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CONVERGENCE AND DECOMPOSITION FOR TENSOR PRODUCTS OF HILBERT SPACE OPERATORS

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Abstract. It is shown that convergence of sequences of Hilbert space operators is preserved by tensor product and the converse holds in case of convergence to zero under the semigroup assumption. In particular, unlike ordinary product of operators, weak convergence is preserved by tensor product. It is also shown that a tensor product of operators is a unilateral shift if and only if it coincides with a tensor product of a unilateral shift and an isometry. These results lead to a decomposition of a tensor product of contractions into an orthogonal direct sum of tensor products of class $\mathcal{C}_{00}$, strongly stable tensor products, unilateral shift tensor products, and a unitary tensor product.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be nonzero complex Hilbert spaces. We shall consider the concept of tensor product space in terms of the single tensor product of vectors as a conjugate bilinear functional on the Cartesian product of $\mathcal{H}$ and $\mathcal{K}$. (See e.g., [4], [10] and [11] — for an abstract approach see e.g., [1] and [14].) The single tensor product of $x \in \mathcal{H}$ and $y \in \mathcal{K}$ is a conjugate bilinear functional $x \otimes y \colon \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ defined by $(x \otimes y)\,(u, v) = \langle x\,; u\rangle\,\langle y\,; v\rangle$ for every $(u, v) \in \mathcal{H} \times \mathcal{K}$. The collection of all (finite) sums of single tensors $x_i \otimes y_i$ with $x_i \in \mathcal{H}$ and $y_i \in \mathcal{K}$, denoted by $\mathcal{H} \otimes \mathcal{K}$, is a complex linear space equipped with an inner product $\langle\ ;\ \rangle \colon (\mathcal{H} \otimes \mathcal{K}) \times (\mathcal{H} \otimes \mathcal{K}) \to \mathbb{C}$ defined, for arbitrary $\sum_{i=1}^{N} x_i \otimes y_i$ and $\sum_{j=1}^{M} w_j \otimes z_j$ in $\mathcal{H} \otimes \mathcal{K}$, by

$$\Big\langle \sum_{i=1}^{N} x_i \otimes y_i\,; \sum_{j=1}^{M} w_j \otimes z_j \Big\rangle = \sum_{i=1}^{N}\sum_{j=1}^{M} \langle x_i\,; w_j\rangle\,\langle y_i\,; z_j\rangle$$

(the same notation for the inner products on $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{H} \otimes \mathcal{K}$). By an operator we mean a bounded linear transformation of a normed space into itself. Let $\mathcal{B}[\mathcal{H}]$, $\mathcal{B}[\mathcal{K}]$ and $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ be the normed algebras of all operators on $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{H} \otimes \mathcal{K}$. The tensor product on $\mathcal{H} \otimes \mathcal{K}$ of two operators T in $\mathcal{B}[\mathcal{H}]$ and S in $\mathcal{B}[\mathcal{K}]$ is the operator $T \otimes S \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ defined by

$$(T \otimes S) \sum_{i=1}^{N} x_i \otimes y_i = \sum_{i=1}^{N} Tx_i \otimes Sy_i \quad \text{for every} \quad \sum_{i=1}^{N} x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{K},$$

which lies in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. The completion of the inner product space $\mathcal{H} \otimes \mathcal{K}$, denoted by $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$, is the tensor product space of $\mathcal{H}$ and $\mathcal{K}$. The extension of $T \otimes S$ over the Hilbert space $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$, denoted by $T \,\widehat{\otimes}\, S$, is the tensor product of T and S on the tensor product space, which lies in $\mathcal{B}[\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}]$. For an expository paper containing the essential properties of tensor products needed here, the reader is referred to [6].

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It is exhibited in Theorem 4 a decomposition of a tensor product contraction $T \mathbin{\widehat{\otimes}} S$ into an orthogonal direct sum of tensor products of class $\mathcal{C}_{00}$, strongly stable tensor products, unilateral shift tensor products, and a unitary tensor product. This is done after showing in Theorem 1 that weak, strong and uniform convergences are preserved by tensor product. The case of convergence to zero is considered in Theorem 2, and the converse is investigated in Theorem 3 under the semigroup assumption (i.e., for power sequences). The above mentioned decomposition is also based on Lemma 1, which ensures that a tensor product is a unilateral shift if and only if it coincides with a tensor product of a unilateral shift and an isometry.

