Acta Scientiarum Mathematicarum (*Szeged*) **75** (2009) 679–692

INVARIANT SUBSPACES OF MULTIPLE TENSOR PRODUCTS

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Abstract. Regular subspaces are tensor products of subspaces. The structure of regular subspaces that are invariant or reducing for the tensor product of a finite collection of Hilbert space operators is entirely characterized. Necessary and sufficient conditions for a multiple tensor product of operators to be a unilateral shift are established, and it is proved that a multiple tensor product of operators is a completely nonunitary contraction if and only if each factor is a contraction, being one of them completely nonunitary.

1. Introduction

Consider the Hilbert space $\bigotimes_{i=1}^m \mathcal{H}_i$ consisting of the tensor product of a finite collection of Hilbert spaces $\mathcal{H}_i$ and the tensor product $\bigotimes_{i=1}^m A_i$ of operators A_i on $\mathcal{H}_i$. A subspace of $\bigotimes_{i=1}^m \mathcal{H}_i$ will be called *regular* if it is the tensor product $\bigotimes_{i=1}^m \mathcal{M}_i$ of subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$. In Section 3 we focus on regular subspaces that are invariant or reducing for $\bigotimes_{i=1}^m A_i$. These are explored in Theorem 1, and reducing irregular subspaces are presented in Corollary 1. In Section 4 we investigate tensor products of lattices of invariant subspaces. These are characterized in Corollary 2.

Sections 5 and 6 deal with applications. Theorem 2 describes unilateral shift tensor products, where it is shown that $\bigotimes_{i=1}^m A_i$ is a unilateral shift if and only if $\bigotimes_{i=1}^m A_i = \bigotimes_{i=1}^m V_i$, where each V_i is an isometry, at least one of them is a unilateral shift. Finally, in Theorem 3, completely nonunitary tensor product contractions are characterized, where it is proved that $\bigotimes_{i=1}^m A_i$ is a completely nonunitary contraction if and only if at least one of the contractions A_i is completely nonunitary.

2. Preliminaries

By an operator A on a Hilbert space $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ be the normed algebra of all operators on $\mathcal{H}$, and let $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stand for kernel and range of $A \in \mathcal{B}[\mathcal{H}]$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is a *closed* linear manifold of $\mathcal{H}$, which is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$, invariant for an operator A (or A-invariant) if $A(\mathcal{M}) \subseteq \mathcal{M}$, and reducing for A (or $\mathcal{M}$ reduces A) if $\mathcal{M}$ is invariant for both A and A^* (where A^* denotes the adjoint of A); equivalently, if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are A-invariant (where $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$ stands for the orthogonal complement of $\mathcal{M}$). An operator A is reducible if it has a nontrivial reducing subspace, otherwise it is called irreducible. If $\{\mathcal{H}_i\}_{i=1}^m$ is a finite collection of Hilbert spaces, then their orthogonal direct sum is denoted by $\bigoplus_{i=1}^m \mathcal{H}_i$, which is again a Hilbert space. If $\{A_i\}_{i=1}^m$ is a finite collection of operators with A_i in $\mathcal{B}[\mathcal{H}_i]$, then their direct sum is denoted by $\bigoplus_{i=1}^m A_i$, which is an operator in $\mathcal{B}[\bigoplus_{i=1}^m \mathcal{H}_i]$.

Date: September 30, 2008; revised: March 20, 2008.
2000 *Mathematics Subject Classification.* Primary 47A80; Secondary 47A15.
Keywords. Tensor product, Hilbert space operators, invariant subspaces.

Let $\mathcal{H}$ and $\mathcal{K}$ be nonzero complex Hilbert spaces. We consider the concept of tensor product space $\mathcal{H}\otimes\mathcal{K}$ in terms of the single tensor product of vectors as a conjugate bilinear functional on the Cartesian product of $\mathcal{H}$ and $\mathcal{K}$. (See e.g., [3], [10] and [11] — for an abstract approach see e.g., [1] and [14].) Let $A\otimes B$ in $\mathcal{B}[\mathcal{H}\otimes\mathcal{K}]$ stand for the tensor product of two operators A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$. For an expository paper on tensor product see e.g., [5]. The tensor product of a pair of Hilbert spaces and of a pair of operators is naturally extended to a finite collection $\{\mathcal{H}_i\}_{i=1}^m$ $(m\geq 2)$ of complex Hilbert spaces and to a finite collection $\{A_i\}_{i=1}^m$ of operators with A_i in $\mathcal{B}[\mathcal{H}_i]$, where $\bigotimes_{i=1}^m\mathcal{H}_i$ denotes the tensor product space of $\{\mathcal{H}_i\}_{i=1}^m$ and $\bigotimes_{i=1}^m A_i$ in $\mathcal{B}[\bigotimes_{i=1}^m\mathcal{H}_i]$ stands for tensor product of the operators $\{A_i\}_{i=1}^m$.

3. Invariant Subspaces

Consider a finite collection $\{\mathcal{H}_i\}_{i=1}^m$ of complex Hilbert spaces. A subspace of the orthogonal direct sum $\bigoplus_{i=1}^m\mathcal{H}_i$ is *intrinsic* if it is of the form $\bigoplus_{i=1}^m\mathcal{M}_i$, where each $\mathcal{M}_i$ is a subspace of $\mathcal{H}_i$. Otherwise it is said to be *extrinsic*. This notion can be brought to tensor product spaces yielding the concept of *regular* subspaces.

Definition 1. A subspace of a tensor product space $\bigotimes_{i=1}^m\mathcal{H}_i$ is *regular* if it is of the form $\bigotimes_{i=1}^m\mathcal{M}_i$, where each $\mathcal{M}_i$ is a subspace of $\mathcal{H}_i$. Otherwise it is *irregular*.

The direct sum $\bigoplus_{i=1}^m\mathcal{M}_i$ of subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ is a subspace of the direct sum space $\bigoplus_{i=1}^m\mathcal{H}_i$, and $\bigoplus_{i=1}^m\mathcal{M}_i$ is invariant (reducing) for the direct sum $\bigoplus_{i=1}^m A_i$ of operators A_i on $\mathcal{H}_i$ if and only if each $\mathcal{M}_i$ is invariant (reducing) for A_i. The tensor product $\bigotimes_{i=1}^m\mathcal{M}_i$ of subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ is a subspace of the tensor product space $\bigotimes_{i=1}^m\mathcal{H}_i$. Regular invariant and reducing subspaces are characterized as follows.

