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UNITARY EQUIVALENCE AND TRANSLATION REPRESENTATION IN WAVELET THEORY

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Abstract. Unitary Equivalence and Translation Representation play a key role in the Lax-Phillips Scattering Theory. In this paper we show that Translation Representation also plays an important role in Wavelet Theory, for Discrete Multi-Resolution Approximation as well as for Continuous Multi-Translation Approximation — to be defined in the paper.

1. INTRODUCTION

At the heart of Wavelet Theory is a chain of nested subspaces of the functions space $\mathcal{L}^2(\mathbb{R})$, called Multi-Resolution Approximation (MRA), generated from a "resolution-2^0" subspace $\mathcal{V}$ by the discrete unitary group $[D^m]$ of the scale-by-2 operator D. Moreover $\mathcal{V}$ is actually an outgoing (incoming) subspace with respect to $[D^m]$.

The concept of outgoing (incoming) subspace, with respect to either a discrete group, or a continuous one-parameter group, of Hilbert space unitary operators, originated from the Lax-Phillips Scattering Theory in which Translation Representation plays a key role.

In this paper we study applications of Translation Representation to Wavelet Theory. It will be shown that it plays an important role in the Discrete Multi-Resolution Approximation (DMRA) and, particularly, in the associated Continuous Multi-Translation Approximation (CMTA) — to be defined below.

We must note that the "object" that connects Wavelet Theory and Scattering Theory is a group of Hilbert space shift operators. This will be shown below.

We begin in Section 2 by showing that the well known Discrete Multi-Resolution Approximation is actually generated from a specific outgoing (incoming) subspace $\mathcal{O}$ (or $\mathcal{I}$) with respect to the scale-by-2 operator D. Then by Discrete Translation Representation, the space $\mathcal{L}^2(\mathbb{R})$ "goes" into — i.e., "isomorphically onto" — the space of approximations — of a signal at resolution-2^0 — to all nonzero resolutions of the signal. We then turn to Continuous Translation Representation. This leads to the formulation of Continuous Multi-Translation Approximation over the signal space $\mathcal{L}^2(\mathbb{R})$ as well as over the Translation Representation "Hilbert over Hilbert" space $\mathcal{L}^2(\mathbb{R};\mathcal{W})$ — i.e., the space of $\mathcal{L}^2(\mathbb{R})$ functions taking values in an auxiliary Hilbert space $\mathcal{W}$.

We close the paper with some historical notes and remarks on the Discrete Wavelet Transforms, as well as on the time-steps decomposition which leads to "elementary" Translation Representation for wavelets.

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2. MAIN RESULTS

Let $\mathcal{H}$ be a separable complex Hilbert space on which there is a discrete group of operators $[U^m] := \{U^m,\ m \in \mathbb{Z}\}$ generated from a unitary operator U, where $\mathbb{Z}$ denotes the set of all integers.

2.1. Discrete Translation Representation and Discrete Multi-Resolution Approximation.

We now recall the concept of outgoing (incoming) subspace [12] with respect to $[U^m]$ [12]. This leads to Discrete Translation Representations of $\mathcal{H}$ and of an outgoing (incoming) subspace as well as that of the unitary operator U.

Definition 1. A subspace $\mathcal{O}$ (or $\mathcal{I}$) of $\mathcal{H}$ is an outgoing (incoming) subspace with respect to $[U^m]$ if the following properties are satisfied.

(i) $U\mathcal{O} \subset \mathcal{O} \quad \big(U^*\mathcal{I} \subset \mathcal{I}\big)$.

(ii) $\overline{\bigcup}_{m\in\mathbb{Z}} U^m\mathcal{O}$ (or $\mathcal{I}$) $= \mathcal{H}$.

(iii) $\bigcap_{m\in\mathbb{Z}} U^m\mathcal{O}$ (or $\mathcal{I}$) $= \{0\}$.

It follows from property (i) that an outgoing subspace $\mathcal{O}$ is U-invariant. Moreover, the chain of subspaces $\{U^m\mathcal{O}\} := \{U^m\mathcal{O}, m \in \mathbb{Z}\}$ "generated" from $\mathcal{O}$ is "decreasingly" nested

$$U^{m+1}\mathcal{O} \subset U^m\mathcal{O}, \quad m \in \mathbb{Z}. \tag{2.1}$$

We must note that the closed union of a decreasingly nested sequence of subspaces coincides with its closed span, hence (2.1) makes sense only because of property (i).

Let $\mathcal{W}$ be the orthogonal complement of $U\mathcal{O}$ in $\mathcal{O}$, that is,

$$\mathcal{W} := \mathcal{O} \ominus U\mathcal{O}. \tag{2.2}$$

Then it can be shown that [12]

$$\mathcal{O} = \bigoplus_{m=0}^{\infty} U^m\mathcal{W} \tag{2.3}$$

and

$$\mathcal{H} = \bigoplus_{m\in\mathbb{Z}} U^m\mathcal{W}, \tag{2.4}$$

where

$$U^m\mathcal{W} \perp U^{m'}\mathcal{W}, \quad m \neq m'. \tag{2.5}$$

Hence $\mathcal{W}$ is called *wandering* for U [4, 6, 9, 21]. Moreover, because of (2.4), it is also a "generating" wandering subspace.