2. Convergence

A sequence $\{T_n\}$ of operators in $\mathcal{B}[\mathcal{H}]$ converges uniformly, or strongly, or weakly to an operator T in $\mathcal{B}[\mathcal{H}]$ if $\|T_n - T\| \to 0$, or $\|(T_n - T)x\| \to 0$ for every x in $\mathcal{H}$, or $\langle T_n x\,; y\rangle \to 0$ for every x and y in $\mathcal{H}$ (equivalently, $\langle T_n x\,; x\rangle \to 0$ for every x in the complex Hilbert space $\mathcal{H}$), and these will be denoted by $T_n \xrightarrow{u} T$, or $T_n \xrightarrow{s} T$, or $T_n \xrightarrow{w} T$, respectively. It is bounded if $\sup_n \|T_n\| < \infty$. Clearly,

$$T_n \xrightarrow{u} T \quad\Longrightarrow\quad T_n \xrightarrow{s} T \quad\Longrightarrow\quad T_n \xrightarrow{w} T \quad\Longrightarrow\quad \sup_n \|T_n\| < \infty.$$

Theorem 1. *Let $\{T_n\}$ be a sequence of operators in $\mathcal{B}[\mathcal{H}]$ and let $\{S_n\}$ be a sequence of operators in $\mathcal{B}[\mathcal{K}]$. Let T and S be operators in $\mathcal{B}[\mathcal{H}]$ and in $\mathcal{B}[\mathcal{K}]$.*

(a) *If $T_n \xrightarrow{u} T$ and $S_n \xrightarrow{u} S$, then $T_n \mathbin{\widehat{\otimes}} S_n \xrightarrow{u} T \mathbin{\widehat{\otimes}} S$.*
(b) *If $T_n \xrightarrow{s} T$ and $S_n \xrightarrow{s} S$, then $T_n \mathbin{\widehat{\otimes}} S_n \xrightarrow{s} T \mathbin{\widehat{\otimes}} S$.*
(c) *If $T_n \xrightarrow{w} T$ and $S_n \xrightarrow{w} S$, then $T_n \mathbin{\widehat{\otimes}} S_n \xrightarrow{w} T \mathbin{\widehat{\otimes}} S$.*

Proof. Recall that $T_n \otimes S_n - T \otimes S = T_n \otimes (S_n - S) + (T_n - T) \otimes S$ for each n, which still holds if $\otimes$ is replaced with $\widehat{\otimes}$ (see e.g., [6, Propositions 2(b_1,b_2) 4(b_1,b_2)]).

(a) If $\|T_n - T\| \to O$ (so that $\{T_n\}$ is bounded) and $\|S_n - S\| \to O$, then

$$\|T_n \mathbin{\widehat{\otimes}} S_n - T \mathbin{\widehat{\otimes}} S\| \le \sup_n \|T_n\|\,\|S_n - S\| + \|S\|\,\|T_n - T\|,$$

and hence $\|T_n \mathbin{\widehat{\otimes}} S_n - T \mathbin{\widehat{\otimes}} S\| \to 0$. That is, $T_n \mathbin{\widehat{\otimes}} S_n \xrightarrow{u} T \mathbin{\widehat{\otimes}} S$.

(b) Take an arbitrary $\sum_{i=1}^N x_i \otimes y_i$ in $\mathcal{H} \otimes \mathcal{K}$ and observe that

$$\begin{aligned}&\Big\|(T_n \otimes S_n - T \otimes S)\sum_{i=1}^N x_i \otimes y_i\Big\| \\ &\le \sup_n \|T_n\| \sum_{i=1}^N \|x_i\| \sum_{i=1}^N \|(S_n - S)y_i\| + \|S\| \sum_{i=1}^N \|y_i\| \sum_{i=1}^N \|(T_n - T)x_i\|.\end{aligned}$$

If $S_n \xrightarrow{s} S$ and $T_n \xrightarrow{s} T$, then $\|(T_n \otimes S_n - T \otimes S)\sum_{i=1}^N x_i \otimes y_i\| \to 0$, and so $T_n \otimes S_n \xrightarrow{s} T \otimes S$. Moreover, $\{T_n \mathbin{\widehat{\otimes}} S_n\}$ is bounded (because $\sup_n \|T_n \mathbin{\widehat{\otimes}} S_n\| \le \sup_n \|T_n\| \sup_n \|S_n\| < \infty$). As it is well-known, if a sequence of operators converges strongly in a normed space, and if its extension is bounded in the completion, then convergence holds in the completion of the space. Thus $T_n \mathbin{\widehat{\otimes}} S_n \xrightarrow{s} T \mathbin{\widehat{\otimes}} S$.

(c) Similarly, and applying the Schwarz inequality,

$$\begin{aligned}
&\Big|\Big\langle (T_n\otimes S_n-T\otimes S)\sum_{i=1}^N x_i\otimes y_i\,;\sum_{i=1}^N x_i\otimes y_i\Big\rangle\Big|\\
&\le\ \sup_n\|T_n\|\sum_{i=1}^N\sum_{j=1}^N\|x_i\|\,\|x_j\|\sum_{i=1}^N\sum_{j=1}^N|\langle (S_n-S)y_i\,;y_j\rangle|\\
&\qquad\qquad +\|S\|\sum_{i=1}^N\sum_{j=1}^N\|y_i\|\,\|y_j\|\sum_{i=1}^N\sum_{j=1}^N|\langle (T_n-T)x_i\,;x_j\rangle|.
\end{aligned}$$

