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STABILITIES OF HILBERT SPACE CONTRACTION SEMIGROUPS REVISITED

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Abstract. A necessary and sufficient condition for exponential stability of Hilbert space contraction semigroups is obtained in terms of an inequality involving the dissipative norm associated with the generator of the semigroup and with the Hilbert space norm.

1. Introduction

Throughout this paper $\mathcal{H}$ will stand for a complex Hilbert space, where inner product and norm on $\mathcal{H}$ will be denoted by $\langle\cdot\,;\cdot\rangle$ and $\|\cdot\|$. Let $[T(t)] = \{T(t)\colon\, t \geq 0\}$ be a semigroup of bounded linear operators on $\mathcal{H}$. We will be dealing with C_0-semigroups (i.e., with strongly continuous semigroups) only. In this case, the infinitesimal generator A of $[T(t)]$ is closed and densely defined. Since all semigroups in this paper are strongly continuous, we will omit this attribute from now on. We assume for the most part of this paper that $[T(t)]$ is a *contraction semigroup*, that is, $\|T(t)\| \leq 1$ for all $t \geq 0$ or, equivalently,

$$\|T(t)x\| \leq \|x\| \quad \text{for every } x \in \mathcal{H} \text{ and all } t \geq 0.$$

This implies that the generator A is *dissipative* [5, p. 90], which means that

$$\operatorname{Re}\langle Ax\,;x\rangle \leq 0 \quad \text{for every } x \in \mathcal{D},$$

where $\mathcal{D}$ denotes the domain of A (which is a dense linear manifold of $\mathcal{H}$).

A semigroup $[T(t)]$ is *exponentially stable* (or *e-stable*) if there exist real constants $\alpha > 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{-\alpha t} \quad \text{for every } t \geq 0$$

(equivalently, if $\|T(t)x\| \leq Me^{-\alpha t}\|x\|$ for every $x \in \mathcal{H}$ and every $t \geq 0$), and $[T(t)]$ said to be *uniformly stable* if

$$\lim_{t\to\infty} \|T(t)\| = 0.$$

Exponential stability is equivalent to uniform stability [4]. A semigroup $[T(t)]$ is *strongly stable* (or *s-stable*) if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \quad \text{for every } x \in \mathcal{H}.$$

Necessary and sufficient conditions for e-stability are given in Datko's Theorem [3], which is stated below.

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Theorem 1. *Let $[T(t)]$ be a semigroup with generator A. The following conditions are equivalent.*

(i) $[T(t)]$ *is exponentially stable.*

(ii) $\int_0^\infty \|T(t)x\|^2\,dt < \infty$ *for every* $x \in \mathcal{H}$.

(iii) *There exists a unique positive operator* P (i.e., $P > O$) *on* $\mathcal{H}$ *such that*

$$\langle Px\,;x\rangle = \int_0^\infty \|T(t)x\|^2\,dt \quad \text{for every } x \in \mathcal{H},$$

and it satisfies the Lyapunov equation:

$$2\,\mathrm{Re}\,\langle PAx\,;x\rangle = -\|x\|^2 \quad \text{for every } x \in \mathcal{D}.$$

It is plain that e-stability implies s-stability (since uniform stability implies strong stability). We begin in Section 2 by introducing the concept of *dissipative norm* (denoted by $\|\cdot\|_d$) on the domain $\mathcal{D}$ of the generator A of a contraction semigroup $[T(t)]$. This allows us to obtain a new necessary and sufficient condition for e-stability of Hilbert space contraction semigroups in Theorem 2. Consequences of Theorem 2 are investigated in Section 3, which is closed with a characterization of s-stability for non-e-stable semigroups. Our techniques are direct and elementary.

