq \|T^k x\|\,\|Tx\| \tag{2'}$$

for every unit vector $x \in \mathcal{H}$, then (1′) holds for every unit vector $x \in \mathcal{H}$.

Proof. If (1′) and (2) hold true, then $0 \neq \|T^k x\|$ and

$$\|T^k x\|^k\|Tx\|^k \leq \|T^{k+1}x\|^k \leq \|T^k x\|^{k+1} = \|T^k x\|^k\|T^k x\|,$$

and hence

$$\|Tx\|^k \leq \|T^k x\|$$

for every unit vector $x \in \mathcal{H}$ so that T is (k-1)-paranormal. But if T is (k-1)-paranormal and (2) holds, then Lemma 1 says that T is k-paranormal. Conversely if (2′) and the above inequality hold true for every unit vector $x \in \mathcal{H}$ (i.e., if T is (k-1)-paranormal and (2′) hold true), then

$$\|T^{k+1}x\|^k \le \|T^k x\|^k \|Tx\|^k \le \|T^k x\|^k \|T^k x\| = \|T^k x\|^{k+1}$$

and so (1′) holds true for every unit vector $x \in \mathcal{H}$. But if T is k-paranormal and (2′) holds, then Lemma 1 says that T is (k-1)-paranormal, and so (1′) holds by the above argument. □

Lemma 4. *Take any $T \in \mathcal{B}[\mathcal{H}]$ and an arbitrary integer $k \ge 1$. If*

$$\|T^k x\|^{k+1} \le \|T^{k+1}x\|^k \tag{1}$$

for every unit vector $x \in \mathcal{H}$, and if T^{k+1} is (k-1)-paranormal, then T^k is k-paranormal. Conversely, if

$$\|T^{k+1}x\|^k \le \|T^k x\|^{k+1} \tag{1′}$$

for every unit vector $x \in \mathcal{H}$, and if T^k is k-paranormal, then T^{k+1} is (k-1)-paranormal.

Proof. If (1) holds true, and if T^{k+1} is (k-1)-paranormal, then

$$\|T^k x\|^{k+1} \le \|T^{k+1}x\|^k \le \|T^{(k+1)k}x\| = \|T^{k(k+1)}x\|$$

for every unit vector $x \in \mathcal{H}$, which ensures that T^k is k-paranormal. Conversely, If (1′) holds true, and if T^k is k-paranormal, then

$$\|T^{k+1}x\|^k \le \|T^k x\|^{k+1} \le \|T^{k(k+1)}x\| = \|T^{(k+1)k}x\|$$

for every unit vector $x \in \mathcal{H}$, which ensures that T^{k+1} is (k-1)-paranormal. □

4. Main Results: Invertible k-Paranormal

Note that every operator is trivially 0-paranormal since the inequality that defines a k-paranormal holds trivially for every operator $T \in \mathcal{B}[\mathcal{H}]$ if we set $k = 0$.

Theorem 1. *If $T \in \mathcal{B}[\mathcal{H}]$ is an invertible k-paranormal operator for some integer $k \ge 1$, and if its inverse is (k-1)-paranormal, then T^{-1} is k-paranormal.*

Proof. Let $T \in \mathcal{B}[\mathcal{H}]$ be an invertible operator. If T is k-paranormal, then

$$\|T^j x\|^{k+1} = \|T\,T^{j-1}x\|^{k+1} \le \|T^{k+1}(T^{j-1}x)\|\,\|T^{j-1}x\|^k = \|T^{k+j}x\|\,\|T^{j-1}x\|^k$$

for every $x \in \mathcal{H}$ and every integer $j \in \mathbb{Z}$. Summing up, for each integer $j \in \mathbb{Z}$,

$$\|T^j x\|^{k+1} \le \|T^{k+j}x\|\,\|T^{j-1}x\|^k \tag{$*$}$$

for every $x \in \mathcal{H}$. Put $j = -k$ in ($*$) and get $\|T^{-k}x\|^{k+1} \le \|x\|\,\|T^{-(k+1)}x\|^k$ for every $x \in \mathcal{H}$. Equivalently,

$$\|T^{-k}x\|^{k+1} \le \|T^{-(k+1)}x\|^k \tag{1^*}$$

for every unit vector $x \in \mathcal{H}$. Thus the inequality (1) in Lemma 1 holds for T^{-1}, and so Lemma 1 ensures that, if T^{-1} is (k-1)-paranormal, then T^{-1} is k-paranormal. □

Remark 1. If $T \in \mathcal{B}[\mathcal{H}]$ is an invertible k-paranormal for some $k \geq 1$, then

$$\|T^k x\|^{-1} \leq \|T^{-1} x\|^k$$

for every unit vector $x \in \mathcal{H}$ and therefore, if T^{-1} is (k-1)-paranormal (which completes the hypothesis in Theorem 1), then

$$\|T^k x\|^{-1} \leq \|T^{-1} x\|^k \leq \|T^{-k} x\|$$

for every unit vector $x \in \mathcal{H}$. Indeed, if T is an invertible k-paranormal, then the inequality $(*)$ in the proof of Theorem 1 holds for every $x \in \mathcal{H}$ and every $j \in \mathbb{Z}$. Put $j = 0$ in $(*)$ and get $\|x\|^{k+1} \leq \|T^k x\| \, \|T^{-1} x\|^k$ for every $x \in \mathcal{H}$. Equivalently, $\|T^k x\|^{-1} \leq \|T^{-1} x\|^k$ for every unit vector $x \in \mathcal{H}$.