Equation (2.4) can be characterized as a "wandering subspace" definition of a bilateral shift whose *multiplicity* is $\dim\mathcal{W}$. Therefore, since (2.4) is a consequence of Definition 1, it follows that a unitary operator for which there is an outgoing (incoming) subspace is a bilateral shift [2], see also [11].

More is true [12].

Proposition 1. *Let $\mathcal{O}$ be an outgoing subspace with respect to the group of bilateral shifts $[U^m]$ over $\mathcal{H}$. Then the operator Ω_d defined by*

$$\Omega_d :\ \mathcal{H} \to \ell^2(\mathbb{R};\mathcal{W}), \quad \Omega_d\, h = \{w_m\}, \quad h \in \mathcal{H}, \quad w_m \in \mathcal{W}, \tag{2.6}$$

where

$$h = \sum_{m\in\mathbb{Z}} U^m w_m, \quad w_m \in \mathcal{W}, \quad \textit{and} \quad \sum_{m\in\mathbb{Z}} ||w_m||^2 = ||h||^2. \tag{2.7}$$

is unitary and is such that

$$\mathcal{H} \leftrightarrow \ell^2(\mathbb{R};\mathcal{W}), \tag{2.8}$$

$$\mathcal{O} \leftrightarrow \ell^2([0,\infty);\mathcal{W}), \tag{2.9}$$

$$U \leftrightarrow S, \qquad \Omega_d\, U = S\,\Omega_d, \tag{2.10}$$

where $\leftrightarrow$ *means unitary equivalent, while* S *is the bilateral shift on* $\ell^2(\mathbb{R};\mathcal{W})$ *defined by*

$$S\{w_m\} := \{w_{m-1}\}. \tag{2.11}$$

Equations (2.8) - (2.10), because of the "translation" action of S on the subscript m in (2.11), are referred to as Discrete Translation Representations of $\mathcal{H}$, $\mathcal{O}$ and U, respectively. The important point here is how to characterize the operator Ω_d which "carries out" the unitary equivalence.

Similar results can be stated for an incoming subspace $\mathcal{I}$. In the following we will be mostly dealing with outgoing subspaces. We now connect the above to Discrete Multi-Resolution Approximation [14, 18], see also [8, 15].

Consider the outgoing subspace set up with $\mathcal{H} := \mathcal{L}^2(\mathbb{R})$ and with $U := D$ — the scale-by-2 operator on $\mathcal{L}^2(\mathbb{R})$ defined by

$$Df(\cdot) := \sqrt{2}\, f\big(2(\cdot)\big). \tag{2.12}$$

Moreover, the outgoing subspace $\mathcal{O}$ — called *wavelet outgoing subspace* — is explicitely given by

$$\mathcal{O} := \bigvee_{n\in\mathbb{Z}} \phi\big((\cdot) - n\big) \tag{2.13}$$

for some unit function $\phi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ — called *scaling function*, see for instance, [8, 15] — which is such that its integral translates $\phi\big((\cdot)-n\big)$ are orthogonal

$$\phi\big((\cdot) - n\big) \perp \phi\big((\cdot) - n'\big), \quad n \neq n'. \tag{2.14}$$

Definition 2. The chain of subspaces $\{D^m\mathcal{O}\} := \{D^m\mathcal{O},\ m \in \mathbb{Z}\}$ with the outgoing subspace $\mathcal{O}$ defined in terms of a scaling function $\phi(\cdot)$ is called a Discrete Multi-Resolution Approximation.

More is true, see for instance [8, 15].

Proposition 2. *Given a DMRA* $\{D^m\mathcal{O}\}$ *— with scaling function* $\phi(\cdot)$ *— over* $\mathcal{L}^2(\mathbb{R})$*. Then there exists a wavelet* $\psi(\cdot)$ *such that the wavelet functions*

$$\psi_{m,n}(\cdot) := D^m\psi\big((\cdot) - n\big), \quad (m,n) \in \mathbb{Z}^2 \tag{2.15}$$

form an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$*, and the subspace* $\mathcal{W}$ *is expressed explicitely in terms of* $\psi\big((\cdot)-n\big)$ *as*

$$\mathcal{W} := \bigvee_{n\in\mathbb{Z}} \psi\big((\cdot) - n\big). \tag{2.16}$$

It follows from (2.4) and (2.7) that, for any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$,

$$f(\cdot) = \sum_{m\in\mathbb{Z}} D^m w_m(\cdot), \quad w_m(\cdot) \in \mathcal{W}, \quad \text{and} \quad \sum_{m\in\mathbb{Z}} ||w_m(\cdot)||^2 = ||f(\cdot)||^2. \tag{2.17}$$