Theorem 1. *Take any integer $m\geq 2$. For each integer $i\in[1,m]$ let A_i be an operator on a Hilbert space $\mathcal{H}_i$ and let $\mathcal{M}_i$ be a subspace of $\mathcal{H}_i$. Consider the tensor product $\bigotimes_{i=1}^m A_i$ of $\{A_i\}_{i=1}^m$ on the tensor product space $\bigotimes_{i=1}^m\mathcal{H}_i$.*

- (a_1) *If each $\mathcal{M}_i$ is invariant (reducing) for A_i, then $\bigotimes_{i=1}^m\mathcal{M}_i$ is an invariant (reducing) subspace for $\bigotimes_{i=1}^m A_i$.*
- (a_2) *If $\bigotimes_{i=1}^m\mathcal{M}_i$ is invariant for $\bigotimes_{i=1}^m A_i$, then one of the subspaces $\mathcal{M}_i$ is invariant for A_i.*
- (a_3) *If $\bigotimes_{i=1}^m\mathcal{M}_i$ reduces $\bigotimes_{i=1}^m A_i$, then one of the subspaces $\mathcal{M}_i$ reduces A_i, or one of the subspaces $\mathcal{M}_i$ is invariant for A_i and the orthogonal complement $\mathcal{M}_j^\perp$ of another subspace $\mathcal{M}_j$ with $j\neq i$ is invariant for A_j.*
- (a_4) *If $\bigotimes_{i=1}^m\mathcal{M}_i$ is invariant (reducing) for $\bigotimes_{i=1}^m A_i$ and if each $\mathcal{M}_i\not\subseteq\mathcal{N}(A_i)$, then each $\mathcal{M}_i$ is invariant (reducing) for A_i. Particular case:*
  - (a_4') *If $\bigotimes_{i=1}^m\mathcal{M}_i$ is nonzero and invariant (reducing) for $\bigotimes_{i=1}^m A_i$ and if every A_i is injective, then each $\mathcal{M}_i$ is invariant (reducing) for A_i.*
- (b) *One of the subspaces $\mathcal{M}_i$ is nontrivial and the others $\{\mathcal{M}_j\}_{i\neq j=1}^m$ are nonzero if and only if $\bigotimes_{i=1}^m\mathcal{M}_i$ is nontrivial.*
- (c_1) *If each $\mathcal{M}_i$ is A_i-invariant, then*

$$\Big(\bigotimes_{i=1}^m A_i\Big)\Big|_{\bigotimes_{i=1}^m\mathcal{M}_i}=\bigotimes_{i=1}^m A_i|_{\mathcal{M}_i}.$$

(c_2) *If* $\bigotimes_{i=1}^m \mathcal{M}_i$ *is nonzero and* $\bigotimes_{i=1}^m A_i$*-invariant, and if each* A_i *is injective, then*

$$\Big(\bigotimes_{i=1}^m A_i\Big)\Big|_{\bigotimes_{i=1}^m \mathcal{M}_i} = \bigotimes_{i=1}^m A_i|_{\mathcal{M}_i}.$$

Proof. Suppose all A_i are nonzero (otherwise the results are trivial). Assertion (a_1) is readily verified since a subspace $\mathcal{M}$ of a Hilbert space is invariant (or reducing) for a nonzero operator A if and only if the unique orthogonal projection P with $\mathcal{R}(P) = \mathcal{M}$ is such that $PAP = AP$ (or such that $PA = AP$); see e.g., [9, Theorem 0.1]. Conversely, if $\bigotimes_{i=1}^m \mathcal{M}_i$ is invariant (or reducing) for $\bigotimes_{i=1}^m A_i$, then there exists an orthogonal projection E on $\bigotimes_{i=1}^m \mathcal{H}_i$ with $\mathcal{R}(E) = \bigotimes_{i=1}^m \mathcal{M}_i$ and such that $E\big(\bigotimes_{i=1}^m A_i\big)E = \big(\bigotimes_{i=1}^m A_i\big)E$ [or such that $E\big(\bigotimes_{i=1}^m A_i\big) = \big(\bigotimes_{i=1}^m A_i\big)E$]. Consider the projections P_i onto $\mathcal{M}_i$. Since E is unique and since $\mathcal{R}\big(\bigotimes_{i=1}^m P_i\big) = \bigotimes_{i=1}^m \mathcal{M}_i$, it follows that $E = \bigotimes_{i=1}^m P_i$. Thus

$$\bigotimes_{i=1}^m P_i A_i P_i = E\Big(\bigotimes_{i=1}^m A_i\Big)E = \Big(\bigotimes_{i=1}^m A_i\Big)E = \bigotimes_{i=1}^m A_i P_i$$

whenever $\bigotimes_{i=1}^m \mathcal{M}_i$ is invariant for $\bigotimes_{i=1}^m A_i$, or

$$\bigotimes_{i=1}^m P_i A_i = E\Big(\bigotimes_{i=1}^m A_i\Big) = \Big(\bigotimes_{i=1}^m A_i\Big)E = \bigotimes_{i=1}^m A_i P_i$$

whenever $\bigotimes_{i=1}^m \mathcal{M}_i$ is reducing for $\bigotimes_{i=1}^m A_i$. In any case, if $A_iP_i = O$ for some i, then $\mathcal{M}_i$ is A_i-invariant since $\mathcal{M}_i = \mathcal{R}(P_i) \subseteq \mathcal{N}(A_i)$. If $\bigotimes_{i=1}^m \mathcal{M}_i$ reduces $\bigotimes_{i=1}^m A_i$, then $A_iP_i = O$ also implies that $P_jA_j = O$, that is, $A_j^*P_j = O$, for some j, so that either $\mathcal{M}_i$ reduces A_i (if $j = i$) or $\mathcal{M}_i$ is A_i-invariant and $\mathcal{M}_j^\perp$ is A_j-invariant (if $j \neq i$). On the other hand, suppose $A_iP_i \neq O$ for every i. This means that $\mathcal{M}_i = \mathcal{R}(P_i) \not\subseteq \mathcal{N}(A_i)$ for every i. In particular, this happens if every A_i is injective and $\bigotimes_{i=1}^m \mathcal{M}_i$ is nonzero, since $P_i \neq O$ if $\mathcal{M}_i \neq \{0\}$. In this case (i.e., if $A_iP_i \neq O$ — cf. [12, Proposition 2.1]), for each i,

