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REGULAR SUBSPACES OF TENSOR PRODUCTS

CARLOS S. KUBRUSLY

Abstract. The concept of regular subspaces of a tensor product space and the structure of regular invariant and reducing subspaces for tensor products of Hilbert space operators are explored. Preservation by tensor product is investigated. It is shown how transitiveness and reducibility, as well as the properties of being hereditarily normaloid and totally hereditarily normaloid travel between a pair of Hilbert space operators $\{A, B\}$ and their tensor product $A \otimes B$. The case of completely nonunitary tensor products is also considered.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be nonzero complex Hilbert spaces. The single tensor product of $x \in \mathcal{H}$ and $y \in \mathcal{K}$ is a conjugate bilinear functional $x \otimes y \colon \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ defined by $(x \otimes y)(u, v) = \langle x\,; u\rangle \langle y\,; v\rangle$ for every $(u, v) \in \mathcal{H} \times \mathcal{K}$. The tensor product space $\mathcal{H} \otimes \mathcal{K}$ is the completion of the inner product space consisting of all (finite) sums of single tensors $x_i \otimes y_i$ with $x_i \in \mathcal{H}$ and $y_i \in \mathcal{K}$, which is a Hilbert space with respect to the inner product

$$\Big\langle \sum_i x_i \otimes y_i\,; \sum_j w_j \otimes z_j \Big\rangle = \sum_i \sum_j \langle x_i\,; w_j\rangle \langle y_i\,; z_j\rangle$$

for every $\sum_i x_i \otimes y_i$ and $\sum_j w_j \otimes z_j$ in $\mathcal{H} \otimes \mathcal{K}$. By an operator on a normed space $\mathcal{X}$ we mean a bounded linear transformation of $\mathcal{X}$ into itself. Let $\mathcal{B}[\mathcal{X}]$ be the normed algebra of all operators on $\mathcal{X}$. The tensor product of two operators A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$ is the transformation $A \otimes B \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ defined by

$$(A \otimes B) \sum_i x_i \otimes y_i = \sum_i Ax_i \otimes By_i \quad \text{for every} \quad \sum_i x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{K},$$

which is an operator in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. For an expository paper containing the essential properties of tensor products needed here, the reader is referred to [8].

There are a number of properties that survive when taking the tensor product, and several more that do not. A property that is not preserved by tensor product is irreducibility. The concept of regular subspaces of a tensor product space $\mathcal{H} \otimes \mathcal{K}$ is defined in Section 2, and it is shown in Section 3 how transitiveness and reducibility may be transferred from operators A and B to their tensor product $A \otimes B$.

On the other hand, some properties of A and B are preserved when taking the tensor product. For instance, if A and B are nonnegative, self-adjoint, normal, quasinormal, subnormal, hyponormal, quasihyponormal, or semi-quasihyponormal, then so is $A \otimes B$ (see e.g., [8]); and the converse holds in many cases: if $A \otimes B$ is

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normal, quasinormal, subnormal or hyponormal, then so are A and B (if they are nonzero) [15]. Preservation by tensor product has also been verified for other classes of close to normal operators (see e.g., [3], [6], [9], [13] and [17]). However, such a preservation may fail for some important classes: the properties of being paranormal or spectraloid are not preserved when taking tensor products [14, pp.629,631].

Recall the following standard definitions. An operator $T \in \mathcal{B}[\mathcal{H}]$ is hyponormal if $TT^* \leq T^*T$, paranormal if $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$ for every x in $\mathcal{H}$, and normaloid if $r(T) = \|T\|$ (where $r(T)$ denotes the spectral radius of T and $\|T\|$ denotes the norm of T in $\mathcal{B}[\mathcal{H}]$). An operator is hereditarily normaloid (abbreviated: HN) if every part of it is normaloid, and totally hereditarily normaloid (abbreviated: THN) if it is hereditarily normaloid and every invertible part of it has a normaloid inverse. These classes of operators are related by proper inclusion [4]:

$$\text{Hyponormal} \;\subset\; \text{Paranormal} \;\subset\; \text{THN} \;\subset\; \text{HN} \;\subset\; \text{Normaloid}.$$

As we commented above, hyponormality is preserved by tensor product but paranormality is not. Normaloidness is preserved by tensor product as well, and it is shown in Section 4 how the properties of being hereditarily normaloid, or totally hereditarily normaloid, travel between the tensor product and each of the factors.