2. Exponential Stability of Contraction Semigroups

First observe that in order to have e-stability or s-stability it is necessary to exclude those contraction semigroups $[T(t)]$ which admit the invariant subspace

$$\mathcal{M}_T = \big\{x \in \mathcal{H}\colon\; \|T(t)x\| = \|x\| \text{ for every } t \geq 0\big\}.$$

This leads to contraction semigroups for which $\mathcal{M}_T = \{0\}$. Among those contraction semigroups for which $\mathcal{M}_T = \{0\}$, we find the *proper contraction semigroups*, that is, semigroups of contractions $[T(t)]$ such that [1, 6]

$$\|T(t)x\| < \|x\| \quad \text{for every nonzero } x \in \mathcal{H} \text{ and all } t > 0.$$

Moreover, it will also be necessary to consider the concept of *strictly dissipative* [1] generator A, which means that

$$\mathrm{Re}\,\langle Ax\,;x\rangle < 0 \quad \text{for every nonzero } x \in \mathcal{D}.$$

If the generator A of a contraction semigroup $[T(t)]$ is strictly dissipative, then $[T(t)]$ is proper [1]. Therefore, the statements *"contraction semigroups with strictly dissipative generator"* and *"proper contractions semigroups with strictly dissipative generator"* coincide. Also note that the concept of strictly dissipative can be seen in the Lyapunov equation with respect to an inner product $\langle\cdot\,;\cdot\rangle_P$ such that

$$2\,\mathrm{Re}\,\langle Ax\,;x\rangle_P = 2\,\mathrm{Re}\,\langle PAx\,;x\rangle = -\|x\|^2 < 0 \quad \text{for every nonzero } x \in \mathcal{D}.$$

We now consider a property of contraction semigroups that plays a crucial role in the sequel. Let A be the dissipative generator of a contraction semigroup $[T(t)]$, and let $\langle\cdot\,;\cdot\rangle_d$ denote the sesquilinear form defined on $\mathcal{D}$ as follows [5, 8, 9].

$$\langle x\,;y\rangle_d = -\langle Ax\,;y\rangle - \langle x\,;Ay\rangle \quad \text{for every } x, y \in \mathcal{D}.$$

It is easy to see that, if A is strictly dissipative, then $\langle\cdot\,;\cdot\rangle_d$ is an inner product on $\mathcal{D}$. Therefore, if A is strictly dissipative, then we can consider the *dissipative norm* $\|\cdot\|_d$ on $\mathcal{D}$, which is defined, for every nonzero $x\in\mathcal{D}$, by

$$\|x\|_d^2=\langle x\,;x\rangle_d=-\langle Ax\,;x\rangle-\langle x\,;Ax\rangle=-2\,\mathrm{Re}\,\langle Ax\,;x\rangle>0. \tag{2.1}$$

Note that if A is dissipative (but not strictly dissipative), then $\|\cdot\|_d$ is a seminorm on $\mathcal{D}$. The next lemma shows the role played by $\|\cdot\|_d$ in e-stability and s-stability.

Lemma 1. *Let $[T(t)]$ be a contraction semigroup. If the generator A is strictly dissipative, then*

(i) $\int_0^\infty\|T(t)x\|_d^2\,dt\le\|x\|^2<\infty$ *for every* $x\in\mathcal{D}$,

(ii) $\lim_{t\to\infty}\|T(t)x\|^2=\|x\|^2-\int_0^\infty\|T(t)x\|_d^2\,dt$ *for every* $x\in\mathcal{D}$, *and*

(iii) $\lim_{t\to\infty}\|T(t)x\|\le\|x\|<\infty$ *for every* $x\in\mathcal{H}$.

Proof. Let $[T(t)]$ be a contraction semigroup with a strictly dissipative generator A, and take an arbitrary $x\in\mathcal{D}$.