$$P_i A_i P_i = \alpha_i A_i P_i$$

for nonzero scalars α_i, so that each $\mathcal{M}_i = \mathcal{R}(P_i)$ is A_i-invariant, whenever $\bigotimes_{i=1}^m \mathcal{M}_i$ is $\bigotimes_{i=1}^m A_i$-invariant, or

$$P_i A_i = \beta_i A_i P_i$$

for nonzero scalars β_i, so that each $\mathcal{M}_i = \mathcal{R}(P_i)$ reduces A_i, whenever $\bigotimes_{i=1}^m \mathcal{M}_i$ reduces $\bigotimes_{i=1}^m A_i$. This completes the proof of (a_2), (a_3) and (a_4). Assertions (b) and (c_1) are also readily verified, and so their proofs will be omitted. If $\bigotimes_{i=1}^m \mathcal{M}_i$ is nonzero, then (c_2) is a consequence of (c_1) and (a_4'). □

Both the zero subspace and the whole space are always intrinsic and regular, and therefore extrinsic and irregular subspaces are always nontrivial. Indeed, the zero subspace $\bigoplus_{i=1}^m\{0\}$ of the direct sum $\bigoplus_{i=1}^m \mathcal{H}_i$ is intrinsic as well as the whole space $\bigoplus_{i=1}^m \mathcal{H}_i$. The zero subspace $\{0\}$ of the tensor product space $\bigotimes_{i=1}^m \mathcal{H}_i$ is precisely $\{0\} = \bigotimes_{i=1}^m \mathcal{M}_i$ with one of the $\mathcal{M}_i$ being the zero subspace ($\mathcal{M}_i = \{0\} \subset \mathcal{H}_i$ for some i), which is regular and, again, the whole space $\bigotimes_{i=1}^m \mathcal{H}_i$ is clearly regular.

Remark 1. There exist extrinsic subspaces of a direct sum space that, in addition to being extrinsic, are invariant for the direct sum of Hilbert space operators. For instance, take any nonzero operator C on a Hilbert space $\mathcal{H}$ and consider the set

$$\mathcal{M}_C = \big\{(Cx, \cdots, Cx) \in \textstyle\bigoplus_{i=1}^m \mathcal{H}\colon\; x \in \mathcal{H}\big\},$$

which in fact is a nontrivial linear manifold of the direct sum space $\bigoplus_{i=1}^m \mathcal{H}$ of m copies of $\mathcal{H}$. It is readily verified that if C has a closed range, then $\mathcal{M}_C$ is closed in $\bigoplus_{i=1}^m \mathcal{H}$, and so $\mathcal{M}_C$ is a subspace of $\bigoplus_{i=1}^m \mathcal{H}$, which is clearly *extrinsic*. Moreover, if C commutes with an operator A on $\mathcal{H}$, then the extrinsic subspace $\mathcal{M}_C$ of $\bigoplus_{i=1}^m \mathcal{H}$ is invariant for the direct sum $\bigoplus_{i=1}^m A$ of m copies of A. We consider the existence of irregular reducing subspaces of tensor product spaces in Corollary 1(b) below.

Corollary 1. *Take an arbitrary integer $m \ge 2$. For each integer $i \in [1, m]$ let A_i be an injective operator on a Hilbert space $\mathcal{H}_i$ and let $\mathcal{M}_i$ be a nonzero subspace of $\mathcal{H}_i$. Consider the regular subspace $\bigotimes_{i=1}^m \mathcal{M}_i$ of $\bigotimes_{i=1}^m \mathcal{H}_i$ and the tensor product $\bigotimes_{i=1}^m A_i$ on $\bigotimes_{i=1}^m \mathcal{H}_i$.*

(a) *$\bigotimes_{i=1}^m \mathcal{M}_i$ is $\bigotimes_{i=1}^m A_i$-invariant if and only if each $\mathcal{M}_i$ is A_i-invariant.*

Put $\mathcal{H}_i = \mathcal{H}$ (a Hilbert space of dimension greater than 1) and $A_i = A$ (injective) for all i. Take the tensor product $\bigotimes_{i=1}^m A$ of m copies of $A \in \mathcal{B}[\mathcal{H}]$ on the tensor product space $\bigotimes_{i=1}^m \mathcal{H}$ of m copies of $\mathcal{H}$.

(b) *$\bigotimes_{i=1}^m A$ is reducible and, if A is irreducible, then all nontrivial reducing subspaces of $\bigotimes_{i=1}^m A$ are irregular.*

Proof. The result in (a) is a straightforward application of Theorem 1(a_1, a_4'). The proof of the result in (b) goes as follows. Consider the tensor product $\bigotimes_{i=1}^m A_i$ of m operators $\{A_i\}_{i=1}^m$, each A_i acting on the Hilbert space $\mathcal{H}_i$, and consider the mapping $\Pi\colon \bigotimes_{i=1}^m \mathcal{H}_i \to \bigotimes_{i=m}^1 \mathcal{H}_i$ defined by

$$\Pi\Big(\textstyle\sum_k \bigotimes_{i=1}^m x_{i,k}\Big) = \sum_k \bigotimes_{i=m}^1 x_{i,k}$$

for every $\sum_k \bigotimes_{i=1}^m x_{i,k}$ in $\bigotimes_{i=1}^m \mathcal{H}_i$. This is an invertible linear isometry, thus a unitary transformation of $\bigotimes_{i=1}^m \mathcal{H}_i$ onto $\bigotimes_{i=m}^1 \mathcal{H}_i$, and it is readily verified that

$$\Pi\Big(\bigotimes_{i=1}^m A_i\Big) = \Big(\bigotimes_{i=m}^1 A_i\Big)\Pi$$

(i.e., the tensor product of m operators is unitarily equivalent commutative). Now put $\mathcal{H}_i = \mathcal{H}$ and $A_i = A$ for all i. In this case the unitary operator Π on $\bigotimes_{i=1}^m \mathcal{H}$ also is an involution ($\Pi^2 = I$, and so it is a symmetry — i.e., $\Pi^{-1} = \Pi^* = \Pi$), and the tensor product $\bigotimes_{i=1}^m A$ of m copies of an arbitrary operator A in $\mathcal{B}[\mathcal{H}]$ commutes with the nonscalar normal operator Π in $\mathcal{B}\big[\bigotimes_{i=1}^m \mathcal{H}\big]$. This means that