Also recall that an operator T on $\mathcal{H}$ is strongly stable if $\|T^nx\| \to 0$ for every x in $\mathcal{H}$, and weakly stable if $\langle T^nx\,;y\rangle \to 0$ for every x and y in $\mathcal{H}$ (equivalently, if $\langle T^nx\,;x\rangle \to 0$ for every x in the complex Hilbert space $\mathcal{H}$). These are denoted by $T^n \xrightarrow{s} O$ and $T^n \xrightarrow{w} O$, respectively. Moreover, T is a contraction if $\|T\| \leq 1$, and a contraction is completely nonunitary (abbreviated: c.n.u.) if the restriction of it to every nonzero reducing subspace is not unitary. If T is a contraction, then

$$T^n \xrightarrow{s} O \quad\Longrightarrow\quad T \text{ is c.n.u.} \quad\Longrightarrow\quad T^n \xrightarrow{w} O.$$

(The first implication is trivially verified and the second is a consequence of the Foguel decomposition for contractions — see e.g., [5, p.55] or [7, p.106].) It was shown in [3] and [9] that if A and B are contractions, then $A \otimes B$ is strongly (weakly) stable whenever one of A or B is strongly (weakly) stable; and the converse holds for strong stability. (For weak stability, the converse requires additional assumption — [12, Theorem 3].) Preservation of the property of being completely nonunitary will be considered in Section 5, where it is given a new proof that $A \otimes B$ is completely nonunitary if and only if one of A or B is completely nonunitary.

2. Regular Subspaces

By a subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ we mean a *closed* linear manifold of $\mathcal{H}$. It is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$, and T-invariant — or invariant for an operator T in $\mathcal{B}[\mathcal{H}]$ — if $T(\mathcal{M}) \subseteq \mathcal{M}$. It is said to be a reducing subspace for T — or to reduce the operator T in $\mathcal{B}[\mathcal{H}]$ — if both $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$ are T-invariant (equivalently, if $\mathcal{M}$ invariant for both T and its adjoint T^*).

Regular subspaces of tensor product spaces were introduced in [10] as follows.

Definition 1. A subspace of $\mathcal{H} \otimes \mathcal{K}$ is *regular* if it is of the form $\mathcal{M} \otimes \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively. Otherwise it is called *irregular*.

Let $\mathcal{M}$ be any nonzero subspace of a Hilbert space $\mathcal{H}$ and let $\{h_\gamma\}_{\gamma\in\Gamma'}$ be an orthonormal basis for $\mathcal{M}$ so that, with $\bigvee$ denoting *closure of span*,

$$\mathcal{M} = \bigvee_{\gamma\in\Gamma'} h_\gamma,$$

where $\{h_\gamma\}_{\gamma\in\Gamma'}$ is a subset of an orthonormal basis $\{h_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$ with $\Gamma'\subseteq\Gamma$. Let $\mathcal{S}$ be an arbitrary nonzero subspace of the tensor product space $\mathcal{H}\otimes\mathcal{K}$. Thus,

$$\mathcal{S} \;=\; \bigvee_{\delta\in\Delta'} f_\delta$$

for an orthonormal basis $\{f_\delta\}_{\delta\in\Delta}$ for $\mathcal{H}\otimes\mathcal{K}$ with $\Delta'\subseteq\Delta$. If $\{h_\gamma\}_{\gamma\in\Gamma}$ and $\{k_\lambda\}_{\lambda\in\Lambda}$ are orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, then $\{h_\gamma\otimes k_\lambda\}_{(\gamma,\lambda)\in\Gamma\times\Lambda}$ is an orthonormal basis for $\mathcal{H}\otimes\mathcal{K}$ (see e.g., [18, Theorem 3.12(b)]). Hence a nonzero subspace $\mathcal{S}$ of $\mathcal{H}\otimes\mathcal{K}$ may be given by

$$\mathcal{S} \;=\; \bigvee_{(\gamma,\lambda)\in(\Gamma\times\Lambda)'} h_\gamma\otimes k_\lambda$$

for some orthonormal bases $\{h_\gamma\}_{\gamma\in\Gamma}$ and $\{k_\lambda\}_{\lambda\in\Lambda}$ for $\mathcal{H}$ and $\mathcal{K}$, where $(\Gamma\times\Lambda)'$ is a subset of the double index set $\Gamma\times\Lambda$, which does not necessarily coincides with any rectangle $\Gamma'\times\Lambda'$ of subsets Γ' and Λ' of the index sets Γ and Λ.

Regular subspaces of tensor product spaces can also be characterized in Lemma 1 below. Its proof is straightforward, hence omitted.

Lemma 1. *A subspace $\mathcal{S}$ of $\mathcal{H}\otimes\mathcal{K}$ is regular if and only if it is either zero or*

$$\mathcal{S} \;=\; \bigvee_{(\gamma,\lambda)\in\Gamma'\times\Lambda'} h_\gamma\otimes k_\lambda,$$

for orthonormal bases $\{h_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$ and $\{k_\lambda\}_{\lambda\in\Lambda}$ for $\mathcal{K}$ with $\Gamma'\subseteq\Gamma$ and $\Lambda'\subseteq\Lambda$.

3. Regular Invariant and Reducing Subspaces

We borrow the next result from [10] since it will play a central role in the forthcoming sections. It explores the structure of regular invariant and regular reducing subspaces of tensor products.