Thus $|\langle (T_n\otimes S_n-T\otimes S)\sum_{i=1}^N x_i\otimes y_i\,;\sum_{i=1}^N x_i\otimes y_i\rangle|\ \to\ 0$, whenever $S_n\xrightarrow{w}S$ and $T_n\xrightarrow{w}T$, and so $T_n\otimes S_n\xrightarrow{w}T\otimes S$. The same argument applies for weak convergence so that $T_n\,\widehat{\otimes}\,S_n\xrightarrow{w}T\,\widehat{\otimes}\,S$. □

Remark 1. The result in Theorem 1(c) does not mirror the ordinary product counterpart. Indeed, $T_n\xrightarrow{w}T$ and $S_n\xrightarrow{w}S$ do not imply $T_nS_n\xrightarrow{w}TS$ (in fact, even $T_n\xrightarrow{s}T$ and $S_n\xrightarrow{w}S$ do not imply $T_nS_n\xrightarrow{w}TS$). Sample: if V is a unilateral shift, put $T_n^*=S_n=V^n$ so that $T_n\xrightarrow{s}O$, $S_n\xrightarrow{w}O$, but $T_nS_n=I$ for every n.

Theorem 2. *Let $\{T_n\}$ and $\{S_n\}$ be sequences of operators in $\mathcal{B}[\mathcal{H}]$ and $\mathcal{B}[\mathcal{K}]$, respectively. If one of them converges to zero uniformly (strongly, weakly) and the other is bounded, then $\{T_n\,\widehat{\otimes}\,S_n\}$ converges to zero uniformly (strongly, weakly).*

Proof. If $\|T_n\|\to 0$ and $\sup_n\|S_n\|<\infty$ (or vice versa), then $\|T_n\,\widehat{\otimes}\,S_n\|\to 0$ because $\|T_n\,\widehat{\otimes}\,S_n\|=\|T_n\otimes S_n\|=\|T_n\|\,\|S_n\|$ for every $n\ge 1$, which proves the claimed result for uniform convergence. For strong and weak convergences take an arbitrary vector $\sum_{i=1}^N x_i\otimes y_i$ in $\mathcal{H}\otimes\mathcal{K}$. Note that

$$\Big\|(T_n\otimes S_n)\sum_{i=1}^N x_i\otimes y_i\Big\|\le\sup_n\|S_n\|\sum_{i=1}^N\|T_nx_i\|\sum_{i=1}^N\|y_i\|.$$

If $\{T_n\}$ converges strongly to zero and if $\{S_n\}$ is bounded (or vice versa), then $\big\|(T_n\otimes S_n)\sum_{i=1}^N x_i\otimes y_i\big\|\to 0$. Applying the same argument in the proof of Theorem 1(b) we get $T_n\,\widehat{\otimes}\,S_n\xrightarrow{s}O$. Similarly,

$$\Big|\Big\langle (T_n\otimes S_n)\sum_{i=1}^N x_i\otimes y_i\,;\sum_{i=1}^N x_i\otimes y_i\Big\rangle\Big|\le\sup_n\|S_n\|\sum_{i=1}^N\sum_{j=1}^N|\langle T_nx_i\,;x_j\rangle|\sum_{i=1}^N\sum_{j=1}^N\|y_i\|\,\|y_j\|.$$

If $\{T_n\}$ converges weakly to zero and if $\{S_n\}$ is bounded (or vice versa), then $\big\langle (T_n\otimes S_n)\sum_{i=1}^N x_i\otimes y_i\ ;\ \sum_{i=1}^N x_i\otimes y_i\big\rangle\to 0$. Again, applying the same argument in the proof of Theorem 1(c), it follows that $T_n\,\widehat{\otimes}\,S_n\xrightarrow{w}O$. □

Remark 2. A tensor product sequence $\{T_n\,\widehat{\otimes}\,S_n\}$ may converge in every topology and both sequences $\{T_n\}$ and $\{S_n\}$ may not converge in any topology (actually, both sequences may not even be bounded). For instance, put $T_n=nI$ if n is odd and $T_n=O$ if n is even, and $S_n=O$ if n is odd and $S_n=nI$ if n is even, so that $\|T_n\,\widehat{\otimes}\,S_n\|=\|T_n\|\|S_n\|=0$. However, in general, we always have

$$(*)\qquad \inf_n\|T_n\|\sup_n\|S_n\|\le\sup_n\big(\|T_n\|\|S_n\|\big)=\sup_n\|T_n\,\widehat{\otimes}\,S_n\|$$

(if we declare that $0 \cdot \infty = 0$). Theorem 3 below shows that, unlike the above example, convergence to zero of power sequences (or, equivalently, of sequences having the semigroup property) is transferred from the tensor product to one of the factors. First we consider the following auxiliary result.