(i) $\bigotimes_{i=1}^m A$ is reducible for every Hilbert space operator A,

which is a classical consequence of the Spectral Theorem. Hence the tensor product $\bigotimes_{i=1}^m A$ always has a nontrivial reducing subspace. Now suppose A is injective.

(ii) If A is irreducible, then $\bigotimes_{i=1}^m A$ has no nontrivial regular reducing subspace.

This follows from Theorem 1(a_4'). □

4. Intrinsic and Regular Lattices

Take a finite collection of Hilbert spaces $\{\mathcal{H}_i\}_{i=1}^m$ and consider the Hilbert spaces made up of their orthogonal direct sum, $\bigoplus_{i=1}^m \mathcal{H}_i$, and of their tensor product, $\bigotimes_{i=1}^m \mathcal{H}_i$. Let $\mathrm{Lat}(A)$ stand for the lattice of all invariant subspaces for an arbitrary operator A, and take operators A_i in $\mathcal{B}[\mathcal{H}_i]$ for each integer $i \in [1, m]$.

Intrinsic Lattices: Let $\bigoplus_{i=1}^m \mathrm{Lat}(A_i)$ be the collection of all intrinsic subspaces $\bigoplus_{i=1}^m \mathcal{M}_i \subseteq \bigoplus_{i=1}^m \mathcal{H}_i$ made up of A_i-invariant subspaces $\mathcal{M}_i$,

$$\bigoplus_{i=1}^m \mathrm{Lat}(A_i) = \Big\{ \bigoplus_{i=1}^m \mathcal{M}_i \subseteq \bigoplus_{i=1}^m \mathcal{H}_i :\ \mathcal{M}_i \in \mathrm{Lat}(A_i) \Big\}.$$

Let $\mathrm{ILat}\big(\bigoplus_{i=1}^m A_i\big)$ denote the collection of all intrinsic invariant subspaces for the direct sum $\bigoplus_{i=1}^m A_i$ on $\bigoplus_{i=1}^m \mathcal{H}_i$,

$$\mathrm{ILat}\Big(\bigoplus_{i=1}^m A_i\Big) = \Big\{ \bigoplus_{i=1}^m \mathcal{M}_i \subseteq \bigoplus_{i=1}^m \mathcal{H}_i :\ \bigoplus_{i=1}^m \mathcal{M}_i \in \mathrm{Lat}\Big(\bigoplus_{i=1}^m A_i\Big) \Big\}.$$

This is a lattice. Indeed, consider the lattice $\mathrm{Lat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ of all subspaces of the direct sum $\bigoplus_{i=1}^m \mathcal{H}_i$ and let $\mathrm{ILat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ be the subcollection of $\mathrm{Lat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ consisting of all intrinsic subspaces of $\bigoplus_{i=1}^m \mathcal{H}_i$. It is not difficult to verify that $\mathrm{ILat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$ is sublattice of $\mathrm{Lat}\big(\bigoplus_{i=1}^m \mathcal{H}_i\big)$. Since

$$\mathrm{ILat}\Big(\bigoplus_{i=1}^m A_i\Big) = \mathrm{ILat}\Big(\bigoplus_{i=1}^m \mathcal{H}_i\Big) \cap \mathrm{Lat}\Big(\bigoplus_{i=1}^m A_i\Big),$$

it follows that $\mathrm{ILat}\big(\bigoplus_{i=1}^m A_i\big)$ is a sublattice of $\mathrm{Lat}\big(\bigoplus_{i=1}^m A_i\big)$. Moreover, it is readily verified that an intrinsic subspace $\mathcal{I} = \bigoplus_{i=1}^m \mathcal{M}_i$ lies in $\mathrm{ILat}\big(\bigoplus_{i=1}^m A_i\big)$ if and only if $\mathcal{I} \in \bigoplus_{i=1}^m \mathrm{Lat}(A_i)$. Therefore (and recalling Remark 1),

$$\bigoplus_{i=1}^m \mathrm{Lat}(A_i) = \mathrm{ILat}\Big(\bigoplus_{i=1}^m A_i\Big) \subsetneq \mathrm{Lat}\Big(\bigoplus_{i=1}^m A_i\Big).$$

Regular Lattices: Since $\{0\} \in \mathrm{Lat}(A_i)$ for every $i \in [1, m]$, it follows that if $\mathcal{M}_{i_0} = \{0\}$ for some $i_0 \in [1, m]$, then $\mathcal{M}_{i_0} \in \mathrm{Lat}(A_{i_0})$ and $\bigotimes_{i=1}^m \mathcal{M}_i = \{0\} = \bigotimes_{i=1}^m \{0\}$ is a regular subspace made up of A_i-invariant subspaces even if $\mathcal{M}_i \notin \mathrm{Lat}(A_i)$ for every $i \neq i_0$. So it is convenient to exclude zero subspaces when defining the tensor product space counterpart. Thus let $\bigotimes_{i=1}^m \mathrm{Lat}(A_i)$ be the collection of all nonzero regular subspaces $\bigotimes_{i=1}^m \mathcal{M}_i \subseteq \bigotimes_{i=1}^m \mathcal{H}_i$ made up of A_i-invariant subspaces $\mathcal{M}_i$,

$$\bigotimes_{i=1}^m \mathrm{Lat}(A_i) = \Big\{ \{0\} \neq \bigotimes_{i=1}^m \mathcal{M}_i \subseteq \bigotimes_{i=1}^m \mathcal{H}_i :\ \{0\} \neq \mathcal{M}_i \in \mathrm{Lat}(A_i) \Big\}.$$