Lemma 2. *Let A and B be nonzero operators on $\mathcal{H}$ and $\mathcal{K}$, let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively, and consider the subspace $\mathcal{M}\otimes\mathcal{N}$ of $\mathcal{H}\otimes\mathcal{K}$.*

(a_1) *If $\mathcal{M}$ is invariant (reducing) for A and $\mathcal{N}$ is invariant (reducing) for B, then $\mathcal{M}\otimes\mathcal{N}$ is invariant (reducing) for $A\otimes B$.*

(a_2) *If $\mathcal{M}\otimes\mathcal{N}$ is invariant for $A\otimes B$, then $\mathcal{M}$ is invariant for A or $\mathcal{N}$ is invariant for B.*

(a_3) *If $\mathcal{M}\otimes\mathcal{N}$ reduces $A\otimes B$, then*

$\mathcal{M}$ reduces A, or

$\mathcal{N}$ reduces B, or

$\mathcal{M}$ is invariant for A and $\mathcal{N}^\perp$ is invariant for B, or

$\mathcal{N}$ is invariant for B and $\mathcal{M}^\perp$ is invariant for A.

(a_4) *If $\mathcal{M}\otimes\mathcal{N}$ is nonzero and invariant (reducing) for $A\otimes B$, and if A and B are injective, then $\mathcal{M}$ is invariant (reducing) for A and $\mathcal{N}$ is invariant (reducing) for B.*

(b) *One of $\mathcal{M}$ or $\mathcal{N}$ is nontrivial and the other is nonzero if and only if $\mathcal{M}\otimes\mathcal{N}$ is nontrivial.*

(c_1) *If $\mathcal{M}$ is A-invariant and $\mathcal{N}$ is B-invariant, then*

$$(A\otimes B)|_{\mathcal{M}\otimes\mathcal{N}} = A|_{\mathcal{M}}\otimes B|_{\mathcal{N}}.$$

(c_2) *If $\mathcal{M}\otimes\mathcal{N}$ is nonzero and $A\otimes B$-invariant, and if A and B are injective, then*

$$(A\otimes B)|_{\mathcal{M}\otimes\mathcal{N}} = A|_{\mathcal{M}}\otimes B|_{\mathcal{N}}.$$

Proof. See [10, Theorem 1] for the case of $m = 2$. □

An operator is transitive if it has no nontrivial invariant subspace, and intransitive if it has a nontrivial invariant subspace. The invariant subspace problem is the open question that asks whether the class of all transitive operators acting on an infinite-dimensional complex separable Hilbert space is nonempty. An operator is reducible if it has a nontrivial reducing subspace, and irreducible otherwise.

Theorem 1. *Let A and B be nonzero operators on $\mathcal{H}$ and $\mathcal{K}$.*

(a) *If one of A or B is intransitive (reducible), then the tensor product $A\otimes B$ has a nontrivial regular invariant (reducing) subspace.*

(b) *The converse holds if A and B are injective.*

Proof. According to Definition 1 a subspace of $\mathcal{H}\otimes\mathcal{K}$ is regular if and only if it is a tensor product $\mathcal{M}\otimes\mathcal{N}$ of subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ and $\mathcal{K}$.

(a) If one of A or B is intransitive (reducible), then there exists a nontrivial invariant (reducing) subspace $\mathcal{M}$ for A or a nontrivial invariant (reducing) subspace $\mathcal{N}$ for B, and so $\mathcal{M}\otimes\mathcal{K}$ or $\mathcal{H}\otimes\mathcal{N}$ is a regular nontrivial invariant (reducing) subspace for $A\otimes B$ by Lemma 2(a_1,b).

(b) Conversely, if A and B are injective and if $A\otimes B$ has a nontrivial regular invariant (reducing) subspace $\mathcal{M}\otimes\mathcal{N}$, then $\mathcal{M}$ is invariant (reducing) for A and $\mathcal{N}$ is invariant (reducing) for B by Lemma 2(a_4), and one of them is nontrivial by Lemma 2(b), and hence one of A or B is intransitive (reducible). □

The whole space $\mathcal{H}\otimes\mathcal{K}$ and the null space $\{0\}$ are regular, and so irregular subspaces are always nontrivial. If A and B are injective, then by Lemma 2(a_1, a_4, b) $\mathcal{M}\otimes\mathcal{N}$ is nontrivial and reduces $A\otimes B$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are nonzero (one of them being nontrivial) and reduce A and B. If both $\mathcal{M}$ and $\mathcal{N}$ are nontrivial, then $A\otimes B$ commutes with a nonscalar normal tensor product $N_A\otimes N_B$ of normal operators N_A and N_B. However, not all normal operators on $\mathcal{H}\otimes\mathcal{K}$ are tensor products of operators. Theorem 1 is applied in Corollary 1 below to verify the existence of reducible tensor products whose nontrivial reducing subspaces are all irregular. A generalization of Corollary 1 was considered in [10]. Nevertheless, we present below the particular case that precisely fits our present needs, and sketch out its proof since part of the proof will be required later in Section 5.

Corollary 1. *Let A be an arbitrary operator on a Hilbert space of dimension greater than* 1.