(i) Recall that $T(t)x$ lies in $\mathcal{D}$ for every $x\in\mathcal{D}$ (i.e., $\mathcal{D}$ is $T[(t)]$-invariant; indeed, A and $T(t)$ commute) so that $AT(t)x$ and $\|T(t)x\|_d$ are both well-defined for every $x\in\mathcal{D}$ and every $t\ge0$. It is well known (see e.g., [5, p. 80]) that, for every $t\ge0$,

$$\frac{d}{dt}\|T(t)x\|^2=2\,\mathrm{Re}\,\langle AT(t)x\,;T(t)x\rangle. \tag{2.2}$$

Since A is strictly dissipative, (2.1) says that the above can be rewritten as

$$\frac{d}{dt}\|T(t)x\|^2=-\|T(t)x\|_d^2$$

for every $t\ge0$. Integrating both sides in $[0,t]$ we get

$$\|T(t)x\|^2-\|x\|^2=-\int_0^t\|T(\tau)x\|_d^2\,d\tau, \tag{2.3}$$

and hence

$$\int_0^t\|T(\tau)x\|_d^2\,d\tau\le\|x\|^2$$

for all $t\ge0$, which ensures the required result in (i).

(ii) The result in (ii) follows from (2.3), according to (i).

(iii) It follows from (ii) that $\lim_{t\to\infty}\|T(t)x\|^2\le\|x\|^2<\infty$ for every $x\in\mathcal{D}$. Extending this by continuity to the whole space $\mathcal{H}$ we get (iii). □

Remark 1. Let $[T(t)]$ be a contraction semigroup with a strictly dissipative generator A. It is worth addressing the following immediate issues related to Lemma 1.

(a) Lemma 1(iii) follows at once from the fact that $[T(t)]$ is a contraction for all $t\ge0$, since $\lim_{t\to\infty}\|T(t)x\|$ exists in $\mathbb{R}$ for every x in $\mathcal{D}$.

(b) Lemma i(ii) actually ensures that $[T(t)]$ is s-stable if and only if [1]

$$\int_0^\infty\|T(t)x)\|_d^2\,dt=\|x\|^2\quad\text{for every } x\in\mathcal{D}.$$

We give in Theorem 2 below a necessary and sufficient condition for e-stability of contraction semigroups, but first we need the following result.

Lemma 2. *If $[T(t)]$ is an e-stable contraction semigroup, then there exists a real number $\alpha > 0$ such that, for every $t \geq 0$ and every $x \in \mathcal{D}$,*

$$\alpha\,\|T(t)x\|^2 \leq -\mathrm{Re}\langle AT(t)x\,;T(t)x\rangle.$$

Proof. Take $x \in \mathcal{H}$ and $t, h \geq 0$ arbitrary. Suppose $[T(t)]$ is an e-stable contraction semigroup so that $\|T(h)\| \leq e^{-\alpha h}$ for some $\alpha > 0$. Hence

$$\begin{aligned}\|T(t+h)x\|^2 - \|T(t)x\|^2 &= \|T(h)T(t)x\|^2 - \|T(t)x\|^2\\ &\leq \|T(h)\|^2\|T(t)x\|^2 - \|T(t)x\|^2\\ &\leq e^{-2\alpha h}\|T(t)x\|^2 - \|T(t)x\|^2\\ &\leq \|T(t)x\|^2(e^{-2\alpha h} - 1) = (e^{-2\alpha h} - e^{-2\alpha 0})\|T(t)x\|^2.\end{aligned}$$

Therefore, according to (2.2),

$$\begin{aligned}2\,\mathrm{Re}\langle AT(t)x\,;T(t)x\rangle &= \frac{d}{dt}\|T(t)x\|^2 = \lim_{h\to 0}\frac{\|T(t+h)x\|^2 - \|T(t)x\|^2}{h}\\ &\leq \lim_{h\to 0}\frac{e^{-2\alpha h} - e^{-2\alpha 0}}{h}\|T(t)x\|^2\\ &= \Big(\frac{d}{dt}e^{-2\alpha t}\Big|_{t=0}\Big)\|T(t)x\|^2 = -2\alpha\,\|T(t)x\|^2,\end{aligned}$$

and so $\alpha\,\|T(t)x\|^2 \leq -\mathrm{Re}\langle AT(t)x\,;T(t)x\rangle$ for every $t \geq 0$ and every $x \in \mathcal{D}$. □