Proposition 1. *If the power sequence $\{T^n \widehat{\otimes} S^n\}$ is bounded, then so is one of the power sequences $\{T^n\}$ or $\{S^n\}$.*

Proof. Since $(T \widehat{\otimes} S)^n = T^n \widehat{\otimes} S^n$ for every $n \geq 0$, the above statement says that *if $T \widehat{\otimes} S$ is power bounded, then so is one of T or S.* Indeed, suppose $T \widehat{\otimes} S$ is power bounded so that $\inf_n \|T^n\| \sup_n \|S^n\| < \infty$ by $(*)$. If one of T or S, say T, is not power bounded, then $\inf_n \|T^n\| \geq 1$ (since $\inf_n \|T^n\| < 1$ implies $\|T^n\| \to 0$; cf. (a) in the proof of Theorem 3 below). Hence $\sup_n \|S^n\| \leq \inf_n \|T^n\| \sup_n \|S^n\|$ and so S is power bounded. Similarly, if S is not power bounded, then T must be. $\square$

Let $\{n\}_{n\geq 0}$ denote the self indexing of the set of all nonnegative integers $\mathbb{N}_0$ equipped with the natural order. We say that a subsequence $\{n_k\}_{k\geq 0}$ of $\{n\}_{n\geq 0}$ is of *bounded increments* if $\sup_{k\geq 0}(n_{k+1} - n_k) < \infty$, and that a Hilbert space operator T is *power incremented* if either the power sequence $\{T^n\}$ converges weakly to zero or there exists a subsequence of bounded increments $\{n_k\}_{k\geq 0}$ of $\{n\}_{n\geq 0}$ such that $\limsup_k |\langle T^{n_k} x\,; y\rangle| > 0$ whenever $\langle T^n x\,; y\rangle \not\to 0$ for some pair of vectors x and y.

Theorem 3. *Let T be an operator in $\mathcal{B}[\mathcal{H}]$ and let S be an operator in $\mathcal{B}[\mathcal{K}]$. Consider the power sequences $\{T^n\}$ and $\{S^n\}$. If $\{T^n \widehat{\otimes} S^n\}$ converges to zero uniformly or strongly, then so does one of the sequences $\{T^n\}$ or $\{S^n\}$. If $\{T^n \widehat{\otimes} S^n\}$ converges to zero weakly, and one of T or S is power incremented, then one of the sequences $\{T^n\}$ or $\{S^n\}$ converges to zero weakly.*

Proof. First recall that $(T \widehat{\otimes} S)^n = T^n \widehat{\otimes} S^n$ for each nonnegative integer n.

Part 1: *Uniform Convergence.*

(a) $$\text{If } \inf_n \|T^n\| < 1, \text{ then } \|T^n\| \to 0.$$

Indeed, if $\inf_n \|T^n\| < 1$, then there is a positive integer n_0 such that $\|T^{n_0}\| < 1$. Thus (with $r(T)$ denoting the spectral radius of any operator T in $\mathcal{B}[\mathcal{H}]$),

$$r(T)^{n_0} = r(T^{n_0}) \leq \|T^{n_0}\| < 1 \quad \Longrightarrow \quad r(T) < 1 \quad \Longleftrightarrow \quad \|T^n\| \to 0.$$

Now suppose $\|T^n \widehat{\otimes} S^n\| \to 0$ and recall that $\|T^n \widehat{\otimes} S^n\| = \|T^n\|\|S^n\|$ for every n.

(b) $$\text{If } \inf_n \|T^n\| > 0, \text{ then } \|S^n\| \to 0.$$

In fact, $\inf_n \|T^n\| > 0$ if and only if $\liminf_n \|T^n\| > 0$ (reason: $\|T^{n+1}\| \leq \|T\|\|T^n\|$). Since $\|T^n\|\|S^n\| \to 0$, it then follows that if $\liminf_n \|T^n\| > 0$, then $\|S^n\| \to 0$. From (a) and (b) we get the claimed result for uniform convergence.

Part 2 : *Strong Convergence.* If the sequence $\{T^n \widehat{\otimes} S^n\}$ converges strongly, then it is bounded (i.e., $T \widehat{\otimes} S$ is power bounded, since $(T \widehat{\otimes} S)^n = T^n \widehat{\otimes} S^n$), and so is one of $\{T^n\}$ and $\{S^n\}$ (i.e., one of T or S is power bounded) by Proposition 1. Thus, with no loss of generality, suppose T is power bounded: $\sup_n \|T^n\| < \infty$.

(c) $$\text{If } \liminf_n \|T^n x\| = 0 \text{ for every } x \in \mathcal{H}, \text{ then } \|T^n x\| \to 0 \text{ for every } x \in \mathcal{H}.$$

Indeed, take any $x \in \mathcal{H}$. If $\liminf_n \|T^n x\| = 0$, then there exists a subsequence $\{\|T^{n_k}x\|\}$ of $\{\|T^n x\|\}$ such that $\lim_k \|T^{n_k}x\| \to 0$. Since $\sup_n \|T^n\| < \infty$, and since $T^{m+n} = T^m T^n$ for every $m \geq 0$ and $n \geq 0$, this ensures that $\|T^n x\| \to 0$. Actually,

$$\|T^n x\| \leq \|T^{n-n_k}\| \, \|T^{n_k}x\| \leq \sup_n \|T^n\| \|T^{n_k}x\| \quad \text{whenever} \quad n \geq n_k.$$

Now suppose $\{T^n \,\widehat{\otimes}\, S^n\}$ converges strongly to zero. Since for each integer $n \geq 0$

$$\|(T \,\widehat{\otimes}\, S)^n x \otimes u\| = \|T^n x \otimes S^n u\| = \|T^n x\| \|S^n u\|$$

for every $x \in \mathcal{H}$ and $u \in \mathcal{K}$, it follows that $\|T^n x\| \|S^n u\| \to 0$ for every $x \in \mathcal{H}$ and $u \in \mathcal{K}$, which ensures the next assertion.