(a) *The tensor product $A \otimes A$ is reducible.*
(b) *If A is injective and irreducible, then all nontrivial reducing subspaces of $A \otimes A$ are irregular.*

Proof. Let A and B be operators acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Consider the mapping $\Pi\colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$ defined by

$$\Pi\Big(\sum_i x_i \otimes y_i\Big) = \sum_i y_i \otimes x_i$$

for every $\sum_i x_i \otimes y_i$ in $\mathcal{H} \otimes \mathcal{K}$. It is readily verified that this is an invertible linear isometry, thus a unitary transformation. If $\mathcal{K} = \mathcal{H}$, then Π in $\mathcal{B}[\mathcal{H} \otimes \mathcal{H}]$ also is an involution (i.e., $\Pi^2 = I$), and so it is a symmetry (i.e., $\Pi^{-1} = \Pi^* = \Pi$) on $\mathcal{H} \otimes \mathcal{H}$.

(a) Consider the tensor product $A \otimes B$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ and let Π be the unitary transformation of $\mathcal{H} \otimes \mathcal{K}$ onto $\mathcal{K} \otimes \mathcal{H}$ defined above so that

$$\Pi\,(A \otimes B) = (B \otimes A)\,\Pi$$

(i.e., the tensor product is unitarily equivalent commutative). In particular, the tensor product of an arbitrary operator A in $\mathcal{B}[\mathcal{H}]$ with itself, $A \otimes A$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{H}]$, commutes with the nonscalar normal operator Π in $\mathcal{B}[\mathcal{H} \otimes \mathcal{H}]$. This means that $A \otimes A$ is reducible (which is a classical consequence of the Spectral Theorem).

(b) Thus the tensor product $A \otimes A$ always has a nontrivial reducing subspace. If A is injective and irreducible (sample: a unilateral shift of multiplicity 1), then $A \otimes A$ has no nontrivial regular reducing subspace by Theorem 1(b), and so all its nontrivial reducing subspaces are irregular. □

4. Regularly Normaloid Tensor Products

After recalling that the property of being normaloid is preserved by tensor product, it is given in Theorem 2 necessary and sufficient conditions on $A \otimes B$ for both operators A and B be hereditarily normaloid (or totally hereditarily normaloid). This shows how hereditary normaloidness and total hereditary normaloidness are transfered between $\{A, B\}$ and their tensor product $A \otimes B$.

Lemma 3. *Let A and B be nonzero operators on $\mathcal{H}$ and $\mathcal{K}$. The tensor product $A \otimes B$ is normaloid if and only if both A and B are normaloid.*

Proof. See e.g., [14, Corollary 6.1]. □

A part of an operator is a restriction of it to an invariant subspace; a nontrivial part is a restriction to a nontrivial invariant subspace. A direct summand of an operator is a restriction of it to a reducing subspace. By a *regular part* of a tensor product of two operators we mean a restriction of the tensor product to a regular invariant subspace. Recall that an operator is called hereditarily normaloid if every part of it is normaloid, and totally hereditarily normaloid if it is hereditarily normaloid and every invertible part of it has a normaloid inverse.

Definition 2. A tensor product is *regularly normaloid* (abbreviated: RN) if every regular part of it is normaloid, and *totally regularly normaloid* (abbreviated: TRN) if it is regularly normaloid and every invertible regular part of it has a normaloid inverse.

Theorem 2. *Let A and B be nonzero operators on $\mathcal{H}$ and $\mathcal{K}$.*

(a_1) *If the tensor product $A\otimes B$ is regularly normaloid, then both A and B are hereditarily normaloid.*

(a_2) *If the tensor product $A\otimes B$ is totally regularly normaloid, and if one of A or B has an invertible part (in particular, if one is invertible), then the other is totally hereditarily normaloid.*

(b) *If A and B are hereditarily normaloid (totally hereditarily normaloid) and injective, then $A\otimes B$ is regularly normaloid (totally regularly normaloid).*

Proof. Let $\mathrm{Lat}(A)$ and $\mathrm{Lat}(B)$ denote the collection of all invariant subspaces for $A\in\mathcal{B}[\mathcal{H}]$ and $B\in\mathcal{B}[\mathcal{K}]$, respectively. Take arbitrary $\mathcal{M}\in\mathrm{Lat}(A)$ and $\mathcal{N}\in\mathrm{Lat}(B)$.

(a_1) By Lemma 2(a_1), $\mathcal{H}\otimes\mathcal{N}$ is a regular invariant subspace for $A\otimes B$. If $A\otimes B$ is regularly normaloid, then $(A\otimes B)|_{\mathcal{H}\otimes\mathcal{N}}$ is normaloid, and so is $A\otimes B|_{\mathcal{N}}$ by Lemma 2(c_1). If $B|_{\mathcal{N}}=O$, then it is trivially normaloid. If $B|_{\mathcal{N}}\neq O$, then (since $A\neq O$), both A and $B|_{\mathcal{N}}$ are normaloid according to Lemma 3. Similarly, If $A\otimes B$ is regularly normaloid, then $(A\otimes B)|_{\mathcal{M}\otimes\mathcal{K}}$ is normaloid and the same argument ensures that, if $A|_{\mathcal{M}}\neq O$, then $A|_{\mathcal{M}}$ and B are normaloid. Therefore,

$$A\otimes B\in\mathrm{RN}\quad\text{implies}\quad A\in\mathrm{HN}\ \text{ and }\ B\in\mathrm{HN}.$$