Let $\mathcal{P}_m$ be the orthogonal projection onto the wandering subspace $D^m\mathcal{W}$ then, see for instance [11],

$$\mathcal{P}_m\{f(\cdot)\} = D^m\,\mathcal{P}_0\,\{D^{*m} f(\cdot)\}, \quad m \in \mathbb{Z}. \tag{2.18}$$

Therefore

$$w_m(\cdot) = D^{*m}\,\mathcal{P}_m\{f(\cdot)\}, \quad m \in \mathbb{Z} \tag{2.19}$$

$$= \mathcal{P}_0\{D^{*m} f(\cdot)\}. \tag{2.20}$$

Hence

$$f(\cdot) = \sum_{m\in\mathbb{Z}} D^m\,\mathcal{P}_0\{D^{*m} f(\cdot)\}. \tag{2.21}$$

As a consequence the unitary operator Ω_d is now defined by

$$\Omega_d :\ \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{R};\mathcal{W}), \quad \Omega_d\, f(\cdot) = \{w_m(\cdot)\} = \big\{\mathcal{P}_0\{D^{*m} f(\cdot)\}\big\}. \tag{2.22}$$

In Wavelet Theory the function $D^m f(\cdot) = \sqrt{2}^{\,m} f\big(2^m(\cdot)\big)$, $m \in \mathbb{Z}$, is, by tradition, defined as the *resolution (or scale)*-2^m of $f(\cdot)$. Then, since $\mathcal{P}_m\{f(\cdot)\}$ is the projection of $f(\cdot)$ onto the subspace $D^m\mathcal{W}$, it is defined as an *approximation at resolution*-2^m to $f(\cdot)$ [8].

We therefore conclude from (2.22) that.

Proposition 3. *A signal $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$, under the action of Ω_d, goes into approximations at resolution-2^0 to the resolutions-2^m of $f(\cdot)$. In other words, the Translation Representation space of the signal space $\mathcal{L}^2(\mathbb{R})$ is the space of approximations to all resolutions of the signal $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$.*

2.2. **Continuous Translation Representation and Continuous Multi-Translation Approximation.** We now consider the case in which, beside a discrete group of unitary operators $[U^m]$, we also have a one-parameter continuous group of unitary operators $[\mathcal{U}(t)] := \{\mathcal{U}(t),\ t \in \mathbb{R}\}$ over $\mathcal{H}$.

We have [12].

Definition 3. A subspace $\mathcal{O}$ (*or* $\mathcal{I}$) of $\mathcal{H}$ is outgoing (incoming) with respect to a group of unitary operators $[\mathcal{U}(t)]$ over $\mathcal{H}$ if the following conditions are satisfied.

(i) $\mathcal{U}(t)\mathcal{O} \subset \mathcal{O},\ t \geq 0, \quad \big(\,\mathcal{U}(t)\mathcal{I} \subset \mathcal{I},\ t < 0\,\big)$.

(ii) $\overline{\bigcup}_t\, \mathcal{U}(t)\mathcal{O}\,(\text{ or } \mathcal{I}) = \mathcal{H}$.

(iii) $\bigcap_t \mathcal{U}(t)\mathcal{O}\,(\text{ or } \mathcal{I}) = \{0\}$.

We must note that, as in the discrete case, Definition 3 implies that the unitary group $[\mathcal{U}(t)]$ is a continuous shift over $\mathcal{H}$ [2].

To proceed we recall the following properties which are shared by a unitary group $[\mathcal{U}(t)]$ and its discrete cogenerator group $[C_{\mathcal{U}}^m]$ [12, 16].

Theorem 1. *Let $[\mathcal{U}(t)]$ be a unitary group — with cogenerator $C_{\mathcal{U}}$ — over $\mathcal{H}$. Then $\mathcal{U}(t)\, C_{\mathcal{U}}^m = C_{\mathcal{U}}^m\, \mathcal{U}(t)$, $t \in \mathbb{R}$, $m \in \mathbb{Z}$. Moreover, an outgoing (incoming) subspace — with respect to $[C_{\mathcal{U}}^m]$ — is also outgoing (incoming) with respect to $[\mathcal{U}(t)]$ and conversely.*

Definition 3 is a "natural" generalization of Definition 1, whether the unitary group $[\mathcal{U}(t)]$ admits a cogenerator or not. However for wavelets we already have a wavelet outgoing subspace $\mathcal{O}$ — defined by (2.13) — with respect to $[D^m]$, and since it is easy to see that the scale-by-2 operator D can also serve as the cogenerator of a unitary group, we propose the following Definition.

Definition 4. Let $\mathcal{O}$ be a wavelet outgoing subspace, which is defined by $\mathcal{O} := \bigvee_{n\in\mathbb{Z}} \phi\big((\cdot) - n\big)$, with respect to the discrete group $[D^m]$ of the scale-by-2 operator D over $\mathcal{L}^2(\mathbb{R})$, as well as with respect to the unitary group $[\mathcal{U}(t)]$ whose cogenerator is D. Then the continuous chain of subspaces $\{\mathcal{O}_t\} := \{\mathcal{U}(t)\mathcal{O},\ t \geq 0\}$ satisfying $\mathcal{O}_{t_2} \subset \mathcal{O}_{t_1},\ \ 0 \leq t_1 < t_2$, is called an outgoing Continuous Multi-Translation Approximation (CMTA) — associated with the DMRA $\{D^m\mathcal{O}\}$.