(d) If $\liminf_n \|T^n x\| > 0$ for some $x \in \mathcal{H}$, then $\|S^n u\| \to 0$ for every $u \in \mathcal{K}$.

From (c) and (d) we get the claimed result for strong convergence.

Part 3 : *Weak Convergence.* Take an arbitrary x in $\mathcal{H}$.

(e) If there exists a subsequence of bounded increments $\{n_k\}_{k\geq 0}$ of $\{n\}_{n\geq 0}$ such that $|\langle T^{n_k}x\,; y\rangle| \to 0$ for every y in $\mathcal{H}$, then $|\langle T^n x\,; x\rangle| \to 0$.

Indeed, take any $x \in \mathcal{H}$. Let $\{n_k\}_{k\geq 0}$ be a subsequence of $\{n\}_{n\geq 0}$, of bounded increments, such that

$$|\langle T^{n_k}x\,; y\rangle| \to 0 \;\text{ as }\; k \to \infty \;\text{ for every }\; y \in \mathcal{H}.$$

Since $T^{m+n} = T^m T^n$ for every $m \geq 0$ and $n \geq 0$, it follows that, for each $j \geq 0$,

$$|\langle T^{n_k+j}x\,; x\rangle| = |\langle T^{n_k}x\,; T^{*j}x\rangle| \to 0 \;\text{ as }\; k \to \infty.$$

However, *If $\{\alpha_n\}_{n\geq 0}$ is a sequence of nonnegative numbers, and if there exists a subsequence of bounded increments $\{n_k\}_{k\geq 0}$ of $\{n\}_{n\geq 0}$ such that $\alpha_{n_k+j} \to 0$ as $k \to \infty$ for every $j \geq 0$, then $\alpha_n \to 0$ as $n \to \infty$.* Thus

$$|\langle T^n x\,; x\rangle| \to 0 \;\text{ as }\; n \to \infty.$$

Now suppose $\{T^n \,\widehat{\otimes}\, S^n\}$ converges weakly to zero and that one of T or S, say T (with no loss of generality), is power incremented. Since, for each integer $n \geq 0$,

$$|\langle (T \,\widehat{\otimes}\, S)^n x \otimes u\,; y \otimes v\rangle| = |\langle T^n \,\widehat{\otimes}\, S^n x \otimes u\,; y \otimes v\rangle| = |\langle T^n x\,; y\rangle| |\langle S^n u\,; v\rangle|$$

for every $x, y \in \mathcal{H}$ and $u, v \in \mathcal{K}$, it follows that $|\langle T^n x\,; y\rangle||\langle S^n u\,; v\rangle| \to 0$ for every x, y in $\mathcal{H}$ and u, v in $\mathcal{K}$.

(f) If for every subsequence of bounded increments $\{n_k\}_{k\geq 0}$ of $\{n\}_{n\geq 0}$ there exists y in $\mathcal{H}$ such that $|\langle T^{n_k}x\,; y\rangle| \not\to 0$, then $|\langle S^n u\,; u\rangle| \to 0$ for every u in $\mathcal{K}$.

Indeed, if the hypothesis in (f) holds, then it holds, in particular, for the whole sequence $\{n\}_{n\geq 0}$ so that $|\langle T^n x\,; y\rangle| \not\to 0$ for some y in $\mathcal{H}$. Hence there exists a subsequence $\{n_k\}_{k\geq 0}$ of $\{n\}_{n\geq 0}$ such that $\liminf_k |\langle T^{n_k}x\,; y\rangle| > 0$. If T is power incremented, then we may assume that $\{n_k\}_{k\geq 0}$ is of bounded increments. Since $|\langle T^n x\,; y\rangle||\langle S^n u\,; v\rangle| \to 0$, it follows that $|\langle T^{n_k}x\,; y\rangle||\langle S^{n_k}u\,; v\rangle| \to 0$, and therefore $|\langle S^{n_k}u\,; v\rangle| \to 0$ for every u, v in $\mathcal{K}$. Thus take an arbitrary u in $\mathcal{K}$. Since $\{n_k\}_{k\geq 0}$ is a subsequence of bounded increments of $\{n\}_{n\geq 0}$ such that $|\langle S^{n_k}u\,; v\rangle| \to 0$ for every v in $\mathcal{K}$, it follows by (e) that