Let $\mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)$ denote the collection of all regular invariant subspaces for the tensor product operator $\bigotimes_{i=1}^m A_i$ on $\bigotimes_{i=1}^m \mathcal{H}_i$:

$$\mathrm{RLat}\Big(\bigotimes_{i=1}^m A_i\Big) = \Big\{ \bigotimes_{i=1}^m \mathcal{M}_i \subseteq \bigotimes_{i=1}^m \mathcal{H}_i :\ \bigotimes_{i=1}^m \mathcal{M}_i \in \mathrm{Lat}\Big(\bigotimes_{i=1}^m A_i\Big) \Big\}.$$

Observe that, if $A_{i_0} = O$ for some $i_0 \in [1, m]$, then $\bigotimes_{i=1}^m A_i = O$, and so every regular subspace $\bigotimes_{i=1}^m \mathcal{M}_i$ of $\bigotimes_{i=1}^m \mathcal{H}_i$ lies in $\mathrm{Lat}\big(\bigotimes_{i=1}^m A_i\big)$ independently of the operators A_i for every $i \neq i_0$. Also note that if $\mathcal{M}_{i_0} \subseteq \mathcal{N}(A_{i_0})$ for some $i_0 \in [1, m]$ (and so $\mathcal{M}_{i_0} \in \mathrm{Lat}(A_{i_0})$), then $\bigotimes_{i=1}^m \mathcal{M}_i \subseteq \mathcal{N}\big(\bigotimes_{i=1}^m A_i\big)$ so that $\bigotimes_{i=1}^m \mathcal{M}_i$ lies in $\mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)$ independently of the subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ for every $i \neq i_0$.

A word on terminology and another on notation. As usual (see e.g., [9, p. 20]), an operator that is not a multiple of the identity is called *nonscalar*, otherwise it is called *scalar*. Let $\subseteq$ and $\subset$ denote ordinary inclusion and proper inclusion, respectively. We write $\subsetneqq$ for an ordinary inclusion that may not be an identity (i.e., where there are known instances for which the reverse inclusion does not hold — this is different from proper inclusion, where the reverse inclusion never holds).

Corollary 2. *Take an arbitrary integer $m \geq 2$ and consider the preceding setup.*

(a) $\bigotimes_{i=1}^m \mathrm{Lat}(A_i) \subsetneqq \mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\} \subsetneqq \mathrm{Lat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$.

(b) *If either every A_i is injective or every A_i is scalar, then* $\bigotimes_{i=1}^m \mathrm{Lat}(A_i) = \mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$.

(c) *If $\bigotimes_{i=1}^m \mathrm{Lat}(A_i) = \mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$, then there is no pair $\{A_{i_1}, A_{i_2}\}$ from $\{A_i\}_{i=1}^m$ containing a noninjective and a nonscalar operator.*

(d_1) *If every A_i is nonscalar, then $\bigotimes_{i=1}^m \mathrm{Lat}(A_i) = \mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$ if and only if every A_i is injective.*

(d_2) *If every A_i is noninjective, then $\bigotimes_{i=1}^m \mathrm{Lat}(A_i) = \mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$ if and only if every A_i is zero.*

Proof. (a) Theorem $1(\mathrm{a}_1)$ ensures the first ordinary inclusion in (a), which may be proper (depending on $\{A_i\}_{i=1}^m$) by Claim 1 below. The second ordinary inclusion in (a) is trivial, and Corollary 1(b) ensures that it may be proper (even if the operators A_i are all nonscalar and injective).

(b) If A_i are all scalar, then the identity in (b) is trivially verified. If A_i are all injective, then any $\mathcal{R}$ in $\mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$ lies in $\bigotimes_{i=1}^m \mathrm{Lat}(A_i)$ by Theorem $1(\mathrm{a}_4')$, and therefore the identity in (b) follows from the first inclusion in (a).

(c) Conversely, (c) is a direct consequence of the following result.

Claim 1: If there exists a pair $\{A_{i_1}, A_{i_2}\}$ from $\{A_i\}_{i=1}^m$ consisting of a noninjective and a nonscalar operator, then $\mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\} \not\subseteq \bigotimes_{i=1}^m \mathrm{Lat}(A_i)$.

Indeed, an operator A is injective if and only if $\mathcal{N}(A) = \{0\}$ and scalar if and only if $\mathrm{Lat}(A) = \mathrm{Lat}(\mathcal{H})$. Thus, if $\mathcal{N}(A_{i_1}) \neq \{0\}$ for some $i_1 \in [1, m]$, then there exists $\mathcal{M}_{i_1} \in \mathrm{Lat}(\mathcal{H}_{i_1})$ such that $\{0\} \neq \mathcal{M}_{i_1} \subseteq \mathcal{N}(A_{i_1})$. Hence $\mathcal{M}_{i_1} \in \mathrm{Lat}(A_{i_1})$. Moreover, $\bigotimes_{i=1}^m \mathcal{M}_i \in \mathcal{N}\big(\bigotimes_{i=1}^m A_i\big)$, and therefore

$$\{0\} \neq \bigotimes_{i=1}^m \mathcal{M}_i \in \mathrm{RLat}\Big(\bigotimes_{i=1}^m A_i\Big)\backslash\{0\}$$

for every $\{0\} \neq \mathcal{M}_i \in \mathrm{Lat}(\mathcal{H}_i)$ with $i \neq i_1$ in $[1, m]$. If $\mathrm{Lat}(A_{i_2}) \neq \mathrm{Lat}(\mathcal{H}_{i_2})$ for some $i_2 \neq i_1$ in $[1, m]$, then take $\{0\} \neq \mathcal{M}_{i_2} \in \mathrm{Lat}(\mathcal{H}_{i_2})\backslash\mathrm{Lat}(A_{i_2})$ so that

$$\bigotimes_{i=1}^m \mathcal{M}_i \notin \bigotimes_{i=1}^m \mathrm{Lat}(A_i).$$

(Reason: if $\mathcal{M}_1 \otimes \mathcal{M}_1 = \mathcal{N}_1 \otimes \mathcal{N}_2 \neq \{0\}$, then we may infer from [12, Proof of Proposition 2.1] that $\mathcal{M}_1 = \mathcal{N}_1$ and $\mathcal{M}_2 = \mathcal{N}_2$, which can be extended to a finite collection of subspaces). This completes the proof of Claim 1.