The next result is an immediate consequence of Theorem 1.

Corollary 1. *If an operator $T \in \mathcal{B}[\mathcal{H}]$ is invertible and k-paranormal for every integer $i \leq k \leq j$, for some integers $2 \leq i \leq j$, and if its inverse is $(i-1)$-paranormal, then T^{-1} is k-paranormal for every integer $i - 1 \leq k \leq j$.*

Theorem 2. *If $T \in \mathcal{B}[\mathcal{H}]$ is an invertible k-paranormal for some $k \geq 1$, and if*

$$\|T^k x\|^{k+1} \leq \|Tx\|^{k+1} \|T^{k+1} x\|^{k-1} \tag{3'}$$

for every unit vector $x \in \mathcal{H}$, then T^{-1} is k-paranormal.

Proof. If T is an invertible k-paranormal, then (1) of Lemma 1 holds for T^{-1}:

$$\|T^{-k} y\|^{k+1} \leq \|T^{-(k+1)} y\|^k \tag{1*}$$

for every unit vector $y \in \mathcal{H}$ (cf. proof of Theorem 1). Now $(3')$ is equivalent to

$$\|T^k x\|^{k+1} \|x\|^{k-1} \leq \|Tx\|^{k+1} \|T^{k+1} x\|^{k-1}$$

for every $x \in \mathcal{H}$. Since T^{k+1} is invertible, take any y in $\mathcal{H} = \mathcal{R}(T^{k+1})$ so that $y = T^{k+1} x$ for some x in $\mathcal{H}$, and hence $x = T^{-(k+1)} y$. Thus, by the above inequality,

$$\|T^{-1} y\|^{k+1} \|T^{-(k+1)} y\|^{k-1} \leq \|T^{-k} y\|^{k+1} \|y\|^{k-1}$$

for every $y \in \mathcal{H}$, which is equivalent to

$$\|T^{-1} y\|^{k+1} \|T^{-(k+1)} y\|^{k-1} \leq \|T^{-k} y\|^{k+1} \tag{3*}$$

for every unit vector $y \in \mathcal{H}$. Since $T^{-(k+1)}$ is invertible, thus injective, it follows by Lemma 2 that (1^*) and (3^*) imply that T^{-1} is k-paranormal. □

Therefore, according to Proposition 1, the subclass of all k-paranormal operators such that their invertible parts (which are k-paranormal) satisfy either the hypothesis of Theorem 1 or condition $(3')$ in Theorem 2 are included in the class of the totally hereditarily normaloid operators.

Remark 2. Put $k=1$ in Theorem 1 and recall that every operator is 0-paranormal. Similarly, if $k=1$ in Theorem 2, then (3′) holds trivially. Thus Theorems 1 and 2 show, in particular (and with different proofs), that the inverse of a paranormal operator is again paranormal. Therefore, an immediate particular case of Theorems 1 and 2 (cf. Proposition 1) leads to the known result that *every paranormal operator is totally hereditarily normaloid*. Moreover, since an operator is paranormal if and only if it is k-paranormal for every $k \geq 1$, it follows that if T is an invertible paranormal operator, then both T and T^{-1} are k-paranormal for every $k \geq 1$.

Open questions: Suppose $k \geq 2$. *Is the inverse of every invertible k-paranormal operator normaloid?* Equivalently (cf. Proposition 1), *is every k-paranormal operator totally hereditarily normaloid? Is the inverse T^{-1} of an invertible k-paranormal operator k-paranormal if and only if T^{-1} is normaloid?*

5. Completeness of the Diagram of Section 1

Posinormal operators were introduced in [19]. An operator T is *posinormal* if there exists a real number α such that $\|T^*x\| \leq \alpha\|Tx\|$ for every $x \in \mathcal{H}$ or, equivalently, if $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$. Thus

$$\text{dominant} \;\rightarrow\; \text{posinormal}.$$

Actually, an operator T is dominant if and only if $\lambda I - T$ is posinormal for every $\lambda \in \mathbb{C}$. If T is posinormal then $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$, and the converse holds if $\mathcal{R}(T)$ is closed. For a survey on posinormal operators see [15]. Posinormal operators are not necessarily normaloid (not even M-hyponormal are normaloid), and normaloid operators are not necessarily posinormal (in fact, not even paranormal operators are posinormal) — see e.g., [15].