We do not in general have an explicit expression for $[\mathcal{U}(t)]$! However it can be shown that $[\mathcal{U}(t)]$ is the strong limit of a linear combination of D^m, $m \in \mathbb{Z}$, [16]. Therefore, since for each $t \in \mathbb{R}$ the scale-by-2 operator D is an instantaneous translation by t [17], the unitary group $[\mathcal{U}(t)]$ — which is already a group of continuous shift operators — "behaves" like a group of "translation" operators. Indeed, as we shall see, $[\mathcal{U}(t)]$ is unitarily equivalent to the group of translations by t units over another Hilbert space. This is the justification for introducing the term "Multi-Translation".

To proceed we recall the following Translation Representation Theorem due to Sinai [19].

Theorem 2. *Let $\mathcal{O}$ (or $\mathcal{I}$) be an outgoing (incoming) subspace with respect to a group of unitary operators $[\mathcal{U}(t)]$, with cogenerator $C_{\mathcal{U}}$, over the complex separable Hilbert space $\mathcal{H}$. Then there exists a unitary operator Ω_o (or Ω_i) taking $\mathcal{H}$ onto the Hilbert space $\mathcal{L}^2(\mathbb{R};\mathcal{W})$ — called "Translation Representation Space"*

$$\Omega_o(or\,\Omega_i)\,\mathcal{H} = \mathcal{L}^2(\mathbb{R};\mathcal{W}), \tag{2.23}$$

where $\mathcal{W}$ is some auxiliary Hilbert space. Moreover,

$$\Omega_o\,\mathcal{O} = \mathcal{L}^2\big([0,\infty);\mathcal{W}\big), \quad \big(\Omega_i\,\mathcal{I} = \mathcal{L}^2((-\infty,0);\mathcal{W})\,\big), \tag{2.24}$$

$$\Omega_o\,\big(or\,\Omega_i\big)\,\mathcal{U}(t) = \mathcal{T}(t)\,\Omega_o\,(or\,\Omega_i), \tag{2.25}$$

where $[\mathcal{T}(t)]$ is the unitary group of translations by t units over $\mathcal{L}^2(\mathbb{R};\mathcal{W})$,

$$\mathcal{T}(t)f(\cdot) = f\big((\cdot) - t\big), \quad t \in \mathbb{R}. \tag{2.26}$$

Moreover,

$$\Omega_o\,\big(or\,\Omega_i\big)\,C_{\mathcal{U}} = \mathcal{C}\,\Omega_o\,(or\,\Omega_i), \tag{2.27}$$

where $\mathcal{C}$ is the cogenerator of $[\mathcal{T}(t)]$. The Continuous Translation Representations:

$$\mathcal{L}^2(\mathbb{R}) \leftrightarrow \mathcal{L}^2(\mathbb{R};\mathcal{W}), \tag{2.28}$$

$$\mathcal{O} \leftrightarrow \mathcal{L}^2\big([0,\infty);\mathcal{W}\big) \quad \big(\mathcal{I} \leftrightarrow \mathcal{L}^2\big((-\infty,0);\mathcal{W}\big)\big), \tag{2.29}$$

$$\mathcal{U}(t) \leftrightarrow \mathcal{T}(t), \tag{2.30}$$

$$C_{\mathcal{U}} \leftrightarrow \mathcal{C}, \tag{2.31}$$

are unique up to an isomorphism of $\mathcal{W}$.

The cogenerator $\mathcal{C}$ of $[\mathcal{T}(t)]$ is a bilateral shift on $\mathcal{L}^2(\mathbb{R};\mathcal{W})$, since it is unitarily equivalent to $C_{\mathcal{U}}$ which is a bilateral shift by virtue of the fact that it admits an outgoing subspace $\mathcal{O}$. Moreover $[\mathcal{C}^m]$ is expressed in terms of $[\mathcal{T}(t)]$ as [16],

$$\mathcal{C}^{\pm m} f(x) = f(x) + 2\int_0^\infty L_m'(2t)\, e^{-t}\, \mathcal{T}(\pm t) f(x)\, dt, \quad m \geq 0, \quad x \in \mathbb{R}, \tag{2.32}$$

where $L_m(\cdot)$, $m \geq 0$, is the Laguerre polynomial of degree m [22].