(a_2) If S and T are Hilbert space operators, then $S\otimes T$ is invertible if and only if both S and T are invertible (indeed, $\sigma(S\otimes T)=\sigma(S)\cdot\sigma(T)$, where $\sigma(\,\cdot\,)$ denotes spectrum — cf. [2]) and $(S\otimes T)^{-1}=S^{-1}\otimes T^{-1}$. Hence, $(A\otimes B)|_{\mathcal{M}\otimes\mathcal{N}}$ is invertible if and only if $A|_{\mathcal{M}}\otimes B|_{\mathcal{N}}$ is invertible by Lemma 2(c_1), and this happens if and only if both $A|_{\mathcal{M}}$ and $B|_{\mathcal{N}}$ are invertible. Moreover, in such a case,

$$(*)\qquad \big[(A\otimes B)|_{\mathcal{M}\otimes\mathcal{N}}\big]^{-1}=(A|_{\mathcal{M}}\otimes B|_{\mathcal{N}})^{-1}=(A|_{\mathcal{M}})^{-1}\otimes(B|_{\mathcal{N}})^{-1}.$$

Suppose $A\otimes B$ is totally regularly normaloid. Then it is regularly normaloid, and so both A and B are hereditarily normaloid according to item (a_1). Suppose $A|_{\mathcal{M}_0}$ is invertible for some $\mathcal{M}_0\in\mathrm{Lat}(A)$. Take an arbitrary $\mathcal{N}$ in $\mathrm{Lat}(B)$. If $B|_{\mathcal{N}}$ is invertible, then $(A\otimes B)|_{\mathcal{M}_0\otimes\mathcal{N}}=A|_{\mathcal{M}_0}\otimes B|_{\mathcal{N}}$ (cf. Lemma 2(c_1)) is invertible and $[(A\otimes B)|_{\mathcal{M}_0\otimes\mathcal{N}}]^{-1}=(A|_{\mathcal{M}_0})^{-1}\otimes(B|_{\mathcal{N}})^{-1}$ by $(*)$. Since $A\otimes B$ is totally regularly normaloid, it follows that $(A|_{\mathcal{M}_0})^{-1}\otimes(B|_{\mathcal{N}})^{-1}$ is normaloid, and so $(A|_{\mathcal{M}_0})^{-1}$ and $(B|_{\mathcal{N}})^{-1}$ are normaloid by Lemma 3. Thus B is totally hereditarily normaloid. Similarly, Suppose $B|_{\mathcal{N}_0}$ is invertible for some $\mathcal{N}_0\in\mathrm{Lat}(B)$. Take an arbitrary $\mathcal{M}$ in $\mathrm{Lat}(A)$. If $A|_{\mathcal{M}}$ is invertible, then the same argument ensures that $(A|_{\mathcal{M}})^{-1}$ and $(B|_{\mathcal{N}_0})^{-1}$ are normaloid. Thus A is totally hereditarily normaloid. In particular, if both A and B have an invertible part (specially, if they are invertible), then

$$A \otimes B \in \text{TRN} \quad \text{implies} \quad A \in \text{THN} \text{ and } B \in \text{THN}.$$

(b) Let $\mathcal{M} \otimes \mathcal{N}$ be an arbitrary regular $A \otimes B$-invariant subspace of $\mathcal{H} \otimes \mathcal{K}$. If $\mathcal{M} = \{0\}$ or $\mathcal{N} = \{0\}$, then $\mathcal{M} \otimes \mathcal{N} = \{0\}$, and hence $(A \otimes B)|_{\mathcal{M} \otimes \mathcal{N}} = O$, which is trivially normaloid and noninvertible. Thus take $\mathcal{M} \neq \{0\}$ and $\mathcal{N} \neq \{0\}$, so that $\mathcal{M} \otimes \mathcal{N} \neq \{0\}$. Suppose A and B are injective (so that $A|_{\mathcal{M}} \neq O$ and $B|_{\mathcal{N}} \neq O$ since $\mathcal{M} \not\subseteq \text{kernel}\,(A) = \{0\}$ and $\mathcal{M} \not\subseteq \text{kernel}\,(B) = \{0\}$). Then $A \otimes B$ is injective (see e.g., [11]). Thus $\mathcal{M} \otimes \mathcal{N} \not\subseteq \text{kernel}\,(A \otimes B) = \{0\}$, and so $(A \otimes B)|_{\mathcal{M} \otimes \mathcal{N}} \neq O$. By Lemma 2($a_4$), $\mathcal{M}$ is A-invariant and $\mathcal{N}$ is B-invariant. If A and B are hereditarily normaloid, then $A|_{\mathcal{M}}$ and $B|_{\mathcal{N}}$ are normaloid, and hence $A|_{\mathcal{M}} \otimes B|_{\mathcal{N}}$ is normaloid by Lemma 3. Therefore, $(A \otimes B)|_{\mathcal{M} \otimes \mathcal{N}}$ is normaloid by Lemma 2(c_2). Thus

$$A \in \text{HN} \text{ and } B \in \text{HN} \quad \text{implies} \quad A \otimes B \in \text{RN}.$$