(d) The result in (c) ensures that if $\bigotimes_{i=1}^m \mathrm{Lat}(A_i) = \mathrm{RLat}\big(\bigotimes_{i=1}^m A_i\big)\backslash\{0\}$ and if all A_i are nonscalar, then all A_i are also injective; if all A_i are noninjective, then all A_i are also scalar, i.e., all A_i are zero. Thus (d_1) and (d_2) follow from (b) and (c). □

5. Applications — Unilateral Shifts

If the orthogonal direct sum $\bigoplus_{i=1}^m A_i$ of a finite collection $\{A_i\}_{i=1}^m$ of operators on $\mathcal{H}_i$ is a unilateral shift, then it is a *shift direct sum.* It can be verified that

> *$\bigoplus_{i=1}^m A_i$ is a shift direct sum if and only if $A_i = S_i$, where each S_i is a unilateral shift.*

Since $\bigoplus_{i=1}^m \mathrm{Lat}(A_i) = \mathrm{ILat}\big(\bigoplus_{i=1}^m A_i\big)$, an intrinsic subspace $\bigoplus_{i=1}^m \mathcal{M}_i$ of the direct sum $\bigoplus_{i=1}^m \mathcal{H}_i$ is invariant (reducing) for $\bigoplus_{i=1}^m A_i$ if and only if each $\mathcal{M}_i$ is invariant (reducing) for A_i. Thus, the above italicized result ensures the next one:

> *An intrinsic invariant (reducing) subspace of a shift direct sum $\bigoplus_{i=1}^m A_i$ is the direct sum of m invariant (reducing) subspaces for m unilateral shifts.*

The tensor product counterpart is not straightforward, as we shall see in this section. Since a unilateral shift (of any multiplicity, acting on a Hilbert space) is injective, Corollary 2(b) characterizes the collection of all regular invariant subspace of the tensor product $\bigotimes_{i=1}^m S_i$ of a finite collection $\{S_i\}_{i=1}^m$ of unilateral shifts,

$$\mathrm{RLat}\Big(\bigotimes_{i=1}^m S_i\Big)\Big\backslash\{0\} = \bigotimes_{i=1}^m \mathrm{Lat}(S_i),$$

once each $\mathrm{Lat}(S_i)$ was fully characterized in [2]. However, by Corollary 1(b),

$$\mathrm{RLat}\Big(\bigotimes_{i=1}^m S_i\Big) \subset \mathrm{Lat}\Big(\bigotimes_{i=1}^m S_i\Big)$$

if $S_i = S$ for all i, where S is a unilateral shift of multiplicity 1 (which is irreducible). Thus a full characterization of the lattice $\mathrm{Lat}\big(\bigotimes_{i=1}^m S_i\big)$ requires the description of all irregular subspaces of tensor products of unilateral shifts,

$$\mathrm{Lat}\Big(\bigotimes_{i=1}^m S_i\Big)\Big\backslash \mathrm{RLat}\Big(\bigotimes_{i=1}^m S_i\Big),$$

and this turns up from [2] as well because, as a particular case of Theorem 2 below, $\bigotimes_{i=1}^m S_i$ is again a unilateral shift.

Theorem 2. *Take an arbitrary integer $m \geq 2$. The tensor product $\bigotimes_{i=1}^m A_i$ is a unilateral shift if and only if $\bigotimes_{i=1}^m A_i = \bigotimes_{i=1}^m V_i$, where each V_i is an isometry being at least one of them a unilateral shift.*

Proof. A Hilbert space operator A is strongly stable if the power sequence $\{A^n\}$ converges strongly to the null operator (i.e., $\|A^n x\| \to 0$ for every x — notation: $A^n \xrightarrow{s} O$). Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and take A and B in $\mathcal{B}[\mathcal{H}]$ and $\mathcal{B}[\mathcal{K}]$. Since a unilateral shift is precisely an isometry whose adjoint is strongly stable [4, Lemma 6.1], the tensor product $A \otimes B$ is a unilateral shift if and only if

$$A \otimes B \text{ is an isometry} \quad \text{and} \quad (A \otimes B)^{*n} \xrightarrow{s} O.$$

But, for any pair of nonzero operators A and B,

$$A \otimes B \text{ is an isometry} \quad \Longleftrightarrow \quad A \otimes B = V \otimes J,$$

where V and J are isometries in $\mathcal{B}[\mathcal{H}]$ and $\mathcal{B}[\mathcal{K}]$ (see e.g., [12, Theorem 2.4]), and

$$(V\otimes J)^{*n}\xrightarrow{\;s\;}O \qquad\Longleftrightarrow\qquad V^{*n}\xrightarrow{\;s\;}O \quad\text{or}\quad J^{*n}\xrightarrow{\;s\;}O$$

(cf. [6, Proposition 1] or [8, Theorems 2 and 3]). Thus $A\otimes B$ is a unilateral shift if and only if $A\otimes B=V\otimes J$, where V and J are isometries and one of them has a strongly stable adjoint, which means that one of the isometries is a unilateral shift. This proves the theorem for $m=2$. If $m\geq 3$ then, for any $j\in[2,m-1]$,

$$\bigotimes_{i=1}^{m}A_i=\bigotimes_{i=1}^{j-1}A_i\otimes A_j\otimes\bigotimes_{i=j+1}^{m}A_i$$

so that the result for $m=2$ extends by induction to any integer $m\geq 3$, because the tensor product of isometries is again an isometry. □

If the tensor product $\bigotimes_{i=1}^m A_i$ of a finite collection $\{A_i\}_{i=1}^m$ of operators acting on Hilbert spaces $\mathcal{H}_i$ is a unilateral shift, then it is called a *shift tensor product.*

Corollary 3. *A regular invariant* (*reducing*) *subspace of a shift tensor product* $\bigotimes_{i=1}^m A_i$ *is the tensor product of invariant* (*reducing*) *subspaces for* m *isometries, being at least one of them a unilateral shift.*