To proceed, consider the space $\mathcal{L}^2\big([0,\infty);\mathcal{W}\big)$ as a subspace of $\mathcal{L}^2(\mathbb{R};\mathcal{W})$. It is clearly invariant for the semigroup $\{\mathcal{T}(t),\, t \geq 0\}$, hence it is also invariant for $\mathcal{C}$ [12, 21] as a unilateral shift — in $\mathcal{L}^2\big([0,\infty);\mathcal{W}\big)$ — whose generating wandering subspace, denoted by $\mathcal{W}a^+$, is [21],

$$\mathcal{W}a^+ = \sqrt{2}\, e^{-x}\, \mathcal{W}, \quad x \geq 0 \tag{2.33}$$

From which we find

$$\mathcal{W}a^+_m := \mathcal{C}^m\, \mathcal{W}a^+ = \mathcal{C}^m\, \sqrt{2}\, e^{-x}\, \mathcal{W}, \quad x \geq 0, \quad m \geq 0 \tag{2.34}$$

or

$$\mathcal{W}a^+_m = \mathcal{C}^m\, Lag_0(x)\mathcal{W} = Lag_m(x)\, \mathcal{W}, \quad x \geq 0, \quad m \geq 0, \tag{2.35}$$

where $Lag_m(x)$, $m \geq 0$, denote the Laguerre functions defined by

$$Lag_0(x) := \sqrt{2}\, e^{-x}\, L_0(2x) = \sqrt{2}\, e^{-x}, \quad x \geq 0, \tag{2.36}$$

$$Lag_m(x) := \mathcal{C}^m\, Lag_0(x), \quad x \geq 0, \quad m > 0 \tag{2.37}$$

$$= \sqrt{2}\, e^{-x}\, L_m(2x), \quad x \geq 0, \quad m > 0. \tag{2.38}$$

It then follows that $\mathcal{L}^2\big([0,\infty);\mathcal{W}\big)$ admits the wandering subspaces decomposition

$$\mathcal{L}^2\big([0,\infty);\mathcal{W}\big) = \bigoplus_{m=0}^{\infty} \mathcal{W}a^+_m = \bigoplus_{m=0}^{\infty} \mathcal{C}^m\, Lag_0(x)\, \mathcal{W}, \quad x \geq 0 \tag{2.39}$$

$$= \bigoplus_{m=0}^{\infty} Lag_m(x)\, \mathcal{W}, \quad x \geq 0. \tag{2.40}$$

Similarly, $\mathcal{L}^2\big((-\infty,0);\mathcal{W}\big)$ as a subspace of $\mathcal{L}^2(\mathbb{R};\mathcal{W})$ is invariant for the semigroup $\{\mathcal{T}(-t), t \geq 0\}$ which is the adjoint semigroup $\{\mathcal{T}^*(t), t \geq 0\}$, hence it is also invariant for $\mathcal{C}^*$, i.e., for $\mathcal{C}^{-1}$, as a unilateral shift on $\mathcal{L}^2\big((-\infty,0);\mathcal{W}\big)$.

We then find, as for the case of the space $\mathcal{L}^2\big([0,\infty);\mathcal{W}\big)$,

$$\mathcal{L}^2\big((-\infty,0);\mathcal{W}\big) = \bigoplus_{m=1}^{\infty} \mathcal{C}^{*\,m}\, Lag_0(-x)\, \mathcal{W}, \quad x < 0 \tag{2.41}$$

$$= \bigoplus_{m=-\infty}^{-1} \mathcal{C}^m\, Lag_0(-x)\, \mathcal{W}, \quad x < 0 \tag{2.42}$$

$$= \bigoplus_{m=-\infty}^{-1} Lag_m(-x)\, \mathcal{W}, \quad x < 0. \tag{2.43}$$

Combining (2.39) and (2.41), then (2.40) and (2.43), we arrive at the following result.

Theorem 3. *The space $\mathcal{L}^2\big(\mathbb{R};\mathcal{W}\big)$ admits the orthogonal representation*

$$\mathcal{L}^2\big(\mathbb{R};\mathcal{W}\big) = \bigoplus_{m=-\infty}^{-1} \mathcal{C}^m\, Lag_0(-x)\,\mathcal{W} \oplus \bigoplus_{m=0}^{\infty} \mathcal{C}^m\, Lag_0(x)\,\mathcal{W} \tag{2.44}$$

$$= \bigoplus_{m=-\infty}^{-1} Lag_m(-x)\,\mathcal{W} \oplus \bigoplus_{m=0}^{\infty} Lag_m(x)\,\mathcal{W}$$

$$= \bigoplus_{m\in\mathbb{Z}} Lag_m(\pm x)\,\mathcal{W}, \tag{2.45}$$

where

$$\begin{aligned} Lag_m(\pm x) &:= \mathcal{C}^m Lag_0(x), \quad x \geq 0, \quad m \geq 0 \\ &:= \mathcal{C}^m Lag_0(-x), \quad x < 0, \quad m < 0. \end{aligned}$$

To proceed let us consider the outgoing subspace $\mathcal{O}$ of (2.3) — with $C_{\mathcal{U}}$ instead of U,

$$\mathcal{O} = \bigoplus_{m=0}^{\infty} C_{\mathcal{U}}^m \mathcal{W} \tag{2.46}$$

and, by Theorem 2, let the unitary operator Ω_o be such that

$$\Omega_o\,\mathcal{O} = \mathcal{L}^2\big([0,\infty);\mathcal{W}\big). \tag{2.47}$$