As we saw in Section 2, all operator classes in the diagram of Section 1 have the property that *every completely nonunitary contraction is of class $\mathcal{C}_{\cdot 0}$*. First we show that such a property cannot be extended from dominant to posinormal contractions, and then that it cannot be extended from k-paranormal to hereditarily normaloid contractions.

Example 1. *There exist completely nonunitary posinormal contractions that are not of class $\mathcal{C}_{\cdot 0}$.* For instance, consider the bilateral weighted shift

$$T = \text{shift}\{\omega_k\}_{k=-\infty}^{\infty}$$

on ℓ^2 with weights $\omega_k = 1$ if $k \leq 0$ and $\omega_k = \frac{1}{2}$ if $k > 0$. This is an invertible contraction. Indeed, the spectrum of T is the annulus

$$\sigma(T) = \left\{\lambda \in \mathbb{C}\colon\; \tfrac{1}{2} \leq |\lambda| \leq 1\right\}$$

and $\|T\| = 1$ (cf. [20, p. 67]). Then T is posinormal (since every invertible operator is posinormal). Moreover, $\prod_{k=0}^{n} \omega_k = (\frac{1}{2})^n \to 0$ as $n \to \infty$, which means that the product $\prod_{k=0}^{\infty} \omega_k$ diverges to 0, and $\prod_{k=-\infty}^{0} \omega_k = 1$. Hence T is of class $\mathcal{C}_{01}$ (cf. [2, p. 181]), and so it is not of class $\mathcal{C}_{\cdot 0}$. Since the contraction T is strongly stable, it is completely nonunitary. Thus T is a completely nonunitary posinormal contraction that is not of class $\mathcal{C}_{\cdot 0}$ (and so not a dominant contraction according to [22]).

Example 2. *There exist completely nonunitary hereditarily normaloid contractions that are not of class* $\mathcal{C}_{\cdot 0}$. In fact, let

$$T = \text{shift}\{\omega_k\}_{k=-\infty}^{\infty}$$

be a bilateral weighted shift on ℓ^2 with weights $\omega_k = 1$ for all k except for $k = 0$ where $\omega_0 = \frac{1}{2}$. This is a nonunitary $\mathcal{C}_{11}$-contraction similar to a unitary operator [13, p. 69]. Moreover, T is an hereditarily normaloid that is not totally hereditarily normaloid. Actually, it is hereditarily normaloid because every $\mathcal{C}_{1\cdot}$-contraction is [6, Proposition 1]; and it is not totally hereditarily normaloid because if an operator is similar to a unitary operator, then it is invertible with a power bounded inverse, and a totally hereditarily normaloid contraction in $\mathcal{C}_{1\cdot}$ with a power bounded inverse must be unitary [6, Proposition 4]. If the contraction T is not completely nonunitary itself, then there exists a nonzero subspace $\mathcal{M}$ of ℓ^2 that reduces T so that, by the well-known Nagy–Foiaş–Langer decomposition for contractions (see e.g., [23, Theorem 3.2] or [13, Theorem 5.1]),

$$T = C \oplus U \quad \text{on} \quad \ell^2 = \mathcal{M}^\perp \oplus \mathcal{M}$$

where $U = T|_{\mathcal{M}}$ is unitary and $C = T|_{\mathcal{M}^\perp}$ is a nonzero completely nonunitary contraction (acting on a nonzero subspace, because T is not unitary), which is hereditarily normaloid (but not totally hereditarily normaloid) since T is, and of class $\mathcal{C}_{11}$ since T is. (Indeed, $C^n v = (T|_{\mathcal{M}^\perp})^n v = T^n|_{\mathcal{M}^\perp} v = T^n v$; similarly, $C^{*n} v = T^{*n} v$, for every $v \in \mathcal{M}^\perp$, because $\mathcal{M}^\perp$ reduces T.) Thus either T or C is a completely nonunitary hereditarily normaloid contraction (not totally hereditarily normaloid) that is not of class $\mathcal{C}_{\cdot 0}$ (and so not a k-paranormal contraction according to [7]).

Recall the following standard concepts. The defect operator of a contraction T is the nonnegative contraction $(I - T^*T)^{\frac{1}{2}}$. A T-invariant subspace $\mathcal{M}$ is a normal subspace for T if the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ is a normal operator in $\mathcal{B}[\mathcal{M}]$. The class of all operators for which normal subspaces are reducing characterizes a class of operators that lies between the dominant and the posinormal operators. Indeed, every normal subspace for a dominant operator reduces it [21], and every operator with closed range for which normal subspaces are reducing is posinormal [15]. We close the paper with a sufficient condition for a completely nonunitary totally hereditarily normaloid contraction to be of class $\mathcal{C}_{\cdot 0}$, which is an immediate consequence of [6, Theorem 1]:

> *Let $T \in \mathcal{B}[\mathcal{H}]$ be a completely nonunitary contraction with a Hilbert–Schmidt defect operator. Suppose T is totally hereditarily normaloid. If normal subspaces of T reduce T, then T is of class $\mathcal{C}_{\cdot 0}$.*

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