The above proof is elementary and "dissipative-free". Otherwise (using the fact that the generator of a contraction semigroup is dissipative), if $[T(t)]$ is an e-stable contraction semigroup, then $\|T(t)\| \leq e^{-\alpha t}$ for every $t \geq 0$, for some $\alpha > 0$. Hence $\|e^{\alpha t}\,T(t)\| \leq 1$ for all $t \geq 0$, which means that the semigroup $[e^{\alpha t}\,T(t)]$ is contractive. Since its generator is $A + \alpha I$ ([5, p. 90]), it follows that $A + \alpha I$ is dissipative on $\mathcal{D}$, and therefore

$$\mathrm{Re}\langle Ax\,;x\rangle + \alpha\,\|x\|^2 = \mathrm{Re}\langle(A+\alpha I)x\,;x\rangle \leq 0 \quad \text{for every } x \in \mathcal{D}.$$

Thus there is an $\alpha > 0$ such that $\alpha\,\|x\|^2 \leq -\mathrm{Re}\langle Ax\,;x\rangle$ for every $x \in \mathcal{D}$, which is the result of Lemma 2 for $t = 0$. However, since $T(t)x$ lies in $\mathcal{D}$ for every $x \in \mathcal{D}$ and every $t \geq 0$, this actually leads to the conclusion of Lemma 2.

Theorem 2. *Let $[T(t)]$ be a contraction semigroup. $[T(t)]$ is e-stable if and only if*

$$\alpha\,\|x\|^2 \leq -\mathrm{Re}\langle Ax\,;x\rangle \quad \text{for every } x \in \mathcal{D}, \text{ for some } \alpha > 0.$$

Proof. If $[T(t)]$ be an e-stable contraction semigroup then, by Lemma 2 (with $t = 0$),

$$\alpha\,\|x\|^2 \leq -\mathrm{Re}\langle Ax\,;x\rangle \quad \text{for every } x \in \mathcal{D}, \text{ for some } \alpha > 0.$$

Conversely, take an arbitrary $x \in \mathcal{D}$ and suppose the above inequality holds. This implies that A is strictly dissipative so that we may consider the norm $\|\cdot\|_d$ defined in (2.1). In terms of $\|\cdot\|_d$, the above inequality can be rewritten as

$$\alpha\,\|x\|^2 \leq \tfrac{1}{2}\|x\|_d^2$$

for some $\alpha > 0$. Recall again that $T(t)x$ lies in $\mathcal{D}$ for every $x \in \mathcal{D}$. Therefore,

$$\|T(t)x\|^2 \le \tfrac{1}{2\alpha}\,\|T(t)x\|_d^2$$

for every $t \ge 0$. Thus, integrating on $[0,\infty)$ and using Lemma 1(i) — since $[T(t)]$ is a contraction semigroup with a strictly dissipative generator A,

$$\int_0^\infty \|T(t)x\|^2\,dt \le \tfrac{1}{2\alpha}\int_0^\infty \|T(t)x\|_d^2\,dt \le \tfrac{1}{2\alpha}\,\|x\|^2$$

for every $x \in \mathcal{D}$. Extending by continuity (since integral is a continuous functional) from the dense domain $\mathcal{D}$ to the whole space $\mathcal{H}$ we get

$$\int_0^\infty \|T(t)x\|^2\,dt \le \tfrac{1}{2\alpha}\,\|x\|^2 < \infty$$

for every $x \in \mathcal{H}$, which implies that $[T(t)]$ is e-stable according to Theorem 1. □

3. Applications

Corollaries of Theorem 2 are discussed in this section. We begin with a straightforward consequence of Theorem 2 in Corollary 1, and close the section with Corollary 4 that characterizes s-stability for non-e-stable contractive semigroups with a strictly dissipative generator.

Corollary 1. *If $[T(t)]$ is e-stable contractive, then A is strictly dissipative.*

In fact, if $[T(t)]$ is e-stable contractive, then $0 < \alpha \le -\sup_{\|x\|=1} -\mathrm{Re}\langle Ax\,;x\rangle$.