Then by (2.46)

$$\Omega_o\,\mathcal{O} = \bigoplus_{m=0}^{\infty} \Omega_o\, C_{\mathcal{U}}^m\,\mathcal{W}. \tag{2.48}$$

This becomes, by (2.31) and (2.47),

$$\Omega_o\,\mathcal{O} = \bigoplus_{m=0}^{\infty} \mathcal{C}^m\,\Omega_o\,\mathcal{W} = \mathcal{L}^2\big([0),\infty);\mathcal{W}\big). \tag{2.49}$$

It then follows from this and from (2.39) that

$$\bigoplus_{m=0}^{\infty} \mathcal{C}^m\,\Omega_o\mathcal{W} = \bigoplus_{m=0}^{\infty} \mathcal{C}^m\, Lag_0(x)\,\mathcal{W}, \quad x \geq 0. \tag{2.50}$$

Therefore

$$\mathcal{C}^m\,\Omega_o\,\mathcal{W} = \mathcal{C}^m\, Lag_0(x)\,\mathcal{W} = Lag_m(x)\mathcal{W}, \quad x \geq 0, \quad m \geq 0. \tag{2.51}$$

This can also be rewritten as

$$\Omega_o\, C_{\mathcal{U}}^m\,\mathcal{W} = Lag_m(x)\,\mathcal{W}, \quad x \geq 0, \quad m \geq 0. \tag{2.52}$$

In exactly the same way we obtain

$$\Omega_o\, C_{\mathcal{U}}^{*\,m}\,\mathcal{W} = Lag_m(-x)\,\mathcal{W}, \quad x < 0, \quad m > 0. \tag{2.53}$$

We summarize the above results in the following Proposition.

Proposition 4. *The unitary operator Ω_o is characterized by the following equations.*

$$\Omega_o\, C_{\mathcal{U}}^{m}\, \mathcal{W} \;=\; Lag_m(x)\, \mathcal{W}, \quad x \geq 0, \quad m \geq 0. \tag{2.54}$$

$$\Omega_o\, C_{\mathcal{U}}^{*\,m}\, \mathcal{W} \;=\; Lag_m(-x)\, \mathcal{W}, \quad x < 0, \quad m > 0. \tag{2.55}$$

We now apply the above to the CMTA of Definition 4 — with the wavelet outgoing subspace $\mathcal{O} := \bigvee_{n\in\mathbb{Z}} \phi\big((\cdot)-n\big)$ and with $C_{\mathcal{U}} := D$.

First, we have $\mathcal{O}_{t+\tau} \subset \mathcal{O}_t$ for every $t, \tau > 0$. Therefore, we can write

$$\mathcal{O}_t = \mathcal{O}_{t+\tau} \;\oplus\; \widehat{\mathcal{O}}_t, \tag{2.56}$$

where $\widehat{\mathcal{O}}_t$ is the orthogonal complement of $\mathcal{O}_{t+\tau}$ in $\mathcal{O}_t$.

Next, by definition,

$$\mathcal{O}_t = \mathcal{U}(t)\mathcal{O} = \bigvee_{n\in\mathbb{Z}} \mathcal{U}(t)\, \phi\big((\cdot)-n\big), \quad t \geq 0. \tag{2.57}$$

Then by Sinai's Theorem

$$\Omega_o\, \mathcal{O} \;=\; \mathcal{L}^2\big([0,\infty);\mathcal{W}\big), \tag{2.58}$$

$$\Omega_o\, \mathcal{O}_t, \;=\; \Omega_o\, \mathcal{U}(t)\mathcal{O} = \mathcal{T}(t)\, \Omega_o \mathcal{O}, \quad t \geq 0, \tag{2.59}$$

$$= \; T(t)\, \mathcal{L}^2\big([0,\infty);\mathcal{W}\big), \quad \text{by (2.58)}, \tag{2.60}$$

$$= \; \mathcal{L}^2\big([t,\infty);\mathcal{W}\big), \quad t \geq 0. \tag{2.61}$$

That is

$$\mathcal{O}_t \;\leftrightarrow\; \mathcal{L}^2\big([t,\infty);\mathcal{W}\big), \quad t \geq 0. \tag{2.62}$$

Similarly

$$\mathcal{O}_{t+\tau} \quad\leftrightarrow\quad \mathcal{L}^2\big([t+\tau,\infty);\mathcal{W}\big), \quad t,\, \tau \geq 0. \tag{2.63}$$

Consequently

$$\widehat{\mathcal{O}}_t \;\leftrightarrow\; \mathcal{L}^2\big((t,t+\tau);\mathcal{W}\big), \quad t,\, \tau \geq 0. \tag{2.64}$$