Again, if $(A \otimes B)|_{\mathcal{M} \otimes \mathcal{N}}$ is invertible, then so are $A|_{\mathcal{M}}$ and $B|_{\mathcal{N}}$. If A and B are totally hereditarily normaloid, then $(A|_{\mathcal{M}})^{-1}$ and $(B|_{\mathcal{N}})^{-1}$ are normaloid, and so is $[(A \otimes B)|_{\mathcal{M} \otimes \mathcal{N}}]^{-1}$ by $(*)$ and Lemma 3. Then the above implication ensures that

$$A \in \text{THN} \text{ and } B \in \text{THN} \quad \text{implies} \quad A \otimes B \in \text{TRN}. \qquad \square$$

The next result gives a complete characterization of unitary tensor products. This is applied to prove the forthcoming Corollary 2 and Theorem 3.

Lemma 4. *Consider the tensor product* $A \otimes B \in \mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ *of a pair of nonzero operators* $A \in \mathcal{B}[\mathcal{H}]$ *and* $B \in \mathcal{B}[\mathcal{K}]$.

(a) $A \otimes B$ *is unitary* (*an isometry*) *if and only if both* A *and* B *are nonzero multiples of unitary operators* (*isometries*) *with* $\|A\| = \|B\|^{-1}$.

(b) *In particular, if* A *and* B *are unitary operators* (*isometries*), *then* $A \otimes B$ *is unitary* (*an isometry*) *and, conversely, if* $A \otimes B$ *is unitary* (*an isometry*), *then there are unitary operators* (*isometries*) $U \in \mathcal{B}[\mathcal{H}]$ *and* $V \in \mathcal{B}[\mathcal{K}]$ *such that* $A \otimes B = U \otimes V$.

(c) *Moreover, if* A *and* B *are contractions, then* $A \otimes B$ *is unitary* (*an isometry*) *if and only if both* A *and* B *are unitary operators* (*isometries*).

Proof. The result in (a) is a straightforward application of [15, Theorem 2.4(a)]. The results in (b) and (c) are readily verified by (a). $\square$

A contraction $T \in \mathcal{B}[\mathcal{H}]$ is of class $\mathcal{C}_{11}$ (or T is a $\mathcal{C}_{11}$-contraction) if both sequences $\{\|T^n x\|\}$ and $\{\|T^{*n} x\|\}$ do not converge to zero for every nonzero x in $\mathcal{H}$. For $\mathcal{C}_{11}$-contractions the properties of being totally regularly normaloid and totally hereditarily normaloid coincide.

Corollary 2. *Let* A *and* B *be Hilbert space contractions. The following assertions are pairwise equivalent.*

(a) *The tensor product* $A \otimes B$ *is unitary.*

(b) *The tensor product* $A \otimes B$ *is totally hereditarily normaloid and of class* $\mathcal{C}_{11}$.

(c) *The tensor product* $A \otimes B$ *is totally regularly normaloid and of class* $\mathcal{C}_{11}$.

Proof. It is clear that *every unitary operator is a totally hereditarily normaloid $\mathcal{C}_{11}$-contraction*. Thus (a) implies (b). Moreover, (b) trivially implies (c). It remains to prove that (c) implies (a). First note that $A \otimes B$ is a contraction because A and B are contractions, since $\|A \otimes B\| = \|A\|\,\|B\|$.

Claim 1. *If $A \otimes B$ is of class $\mathcal{C}_{11}$, then both A and B are of class $\mathcal{C}_{11}$.*

Proof. Indeed, for every $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$\|(A \otimes B)^n x \otimes y\| = \|A^n x \otimes B^n y\| = \|A^n x\|\,\|B^n y\|,$$
$$\|(A \otimes B)^{*n} x \otimes y\| = \|A^{*n} x \otimes B^{*n} y\| = \|A^{*n} x\|\,\|B^{*n} y\|.$$

Claim 2. *Every $\mathcal{C}_{11}$-contraction has an invertible part.*

Proof. If $T \in \mathcal{B}[\mathcal{H}]$ is a $\mathcal{C}_{11}$-contraction, then it is quasisimilar to a unitary operator (see e.g., [7, p.70]), which in turn implies that there exists an increasing sequence $\{\mathcal{M}_n\}$ of T-invariant subspaces that span $\mathcal{H}$ such that each part $T|_{\mathcal{M}_n}$ is similar to a unitary operator [1]. Thus every $T|_{\mathcal{M}_n}$ is an invertible part of T.

Claim 3. *Every totally hereditarily normaloid $\mathcal{C}_{11}$-contraction is unitary.*

Proof. See [4, Proposition 2.5].