Corollary 2. *If $[T(t)]$ is an e-stable contraction semigroup, then*

$$\int_0^\infty \|T(t)x\|_d^2\,dt = \|x\|^2 \quad \text{for every } x \in \mathcal{D}.$$

Proof. If $[T(t)]$ is e-stable, then A is strictly dissipative by Corollary 1. Thus $[T(t)]$ is s-stable with a strictly dissipative A, and so we can apply Lemma 1(ii) — cf. Remark 1(b) — to get the claimed identity. □

Corollary 3. *Let $[T(t)]$ be a contraction semigroup. If*

$$\|x\|^2 = -2\mathrm{Re}\langle Ax\,;x\rangle \quad \text{for every } x \in \mathcal{D},$$

then $[T(t)]$ is e-stable and

$$\int_0^\infty \|T(t)x\|^2\,dt = \|x\|^2 \quad \text{for every } x \in \mathcal{H}. \tag{3.1}$$

In this case $[T(t)]$ is unitarily equivalent to the restriction of the left shift semigroup $[L(t)]$ to an $[L(t)]$-invariant subspace, where each $L(t)$ is defined on the $\mathcal{H}$-valued function space $\mathcal{L}^2([0,\infty);\mathcal{H})$ by

$$L(\tau)f = g \quad \text{with} \quad g(t) = f(t+\tau) \quad \text{for every } t,\tau \ge 0.$$

Proof. If $\frac{1}{2}\|x\|^2 = -\mathrm{Re}\langle Ax\,;x\rangle$ for every $x \in \mathcal{D}$, then $[T(t)]$ is e-stable by Theorem 2, and so Corollary 2 ensures that

$$\int_0^\infty \|T(t)x\|_d^2\,dt = \|x\|^2 \quad \text{for every } x \in \mathcal{D}.$$

Moreover, since A is strictly dissipative by Corollary 1, it follows that the assumption $-2\mathrm{Re}\langle Ax\,;x\rangle = \|x\|^2$ means $\|x\|_d^2 = \|x\|^2$ for every $x \in \mathcal{D}$ by (2.1), which implies that $\|T(t)x\|_d^2 = \|T(t)x\|^2$ for every $t \geq 0$ and every $x \in \mathcal{D}$. Thus the identity in (3.1) holds in $\mathcal{D}$. Since $\mathcal{D}$ is dense in $\mathcal{H}$, this can be extended by continuity to $\mathcal{H}$ so that the identity in (3.1) holds in $\mathcal{H}$. Let $V\colon \mathcal{H} \to \mathcal{R}(V) \subseteq \mathcal{L}^2([0,\infty);\mathcal{H})$ be a linear transformation defined, for each $x \in \mathcal{H}$, by

$$(Vx)(t) = T(t)x \quad \text{for every } t \geq 0$$

(i.e., $Vx = y \in \mathcal{L}^2([0,\infty);\mathcal{H})$ for each $x \in \mathcal{H}$ with $y(t) = T(t)x$ for every $t \geq 0$), where $\mathcal{R}(V) = V(\mathcal{H})$ is the range of V. The identity in (3.1) ensures that V is an isometry — $\|Vx\| = \|x\|$ for every $x \in \mathcal{H}$. Thus the linear manifold $\mathcal{R}(V)$ is closed in $\mathcal{H}$, and $V\colon \mathcal{H} \to \mathcal{R}(V)$ is a unitary transformation (i.e., a surjective isometry) between the Hilbert spaces $\mathcal{H}$ and $\mathcal{R}(V)$. Observe that

$$[VT(\tau)x](t) = T(t)T(\tau)x = T(t+\tau)x = (Vx)(t+\tau) = [L(\tau)Vx](t)$$

for every $t, \tau \geq 0$ and every $x \in \mathcal{H}$, and so V intertwines each $T(\tau)$ to each $L(\tau)$,

$$VT(\tau) = L(\tau)V$$

for every $\tau \geq 0$. Therefore, the subspace $\mathcal{R}(V)$ is invariant for every $L(\tau)$, and the restriction $L(\tau)|_{\mathcal{R}(V)}$ of $L(\tau)$ to $\mathcal{R}(V)$ is unitarily equivalent to $T(\tau)$, that is,

$$T(\tau) = V^* L(\tau)|_{\mathcal{R}(V)} V \quad \text{for every } \tau \geq 0. \qquad \square$$