To proceed, let $\mathcal{P}_t$ be the orthogonal projection from $\mathcal{L}^2(\mathbb{R})$ onto $\mathcal{O}_t$ then by (2.57)

$$\mathcal{P}_t\{f(\cdot)\} = \sum_{n\in\mathbb{Z}} \big\langle f(\cdot), \mathcal{U}(t)\phi\big((\cdot)-n\big)\big\rangle\, \mathcal{U}(t)\phi\big((\cdot)-n\big), \quad t \geq 0, \tag{2.65}$$

which is an *approximation to* $f(\cdot)$ *at* $t > 0$. Then, since $[\mathcal{U}(t)]$ is unitary (2.65) can be rewritten as

$$\mathcal{P}_t\{f(\cdot)\} \;=\; \sum_{n\in\mathbb{Z}} \big\langle \mathcal{U}(t)^* f(\cdot), \phi\big((\cdot)-n\big)\big\rangle\, \mathcal{U}(t)\phi\big((\cdot)-n\big), \quad t \geq 0 \tag{2.66}$$

$$= \; \mathcal{U}(t)\, \mathcal{P}_{\mathcal{O}}\, \{\mathcal{U}(t)^* f(\cdot)\}, \tag{2.67}$$

where $\mathcal{P}_{\mathcal{O}}$ denotes the projection onto the outgoing subspace $\mathcal{O}$ — which is also the projection $\mathcal{P}_{t=0}$.

We note that (2.67) is the continuous analogue of $\mathcal{P}_m\{f(\cdot)\}$ of (2.18).

Next, it follows from (2.66) that

$$\Omega_o \mathcal{P}_t\{f(\cdot)\} = \sum_{n\in\mathbb{Z}} \big\langle f(\cdot), \mathcal{U}(t)\phi\big((\cdot)-n\big)\big\rangle\, \Omega_o \mathcal{U}(t)\phi\big((\cdot)-n\big), \quad t\geq 0 \tag{2.68}$$

$$= \sum_{n\in\mathbb{Z}} \big\langle \Omega_o f(\cdot), \Omega_o \mathcal{U}(t)\phi\big((\cdot)-n\big)\big\rangle\, \Omega_o \mathcal{U}(t)\phi\big((\cdot)-n\big). \tag{2.69}$$

Therefore

$$\Omega_o \mathcal{P}_t = \mathcal{P}_{\Omega_o \mathcal{O}_t}\, \Omega_o, \quad t \geq 0, \tag{2.70}$$

This, by (2.61), means that

$$\Omega_o \mathcal{P}_t = \mathcal{P}_{\mathcal{L}^2([t,\infty);\mathcal{W})}\, \Omega_o, \quad t\geq 0, \tag{2.71}$$

that is,

$$\mathcal{P}_t \;\leftrightarrow\; \mathcal{P}_{\mathcal{L}^2([t,\infty);\mathcal{W})}. \tag{2.72}$$

We note that (2.69) can also be rewritten as

$$\Omega_o \mathcal{P}_t\{f(\cdot)\} = \sum_{n\in\mathbb{Z}} \big\langle \Omega_o f(\cdot), \Omega_o \mathcal{U}(t)\phi\big((\cdot)-n\big)\big\rangle\, \Omega_o \mathcal{U}(t)\phi\big((\cdot)-n\big) \tag{2.73}$$

$$= \sum_{n\in\mathbb{Z}} \big\langle \Omega_o f(\cdot), \mathcal{T}(t)\, \Omega_o\, \phi\big((\cdot)-n\big)\big\rangle\, \mathcal{T}(t)\Omega_o\, \phi\big((\cdot)-n\big) \tag{2.74}$$

$$= \mathcal{P}_{\mathcal{T}(t)\,\Omega_o \mathcal{O}}\, \{\Omega_o f(\cdot)\}, \tag{2.75}$$

which is (2.71) as expected.

In exactly the same way we obtain

$$\mathcal{P}_{t+\tau} \;\leftrightarrow\; \mathcal{P}_{\mathcal{L}^2([t+\tau,\infty);\mathcal{W})}, \quad t,\, \tau \geq 0, \tag{2.76}$$

and

$$\widehat{\mathcal{P}}_t \;\leftrightarrow\; \mathcal{P}_{\mathcal{L}^2((t,t+\tau);\mathcal{W})}, \quad t,\, \tau\geq 0, \tag{2.77}$$

where $\widehat{\mathcal{P}}_t$ is the projection onto $\widehat{\mathcal{O}}_t$.

We are now ready to state the following results.