Proof. This follows from Theorem 2 and Corollary 2(b), recalling that if $\bigotimes_{i=1}^m V_i$ is injective, then so is each V_i (see e.g., [7, Property 6]). □

6. Applications — Completely Nonunitary Contractions

A contraction (i.e., an operator A with $\|A\|\leq 1$) is completely nonunitary if the restriction of it to every nonzero reducing subspace is not unitary. Equivalently, a nonzero contraction A in $\mathcal{B}[\mathcal{H}]$ is not completely nonunitary if there exists a nonzero vector $x\in\mathcal{H}$ such that

$$\|A^n x\|=\|A^{*n}x\|=\|x\| \quad \text{for every positive integer } n.$$

For every contraction A in $\mathcal{B}[\mathcal{H}]$, let $\mathcal{U}_A$ be the collection of all those vectors for which the above identity holds. The Nagy–Foiaş–Langer decomposition for contractions (see e.g., [13, p. 9] or [4, p. 79]) says that $\mathcal{U}_A$ is a reducing subspace for A and that A has a unique decomposition on $\mathcal{H}=\mathcal{U}_A\oplus\mathcal{U}_A^\perp$, namely,

$$A=U_A\oplus C_A,$$

where $U_A=A|_{\mathcal{U}_A}$ is unitary and a $C_A=A|_{\mathcal{U}_A^\perp}$ is completely nonunitary. Thus a contraction A is completely nonunitary if and only if $\mathcal{U}_A=\{0\}$.

Consider a finite collection $\{A_i\}_{i=1}^m$ of operators on a complex Hilbert space $\mathcal{H}_i$. The (orthogonal) direct sum $\bigoplus_{i=1}^m A_i$ is a contraction if and only if every operator A_i is a contraction (since $\big\|\bigoplus_{i=1}^m A_i\big\|=\max_{1\leq i\leq m}\|A_i\|$). If every operator A_i is a contraction, then their tensor product $\bigotimes_{i=1}^m A_i$ is a contraction (since $\big\|\bigotimes_{i=1}^m A_i\big\|=\prod_{i=1}^m\|A_i\|$). It is readily verified that

> $\bigoplus_{i=1}^m A_i$ *is a completely nonunitary contraction if and only if each* A_i *is a completely nonunitary contraction.*

(Reason: $\mathcal{U}_{\bigoplus_{i=1}^m A_i}=\bigoplus_{i=1}^m\mathcal{U}_{A_i}$). The tensor product counterpart is investigated in Theorem 3 below, which is based on the next lemma.

Lemma 1. *If A_i is a contraction on a Hilbert space $\mathcal{H}_i$ for each $i \in [1, m]$, then*

$$\mathcal{U}_{\bigotimes_{i=1}^m A_i} = \bigotimes_{i=1}^m \mathcal{U}_{A_i}.$$

Proof. Let A be a contraction on a Hilbert space $\mathcal{H}$ and let B be a contraction on a Hilbert space $\mathcal{K}$. Consider the contraction $A \otimes B$ on $\mathcal{H} \otimes \mathcal{K}$. Let $\{e_\alpha\}$ and $\{f_\beta\}$ be orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively, so that $\{e_\alpha \otimes f_\beta\}$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{K}$. Take an arbitrary z in $\mathcal{H} \otimes \mathcal{K}$ and consider its Fourier expansion,

$$z = \sum_{\alpha,\,\beta} \lambda_{\alpha,\,\beta}\, e_\alpha \otimes f_\beta = \sum_\alpha e_\alpha \otimes y_\alpha = \sum_\beta x_\beta \otimes f_\beta, \tag{1}$$

with $x_\beta = \sum_\alpha \lambda_{\alpha,\,\beta}\, e_\alpha$ and $y_\alpha = \sum_\beta \lambda_{\alpha,\,\beta}\, f_\beta$ in $\mathcal{H}$ and $\mathcal{K}$ for every α and every β, respectively, where $\{\lambda_{\alpha,\,\beta}\}$ is a square-summable family of scalars. Hence

$$\|z\|^2 = \sum_\alpha \|y_\alpha\|^2 = \sum_\beta \|x_\beta\|^2 \tag{2}$$

because $\{e_\alpha \otimes y_\alpha\}$ and $\{x_\beta \otimes f_\beta\}$ are orthonormal families such that $\|e_\alpha \otimes y_\alpha\| = \|y_\alpha\|$ and $\|x_\beta \otimes f_\beta\| = \|x_\beta\|$. Now suppose z lies in $\mathcal{U}_{A\otimes B}$. Then

$$\|z\| = \|(A \otimes B)^n z\| = \|(A^n \otimes I)(I \otimes B^n)z\| \le \|(I \otimes B^n)z\| \le \|z\| \tag{3}$$

since $(A\otimes B)^n = (A^n \otimes I)(I \otimes B^n) = (I \otimes B^n)(A^n \otimes I)$ for every nonnegative integer n, which is a contraction that is a product of contractions, where I stands either for the identity on $\mathcal{H}$ or for the identity on $\mathcal{K}$. Thus, according to (1), (2) and (3),

$$\begin{aligned}\sum_\alpha \|y_\alpha\|^2 = \|z\|^2 = \|(I \otimes B^n)z\|^2 &= \Big\|(I \otimes B^n) \sum_\alpha e_\alpha \otimes y_\alpha\Big\|^2 \\ &= \Big\|\sum_\alpha e_\alpha \otimes B^n y_\alpha\Big\|^2 = \sum_\alpha \|B^n y_\alpha\|^2.\end{aligned}$$