$$|\langle S^n u\,;u\rangle| \to 0 \;\text{ as }\; n\to\infty.$$

From (e) and (f) we get the claimed result for weak convergence. $\square$

An operator T is uniformly, strongly or weakly stable if the power sequence $\{T^n\}$ converges uniformly, strongly or weakly to zero. Preservation of uniform stability can be viewed as a consequence of the spectrum formula $\sigma(T\,\widehat{\otimes}\,S)=\sigma(T)\cdot\sigma(S)$ for a pair operators [2]. Preservation of strong stability as in [3, Theorem 1] and [7, Proposition 1] is a particular case of Theorems 2 and 3. It still remains open whether weak stability in Theorem 3 holds without the power increment assumption.

3. Decompositions

Let $T^*\in\mathcal{B}[\mathcal{H}]$ denote the adjoint of $T\in\mathcal{B}[\mathcal{H}]$. A contraction is an operator T such that $\|T\|\le 1$. If T is a contraction, then the sequence $\{T^{*n}T^n\}$ converges strongly. Let $A_T\in\mathcal{B}[\mathcal{H}]$ be the strong limit of $\{T^{*n}T^n\}$. The following basic properties of A_T will be required in the sequel (see [5, Chapter 3]): $O\le A_T\le I$ (i.e., A_T is a nonnegative contraction) and $\|A_T\|=1$ if $A_T\ne O$. Moreover, if $A_T=A_{T^*}$, then A_T is a projection, (i.e., $A_T=A_T^2$). Furthermore, a contraction T is strongly stable if and only if $A_T=O$. According to [13] a $\mathcal{C}_{0\cdot}$-contraction (or a contraction of class $\mathcal{C}_{0\cdot}$) is a strongly stable contraction (i.e., a contraction T with $A_T=O$), and a $\mathcal{C}_{00}$-contraction (or a contraction of class $\mathcal{C}_{00}$) is a strongly stable contraction whose adjoint also is strongly stable (i.e., a contraction T with $A_T=A_{T^*}=O$).

Corollary 1. *If T and S are Hilbert space contractions, then*

(a) $$A_{T\,\widehat{\otimes}\,S}=A_T\,\widehat{\otimes}\,A_S.$$

If both A_T and A_S are nonzero, then

(b) $$A_{T\,\widehat{\otimes}\,S}=A_{(T\,\widehat{\otimes}\,S)^*}\quad\text{implies}\quad A_T=A_{T^*}\;\text{ and }\;A_S=A_{S^*},$$

(c) $$A_{T\,\widehat{\otimes}\,S}=A^2_{T\,\widehat{\otimes}\,S}\quad\text{implies}\quad A_T=A_T^2\;\text{ and }\;A_S=A_S^2.$$

Proof. (a) Since $T\,\widehat{\otimes}\,S$ is a contraction in $\mathcal{B}[\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}]$ whenever T and S are contractions in $\mathcal{B}[\mathcal{H}]$ and $\mathcal{B}[\mathcal{K}]$, it follows that the sequence $\{(T\,\widehat{\otimes}\,S)^{*n}(T\,\widehat{\otimes}\,S)^n\}$ converges strongly to a nonnegative contraction $A_{T\,\widehat{\otimes}\,S}$ in $\mathcal{B}[\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}]$. Since

$$T^{*n}T^n\xrightarrow{\;s\;}A_T,\qquad S^{*n}S^n\xrightarrow{\;s\;}A_S,$$

and

$$(T\,\widehat{\otimes}\,S)^{*n}(T\,\widehat{\otimes}\,S)^n=(T^{*n}\,\widehat{\otimes}\,S^{*n})(T^n\,\widehat{\otimes}\,S^n)=T^{*n}T^n\,\widehat{\otimes}\,S^{*n}S^n,$$

it follows by Theorem 1 that

$$(T\,\widehat{\otimes}\,S)^{*n}(T\,\widehat{\otimes}\,S)^n\xrightarrow{\;s\;}A_T\,\widehat{\otimes}\,A_S,$$

and so $A_{T\,\widehat{\otimes}\,S}=A_T\,\widehat{\otimes}\,A_S$ by uniqueness of the strong limit.

(b) Since $(T\,\widehat{\otimes}\,S)^*=T^*\,\widehat{\otimes}\,S^*$ is a contraction, the sequence $\{(T\,\widehat{\otimes}\,S)^n(T\,\widehat{\otimes}\,S)^{*n}\}$ converges strongly to $A_{(T\,\widehat{\otimes}\,S)^*}=A_{T^*\,\widehat{\otimes}\,S^*}=A_{T^*}\,\widehat{\otimes}\,A_{S^*}$ by (a). If $A_{T\,\widehat{\otimes}\,S}=A_{(T\,\widehat{\otimes}\,S)^*}$, then $A_T\,\widehat{\otimes}\,A_S=A_{T^*}\,\widehat{\otimes}\,A_{S^*}$ by (a). If both A_T and A_S are nonzero, then $A_T=\alpha A_{T^*}$ and $A_S=\alpha^{-1}A_{S^*}$ for some nonzero scalar α [12, Proposition 2.1], and $\|A_T\|=\|A_{T^*}\|=1$. Thus $|\alpha|=1$. Since $A_T\ge O$, it follows that $\alpha>0$, and so $\alpha=1$. Thus $A_T=A_{T^*}$ and $A_S=A_{S^*}$.