Therefore, if the contraction $A \otimes B$ is of class $\mathcal{C}_{11}$, then it follows by Claims 1 and 2 that both contractions A and B are of class $\mathcal{C}_{11}$ and both have an invertible part. Thus Theorem 2(a_2) ensures that if (a) holds, that is, if the $\mathcal{C}_{11}$-contraction $A \otimes B$ is totally regularly normaloid, then the $\mathcal{C}_{11}$-contractions A and B are totally hereditarily normaloid, and so A and B are unitary by Claim 3. Hence (c) holds, that is, $A \otimes B$ is unitary according to Lemma 4(b). Outcome: (c) implies (a). □

5. Regularly Nonunitary Tensor Products

Recall that a contraction is completely nonunitary if the restriction of it to every nonzero reducing subspace is not unitary. Equivalently, a nonzero contraction T in $\mathcal{B}[\mathcal{H}]$ is not completely nonunitary if there exists a nonzero vector $x \in \mathcal{H}$ such that

$$\|T^n x\| = \|T^{*n} x\| = \|x\| \quad \text{for every positive integer } n.$$

Definition 3. A tensor product is a *regularly nonunitary contraction* if it is a contraction and the restriction of it to every nonzero regular reducing subspace is not unitary.

Theorem 3. *Let A and B be contractions on $\mathcal{H}$ and $\mathcal{K}$.*

(a) *If the tensor product $A \otimes B$ is a regularly nonunitary contraction, then one of A or B is completely nonunitary.*

(b) *The converse holds if A and B are injective.*

Proof. Let $\mathcal{H}$ and $\mathcal{K}$ be the nonzero Hilbert spaces upon which act the contractions A and B. The tensor product $A \otimes B$ is a contraction because both A and B are. Moreover, one of A or B is null if and only if the tensor product $A \otimes B$ is null, which in this case are completely nonunitary once the Hilbert spaces are all nonzero. Thus suppose A and B are nonzero contractions.

(a) If both contractions A and B are not completely nonunitary, then there exist nonzero vectors $x \in \mathcal{H}$ and $y \in \mathcal{K}$ such that

$$\|A^n x\| = \|A^{*n} x\| = \|x\| \quad \text{and} \quad \|B^n y\| = \|B^{*n} y\| = \|y\|$$

for every positive integer n. Let $\mathcal{M}$ and $\mathcal{N}$ be the sets of all those vectors x and y that satisfy the above equations (i.e., put $\mathcal{M} = \{x \in \mathcal{H}\colon\ \|A^n x\| = \|A^{*n} x\| = \|x\|\}$ and $\mathcal{N} = \{y \in \mathcal{K}\colon\ \|B^n y\| = \|B^{*n} y\| = \|y\|\}$. These are nonzero subspaces of $\mathcal{H}$ and $\mathcal{K}$ (see e.g., [16, p.9] or [7, p.76]). Moreover they reduce A and B and $A|_{\mathcal{M}}$ and $B|_{\mathcal{N}}$ are unitary operators. Thus the regular subspace $\mathcal{M} \otimes \mathcal{N}$ of $\mathcal{H} \otimes \mathcal{K}$ is nonzero (since both $\mathcal{M}$ and $\mathcal{N}$ are), reduces $A \otimes B$ by Lemma 2(a_1), and

$$(A \otimes B)|_{\mathcal{M}\otimes\mathcal{N}} = A|_{\mathcal{M}} \otimes B|_{\mathcal{N}}$$

by Lemma 2(c_1), which is unitary by Lemma 4. Hence $A \otimes B$ is not regularly nonunitary. Therefore, if the tensor product $A \otimes B$ is regularly nonunitary, then one of A or B is completely nonunitary.

(b) Conversely, suppose A and B are injective. Let $\mathcal{M} \otimes \mathcal{N}$ be any nonzero regular subspace of $\mathcal{H} \otimes \mathcal{K}$ that reduces $A \otimes B$. By Lemma 2(a_4 , c_2)

$$(A \otimes B)|_{\mathcal{M}\otimes\mathcal{N}} = A|_{\mathcal{M}} \otimes B|_{\mathcal{N}},$$

where $\mathcal{M}$ reduces A and $\mathcal{N}$ reduces B. If one of the parts $A|_{\mathcal{M}}$ or $B|_{\mathcal{N}}$ is zero, then $(A \otimes B)|_{\mathcal{M}\otimes\mathcal{N}} = O$ is certainly not unitary (since $\mathcal{M} \otimes \mathcal{N}$ is a nonzero subspace). Thus suppose $A|_{\mathcal{M}}$ and $B|_{\mathcal{N}}$ are nonzero. If A is a completely nonunitary contraction, then $A|_{\mathcal{M}}$ is a nonunitary contraction. This implies that $A|_{\mathcal{M}} \otimes B|_{\mathcal{N}}$ is not unitary. Indeed, since $A|_{\mathcal{M}}$ and $B|_{\mathcal{N}}$ are nonzero contractions, it follows by Lemma 4(c) that if $A|_{\mathcal{M}} \otimes B|_{\mathcal{N}}$ is unitary, then $A|_{\mathcal{M}}$ is unitary, which is a contradiction. Hence $(A \otimes B)|_{\mathcal{M}\otimes\mathcal{N}}$ is not unitary whenever A is completely nonunitary. Similarly — exactly the same argument — $(A \otimes B)|_{\mathcal{M}\otimes\mathcal{N}}$ is not unitary whenever B is a completely nonunitary contraction. Therefore, if one of A or B is completely nonunitary, then $A \otimes B$ is a regularly nonunitary contraction. □