Recall the definition of the dissipative norm in (2.1), viz., $\|x\|_d^2 = -2\,\mathrm{Re}\langle Ax\,;x\rangle$. Hence, by Theorem 2, a necessary and sufficient condition for e-stability is:

$$\text{There exists an } \alpha > 0 \text{ such that } \quad 2\alpha\|x\|^2 \leq \|x\|_d^2 \quad \text{for every } x \in \mathcal{D}.$$

Therefore, a necessary and sufficient condition for non-e-stability is:

$$\text{For every } \alpha > 0 \text{ there exists an } x = x_\alpha \in \mathcal{D} \text{ such that } \quad \|x\|_d^2 < 2\alpha\|x\|^2.$$

Thus s-stability of non-e-stable contraction semigroups with strictly dissipative generators follows as a consequence of Theorem 2 and Lemma 1(ii) — see Remark 1(b), as in the next corollary.

Corollary 4. *Let $[T(t)]$ be a contraction semigroup with a strictly dissipative generator A. Suppose that for every $\alpha > 0$ there exists an $x \in \mathcal{D}$ such that*

$$0 < -\mathrm{Re}\langle Ax\,;x\rangle < \alpha\|x\|^2.$$

Equivalently, $\|x\|_d^2 < 2\alpha\|x\|^2$ for every $\alpha > 0$, for some $x \in \mathcal{D}$. If

$$\int_0^\infty \|T(t)x\|_d^2\,dt = \|x\|^2 \quad \text{for every } x \in \mathcal{D},$$

then $[T(t)]$ is an s-stable semigroup that is not e-stable.

4. Final Remarks

This paper is a sequel to our effort to go after s-stability of continuous and discrete Hilbert space contraction semigroups [1, 6, 7]. Our eventual goal is to apply representation of s-stable contraction semigroups to Continuous Wavelet Theory.

Strong stability of continuous and discrete operator semigroups over Banach and Hilbert spaces have been extensively studied. We refer to [2] for a recent and comprehensive discussions on the subject.

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References

1. K.N. Boyadzhiev and N. Levan, *Strong stability of Hilbert space contraction semigroups*, Studia Sci. Math. Hungar. **30** (1995), 165–182.
2. R. Chill and Y. Tomilov, *Stability of operator semigroups: ideas and results*, Perspectives in Operator Theory (Banach Center Publ., Polish Acad. Sci., Warsaw) **75** (2007), 71–109.
3. R. Datko, *Extending a theorem of A.M. Liapunov to Hilbert space*, J. Math. Anal. Appl. **32** (1970), 610–616.
4. R. Datko, *Uniform asymptotic stability of evolutionary process in a Banach space*, SIAM, J. Math. Anal. **3** (1972), 428–445.
5. P.A. Fillmore. *Notes on Operator Theory*, Van Nostrand, New York, 1970.
6. C.S. Kubrusly and N. Levan, *Proper contractions and invariant subspaces*, Int. J. Math. Math. Sci. **28** (2001), 223–230.
7. C.S. Kubrusly and P.C.M. Vieira, *Strong stability for cohyponormal operators*, J. Operator Theory **31** (1994), 123–127.
8. P.D. Lax and R.S. Phillips, *Scattering Theory*, Academic Press, New York, 1967.
9. B.Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.

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If a semigroup is e-stable with $M = 1$ (i.e., if there is an $\alpha > 0$ such that $\|T(t)\| \leq e^{-\alpha t}$ for every $t \geq 0$), then it is called *plain-e-stable*.

(1) Replace “e-stable” by “plain-e-stable” in Lemma 2, and Corollaries 1 and 2.

(2) Rewrite Theorem 2: plain-e-stability implies the displayed inequality, which in turn implies e-stability. (In fact, the displayed inequality implies plain-e-stability, but this was proved only in a subsequent paper: see Corollary 2 in *On exponential stability of contraction semigroups*, Semigroup Forum, 2012, to appear.)

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