(c) If $A_{T \,\widehat{\otimes}\, S} = A^2_{T \,\widehat{\otimes}\, S}$, then $A_T \,\widehat{\otimes}\, A_S = (A_T \,\widehat{\otimes}\, A_S)^2 = A^2_T \,\widehat{\otimes}\, A^2_S$ by (a), which implies that $A_T = \alpha A^2_T$ and $A_S = \alpha^{-1} A^2_S$ for some nonzero scalar α [12, Proposition 2.1], whenever both A_T and A_S are nonzero. In this case, $\|A_T\| = \|A_S\| = 1$ so that $\|A^2_T\| \le 1$, $\|A^2_S\| \le 1$, and hence $\alpha = 1$ because A_T and A_S are nonnegative. Thus $A_T = A^2_T$ and $A_S = A^2_S$. □

Remark 3. Consider the assertions in Corollary 1. According to assertion (a) $A_{T \,\widehat{\otimes}\, S} = O$ if and only if either $A_T = O$ or $A_S = O$. The converse to assertions (b) and (c) hold trivially by (a) — since $(A_T \,\widehat{\otimes}\, A_S)^2 = A^2_T \,\widehat{\otimes}\, A^2_S$. The implications in (b) and (c) do not hold if one of A_T or A_S is zero.

Lemma 1. *A tensor product $T \,\widehat{\otimes}\, S$ in $\mathcal{B}[\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}]$ is a unilateral shift if and only if $T \,\widehat{\otimes}\, S = J \,\widehat{\otimes}\, V$ where J is an isometry in $\mathcal{B}[\mathcal{H}]$ and V is a unilateral shift in $\mathcal{B}[\mathcal{K}]$, or $T \,\widehat{\otimes}\, S = V \,\widehat{\otimes}\, J$ where V is a unilateral shift in $\mathcal{B}[\mathcal{H}]$ and J is an isometry in $\mathcal{B}[\mathcal{K}]$.*

Proof. A unilateral shift is precisely an isometry whose adjoint is strongly stable (cf. [5, Lemma 6.1]). Thus $T \,\widehat{\otimes}\, S$ is a unilateral shift if and only if

$$T \,\widehat{\otimes}\, S \;\text{ is an isometry} \quad \text{and} \quad (T \,\widehat{\otimes}\, S)^{*n} \xrightarrow{s} O.$$

But, for any pair of nonzero operators T and S,

$$T \,\widehat{\otimes}\, S \;\text{ is an isometry} \quad \Longleftrightarrow \quad T \,\widehat{\otimes}\, S = J_1 \,\widehat{\otimes}\, J_2,$$

where, J_1 and J_2 are isometries in $\mathcal{B}[\mathcal{H}]$ and $\mathcal{B}[\mathcal{K}]$ (see e.g., [8, Lemma 4(b)]), and

$$(J_1 \,\widehat{\otimes}\, J_2)^{*n} \xrightarrow{s} O \quad \Longleftrightarrow \quad J_1^{*n} \xrightarrow{s} O \;\text{ or }\; J_2^{*n} \xrightarrow{s} O$$

(Theorems 2 and 3). Thus $T \,\widehat{\otimes}\, S$ is a unilateral shift if and only if $T \,\widehat{\otimes}\, S = J_1 \,\widehat{\otimes}\, J_2$, where J_1 and J_2 are isometries and one of them has a strongly stable adjoint. Equivalently, J_1 and J_2 are isometries and one of them is a unilateral shift. □

The closing result exhibits a decomposition of a tensor product of contractions for which the strong limit $A_{T \,\widehat{\otimes}\, S}$ is a projection. Here $\oplus$ stands for orthogonal direct sum, and $\cong$ stands for unitary equivalence.

Theorem 4. *Let T and S be contractions on $\mathcal{H}$ and $\mathcal{K}$ and consider the tensor product $T \,\widehat{\otimes}\, S$ on $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$.*

(a) $$\text{If} \quad A_{T \,\widehat{\otimes}\, S} = A^2_{T \,\widehat{\otimes}\, S}, \quad \text{then} \quad T \,\widehat{\otimes}\, S \cong B \oplus G \oplus V \oplus U,$$

where B is a $\mathcal{C}_{00}$-contraction, G is a $\mathcal{C}_{0\cdot}$-contraction (i.e., *a strongly stable contraction*), *V is a unilateral shift, and U is a unitary operator.*

(b) $$\text{If} \quad A_{T \,\widehat{\otimes}\, S} = A_{(T \,\widehat{\otimes}\, S)^*}, \quad \text{then} \quad T \,\widehat{\otimes}\, S \cong B \oplus U.$$

Proof. Suppose T and S are contractions. If one of A_T or A_S is zero, then $A_{T \,\widehat{\otimes}\, S}$ is zero, which means that $T \,\widehat{\otimes}\, S$ is strongly stable, and so (a) holds with $T \,\widehat{\otimes}\, S = B \oplus G$ and (b) holds with $T \,\widehat{\otimes}\, S = B$. Thus suppose both A_T and A_S are nonzero.