Theorem 4. *Let $\mathcal{O}$ be the wavelet outgoing subspace with respect to a DMRA $\{D^m\mathcal{W}\}$ as well as with respect to its associated CMTA $\{\mathcal{O}_t\}$, then an approximation to $f(\cdot)\in\mathcal{L}^2(\mathbb{R})$ at any $t>0$ can be expressed as*

$$\mathcal{P}_t\{f(\cdot)\} = \mathcal{P}_{t+\tau}\{f(\cdot)\} + \widehat{\mathcal{P}}_t\{f(\cdot)\}, \quad t\geq 0, \tag{2.78}$$

where the first term on the right hand side represents (as in the discrete case) "coarse" details approximation while the second term is "finer" details approximation. Moreover, approximations in the signal space $\mathcal{L}^2(\mathbb{R})$ go into unitarily equivalent approximations in the Translation Representation Space $\mathcal{L}^2(\mathbb{R};\mathcal{W})$. In other words, for $f(\cdot)\in\mathcal{L}^2(\mathbb{R})$,

$$\mathcal{P}_{\mathcal{L}^2([t,\infty);\mathcal{W})}\{\Omega_o f(\cdot)\} = \mathcal{P}_{\mathcal{L}^2([t+\tau,\infty);\mathcal{W})}\{\Omega_o f(\cdot)\} + \widehat{\mathcal{P}}_{\mathcal{L}^2((t,t+\tau);\mathcal{W})}\{\Omega_o f(\cdot)\}, \tag{2.79}$$

where the unitary equivalence operator Ω_o is characterized by Theorem 2.

3. Concluding Remarks

Let $\{\mathcal{O}_m := D^m\mathcal{O},\ m \in \mathbb{Z}\}$ be the DMRA with the outgoing subspace $\mathcal{O}$ given by (2.13). Then we have the orthogonal decomposition, see for instance, [8, 15]:

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{O}_m \oplus \bigoplus_{\ell=m}^{\infty} \widehat{\mathcal{O}}_\ell, \quad m \in \mathbb{Z}, \tag{3.1}$$

where

$$\widehat{\mathcal{O}}_\ell := \mathcal{O}_\ell \ominus \mathcal{O}_{\ell+1}. \tag{3.2}$$

Moreover,

$$\mathcal{O}_m = \mathcal{O}_{m'} \oplus \bigoplus_{\ell=m'}^{m+1} \widehat{\mathcal{O}}_\ell, \quad \text{for} \quad m > m'. \tag{3.3}$$

From which it is easy to see that

$$\mathcal{P}_m\{f(\cdot)\} = \mathcal{P}_{m'}\{f(\cdot)\} + \sum_{\ell=m'}^{m+1} \widehat{\mathcal{P}}_\ell\{f(\cdot)\}, \quad \text{for} \quad m > m'. \tag{3.4}$$

This results in the well-known Discrete Wavelet Transform, see for instance [8], for approximating $f(\cdot)$ at resolution-2^m by the "coarse" details at resolution-$2^{m'}$ and the "finer" details from resolution-$2^{m'}$ to resolution-2^{m+1}.

Equation (2.78)) is the discrete $\mathcal{L}^2(\mathbb{R})$-analogue of (3.4), while (2.79) is the continuous $\mathcal{L}^2(\mathbb{R};\mathcal{W})$-analogue of (3.4).

The concept of Continuous Multi-Resolution Analysis (CMRA), as a generalization of the concept of DMRA, was first formulated by Antoniou and Gustafson [1], see also [5], and was motivated by the analogy to continuous-parameter regular stationary processes. Their formulation consists of a continuous chain of Hilbert subspaces satisfying properties "similar" to those of a DMRA, as well as some additional properties. Moreover, the continuous group of unitary operators $[\mathcal{U}(t)]$ is, in their formulation, a continuous shift which needs not be generated by the scale-by-2 operator D — as its cogenerator. We must note that [1] and [5] also discuss connections of Wavelet Theory to several areas of Mathematics and Mathematical Physics.

The concept of CMRA was also developed via the theory of Continuous Wavelet Transform in [3] and [20].

Finally, by tradition, the decomposition $\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} D^m\mathcal{W}$, where the wavelet generating wandering subspace $\mathcal{W}$ is defined in terms of a wavelet $\psi(\cdot)$ as $\mathcal{W} := \bigvee_{n\in\mathbb{Z}} \psi\big((\cdot) - n\big)$ is a decomposition at each resolution-2^m. In [13], see also [10], we show that the space $\mathcal{L}^2(\mathbb{R})$ can also be represented by the decomposition with respect to time-steps-n, i.e., it is of the form $\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$, where $\mathcal{H}_n := \bigvee_{m\in\mathbb{Z}} D^m\psi\big((\cdot) - n\big)$, $n \in \mathbb{Z}$. What is interesting about this time-steps decomposition is the fact that the subspaces $\mathcal{H}_n$ are D-reducing and the parts of D on $\mathcal{H}_n$, i.e., $D|\mathcal{H}_n$ are shifts of multiplicity 1. Therefore, Translation Representation can be obtained for $\mathcal{H}_n$, for an outgoing subspace defined on $\mathcal{H}_n$ and for the shift operators $D|\mathcal{H}_n$. These representations are "elementary" compared to those of Theorem 2. In other words they involve shifts of multiplicity 1 instead of shifts of nonfinite multiplicity.

We have not considered the role of the Lax-Phillips scattering operator for Wavelet Theory. For this we refer to the work of Jorgensen [7].

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