This implies that

$$\|y_\alpha\| = \|B^n y_\alpha\| \quad \text{for every positive integer } n \text{ and every index } \alpha$$

because each B^n is a contraction. Since z lies in $\mathcal{U}_{A\otimes B}$, (3) holds if A and B are replaced with A^* and B^*, and so the above identity holds if B is replaced with B^*. Moreover, since $A^n \otimes I$ and $I \otimes B^n$ commute, we get

$$\|z\| = \|(A^n \otimes I)z\| \tag{3'}$$

for every nonnegative integer n by using the same argument applied to yield (3). Thus, reasoning as above, (1), (2) and (3') ensure that

$$\|x_\beta\| = \|A^n x_\beta\| \quad \text{for every positive integer } n \text{ and every index } \beta,$$

and again the above identity also holds if A is replaced with A^*. Hence $x_\beta \in \mathcal{U}_A$ for every β and $y_\alpha \in \mathcal{U}_B$ for every α, and so $z \in (\mathcal{H} \otimes \mathcal{U}_B) \cap (\mathcal{U}_A \otimes \mathcal{K})$ according to (1). However, if $\mathcal{M}$ and $\mathcal{N}$ are subspaces of $\mathcal{H}$ and $\mathcal{K}$, then

$$(\mathcal{H} \otimes \mathcal{N}) \cap (\mathcal{M} \otimes \mathcal{K}) = \mathcal{M} \otimes \mathcal{N}.$$

Indeed, if P and Q are projections onto $\mathcal{M}$ and $\mathcal{N}$, then $(\mathcal{H} \otimes \mathcal{N}) \cap (\mathcal{M} \otimes \mathcal{K}) = \mathcal{R}(I \otimes Q) \cap \mathcal{R}(P \otimes I) = \mathcal{R}\big((I \otimes Q)(P \otimes I)\big) = \mathcal{R}(P \otimes Q) = \mathcal{R}(P) \otimes \mathcal{R}(Q) = \mathcal{M} \otimes \mathcal{N}$ since $I \otimes Q$ and $P \otimes I$ are commuting projections. Thus $z \in (\mathcal{U}_A \otimes \mathcal{U}_B)$.

Then $\mathcal{U}_{A\otimes B} \subseteq \mathcal{U}_A \otimes \mathcal{U}_B$. The reverse inclusion follows at once. Indeed, $\mathcal{U}_A \otimes \mathcal{U}_B$ reduces $A \otimes B$ by Theorem 1(a$_1$) and $(A\otimes B)|_{\mathcal{U}_A\otimes\mathcal{U}_B} = A|_{\mathcal{U}_A} \otimes B|_{\mathcal{U}_B}$ by Theorem 1(c_1), which is unitary (see e.g., [12, Theorem 2.4] or [5, proposition 5]). Therefore

$$\mathcal{U}_{A\otimes B} = \mathcal{U}_A \otimes \mathcal{U}_B.$$

This proves the lemma for $m = 2$, which extends to any integer $m \geq 3$ by using the same argument applied in the proof of Theorem 2. □

Theorem 3. *Take any integer $m \geq 2$ and, for each integer $i \in [1, m]$, let A_i be a contraction on a Hilbert space $\mathcal{H}_i$. The following assertions are pairwise equivalent.*

(a) *The tensor product $\bigotimes_{i=1}^m A_i$ is completely nonunitary.*

(b) *The restriction of the tensor product $\bigotimes_{i=1}^m A_i$ to every nonzero regular reducing subspace is not unitary.*

(c) *One of the contractions A_i is completely nonunitary.*

Proof. Assertion (a) trivially implies (b). Suppose (c) fails so that every contraction A_i is not completely nonunitary, and hence each reducing subspace $\mathcal{U}_{A_i}$ is nonzero and $A_i|_{\mathcal{U}_{A_i}}$ is a unitary operator on $\mathcal{U}_{A_i}$. Thus the regular subspace $\bigotimes_{i=1}^m \mathcal{U}_{A_i}$ of $\bigotimes_{i=1}^m \mathcal{H}_i$ is nonzero (since each $\mathcal{U}_{A_i}$ is), reduces $\bigotimes_{i=1}^m A_i$ by Theorem 1(a$_1$), and

$$\Big(\bigotimes_{i=1}^m A_i\Big)\Big|_{\bigotimes_{i=1}^m \mathcal{U}_{A_i}} = \bigotimes_{i=1}^m A_i|_{\mathcal{U}_{A_i}}$$

by Theorem 1(c$_1$), which is unitary (since it is the tensor product of unitaries) and so (b) fails. Thus (b) implies (c). Lemma 1 ensures that (c) implies (a). Indeed, if $\mathcal{U}_{A_i} = \{0\}$ for some integer $i \in [1, m]$, then $\mathcal{U}_{\bigotimes_{i=1}^m A_i} = \{0\}$. □

Remark 2. Besides showing that (a) and (c) are equivalent,

> $\bigotimes_{i=1}^m A_i$ *is completely nonunitary if and only if one of the contractions A_i is completely nonunitary,*

Theorem 3 also shows that (a) and (b) are equivalent, which means that the property of being completely nonunitary is invariant for the type (regular or irregular) of the reducing subspace. This can be rewritten as follows.

> *There exists a nonzero reducing subspace for $\bigotimes_{i=1}^m A_i$ on which it acts unitarily if and only if there exists a nonzero* regular *reducing subspace $\bigotimes_{i=1}^m \mathcal{M}_i$ for $\bigotimes_{i=1}^m A_i$ on which $\big(\bigotimes_{i=1}^m A_i\big)|_{\bigotimes_{i=1}^m \mathcal{M}_i}$ is unitary.*

Remark 3. The equivalence (a) $\iff$ (c) in Theorem 3 yields another proof for Theorem 2. In fact, a unilateral shift is precisely a completely nonunitary isometry (i.e., a pure isometry), and so $\bigotimes_{i=1}^m A_i$ is a unilateral shift if and only if it is completely nonunitary and $\bigotimes_{i=1}^m A_i = \bigotimes_{i=1}^m V_i$, where each V_i is an isometry (see e.g., [12, Theorem 2.4]). This means by Theorem 3 that $\bigotimes_{i=1}^m A_i = \bigotimes_{i=1}^m V_i$ is completely nonunitary, where each V_i is an isometry and one of them is completely nonunitary. That is, *$\bigotimes_{i=1}^m A_i$ is a unilateral shift if and only if $\bigotimes_{i=1}^m A_i = \bigotimes_{i=1}^m V_i$, where each V_i is an isometry, being at least one of them a unilateral shift.*

Acknowledgment

The author thanks an anonymous referee who brought Lemma 1 to his attention, thus yielding a simpler proof and a sharper statement of Theorem 3.

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