(a) If $A_{T \,\widehat{\otimes}\, S} = A^2_{T \,\widehat{\otimes}\, S}$, then $A_T = A^2_T$ and $A_S = A^2_S$ by Corollary 1. Moreover,

$$\begin{aligned} A_T = A^2_T \quad &\Longrightarrow \quad T = G_T \oplus V_T \oplus U_T, \\ A_S = A^2_S \quad &\Longrightarrow \quad S = G_S \oplus V_S \oplus U_S, \end{aligned}$$

where G_T and G_S are $\mathcal{C}_{0\cdot}$-contractions, V_T and V_S are unilateral shifts, and U_T and U_S are unitary operators [9, Theorem 1]. Therefore (see [6, Eq. (14),(16)]),

$$\begin{aligned} T\,\widehat{\otimes}\,S \cong\; & (G_T\,\widehat{\otimes}\,G_S)\oplus(G_T\,\widehat{\otimes}\,V_S)\oplus(G_T\,\widehat{\otimes}\,U_S)\\ \oplus\; & (V_T\,\widehat{\otimes}\,G_S)\oplus(V_T\,\widehat{\otimes}\,V_S)\oplus(V_T\,\widehat{\otimes}\,U_S)\\ \oplus\; & (U_T\,\widehat{\otimes}\,G_S)\oplus(U_T\,\widehat{\otimes}\,V_S)\oplus(U_T\,\widehat{\otimes}\,U_S), \end{aligned}$$

which yields the decomposition in (a) with

$$B=(V_T\,\widehat{\otimes}\,G_S)\oplus(G_T\,\widehat{\otimes}\,V_S),$$
$$G=(G_T\,\widehat{\otimes}\,G_S)\oplus(G_T\,\widehat{\otimes}\,U_S)\oplus(U_T\,\widehat{\otimes}\,G_S),$$
$$V=(V_T\,\widehat{\otimes}\,V_S)\oplus(V_T\,\widehat{\otimes}\,U_S)\oplus(U_T\,\widehat{\otimes}\,V_S)\quad\text{and}\quad U=U_T\,\widehat{\otimes}\,U_S.$$

Since G_T and G_S are strongly stable, it follows by Theorem 2 that both $V_T\,\widehat{\otimes}\,G_S$ and $G_T\,\widehat{\otimes}\,V_S$ are strongly stable. Since V_S and V_T are unilateral shifts, their adjoint are strongly stable, and another application of Theorem 2 ensure that $(V_T\,\widehat{\otimes}\,G_S)^*$ and $(G_T\,\widehat{\otimes}\,V_S)^*$ are strongly stable. Thus the contractions $V_T\,\widehat{\otimes}\,G_S$ and $G_T\,\widehat{\otimes}\,V_S$ are of class $\mathcal{C}_{00}$, and so is their direct sum B. Theorem 2 also ensures that all the direct summands of G are strongly stable, and so is G itself. Lemma 1 says that all the direct summands of V are unilateral shifts, and so V is a unilateral shift (of higher multiplicity). Finally, U clearly is unitary once U_T and U_S are.

(b) If $A_{T\,\widehat{\otimes}\,S}=A_{(T\,\widehat{\otimes}\,S)^*}$, then $A_T=A_{T^*}$ and $A_S=A_{S^*}$ by Corollary 1. Moreover,

$$A_T=A_{T^*}\quad\Longrightarrow\quad T=B_T\oplus U_T,$$
$$A_S=A_{S^*}\quad\Longrightarrow\quad S=B_S\oplus U_S,$$

where B_T and B_S are $\mathcal{C}_{00}$-contractions, and U_T and U_S are unitary operators [9, Corollary 1]. Thus, as before,

$$T\,\widehat{\otimes}\,S\cong(B_T\,\widehat{\otimes}\,B_S)\oplus(B_T\,\widehat{\otimes}\,U_S)\oplus(U_T\,\widehat{\otimes}\,B_S)\oplus(U_T\,\widehat{\otimes}\,U_S),$$

which yields the decomposition in (b) with

$$B=(B_T\,\widehat{\otimes}\,B_S)\oplus(B_T\,\widehat{\otimes}\,U_S)\oplus(U_T\,\widehat{\otimes}\,B_S)\quad\text{and}\quad U=U_T\,\widehat{\otimes}\,U_S,$$

where B is a $\mathcal{C}_{00}$-contraction because both B_T and B_S are, and U is unitary because both U_T and U_S are. $\square$

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