In fact, it was proved in [10] that the properties of being regularly nonunitary and completely nonunitary coincide (i.e., the property of being completely nonunitary is invariant for the type — regular or irregular — of the reducing subspace). This enables the converse in the preceding theorem to hold without the injectivity assumption. We shall give below a new and simple proof for these facts. The price for such an elementary proof is that it works for separable Hilbert spaces (while the nonelementary proof in [10, Theorem 3] works for general Hilbert spaces).

Corollary 3. *Let A and B are contractions acting on separable Hilbert spaces. The following assertions are pairwise equivalent.*

(a) *The tensor product $A \otimes B$ is completely nonunitary.*

(b) *The tensor product $A \otimes B$ is regularly nonunitary.*

(c) *One of the contractions A or B is completely nonunitary.*

Proof. Assertion (a) trivially implies (b), and (b) implies (c) by Theorem 3(a). To show that (c) implies (a) proceed as follows. Let S be an operator on $\mathcal{H}$ and let T be an operator on $\mathcal{K}$. If $\mathcal{H}$ is a separable Hilbert space, and if I stands for the identity operator on $\mathcal{H}$, then $I \otimes T$ is unitarily equivalent to the countable (orthogonal) direct sum $\bigoplus_k T$ through a unitary transformation $\Phi_{\mathcal{K}}$ that does not depend on T. That is, with $\ell_+^2(\mathcal{K}) = \bigoplus_k \mathcal{K}$, there exists a unitary transformation $\Phi_{\mathcal{K}}\colon \ell_+^2(\mathcal{K}) \to \mathcal{H}\otimes\mathcal{K}$ such that (see e.g., [8, Remark 5]), for every T in $\mathcal{B}[\mathcal{K}]$,

$$\Phi_{\mathcal{K}}\Big(\bigoplus_k T\Big) = (I\otimes T)\,\Phi_{\mathcal{K}}.$$

Moreover, as we saw in the proof of Corollary 1, there also exists a unitary transformation $\Pi\colon \mathcal{H}\otimes\mathcal{K} \to \mathcal{K}\otimes\mathcal{H}$ such that, if I stands now for the identity on $\mathcal{K}$,

$$\Pi^*(I\otimes S) = (S\otimes I)\,\Pi^*$$

for every S in $\mathcal{B}[\mathcal{H}]$. Thus, if $\mathcal{K}$ is separable, then there exists a unitary transformation $\Phi_{\mathcal{H}}\colon \ell_+^2(\mathcal{H}) \to \mathcal{K}\otimes\mathcal{H}$ such that $S\otimes I$ is unitarily equivalent to the countable direct sum $\bigoplus_k S$ through the unitary transformation $\Pi^*\Phi_{\mathcal{H}} : \ell_+^2(\mathcal{H}) \to \mathcal{H}\otimes\mathcal{K}$,

$$\Pi^*\Phi_{\mathcal{H}}\Big(\bigoplus_k S\Big) = (S\otimes I)\,\Pi^*\Phi_{\mathcal{H}},$$

where $\ell_+^2(\mathcal{H}) = \bigoplus_k \mathcal{H}$. Therefore,

$$(**)\qquad S\otimes T = (S\otimes I)\,(I\otimes T) = \Pi^*\Phi_{\mathcal{H}}\Big(\bigoplus_k S\Big)\Phi_{\mathcal{H}}^*\Pi\;\Phi_{\mathcal{K}}\Big(\bigoplus_k T\Big)\Phi_{\mathcal{K}}^*$$

for every S in $\mathcal{B}[\mathcal{H}]$ and every T in $\mathcal{B}[\mathcal{K}]$. Next let A and B be contractions on $\mathcal{H}$ and $\mathcal{K}$, and suppose the contraction $A\otimes B$ is not completely nonunitary, which means that there exists a nonzero vector $z\in\mathcal{H}\otimes\mathcal{K}$ such that

$$\|(A\otimes B)^n z\| = \|(A\otimes B)^{*n} z\| = \|z\|$$

for every positive integer n. Since $\bigoplus_k A^n$ and $\bigoplus_k B^n$ are contractions, and since composition of unitary operators is unitary, if follows by $(**)$ that

$$\begin{aligned}\|(A\otimes B)^n z\| \;=\; \|(A^n\otimes B^n)z\| &= \Big\|\Pi^*\Phi_{\mathcal{H}}\Big(\bigoplus_k A^n\Big)\Phi_{\mathcal{H}}^*\Pi\;\Phi_{\mathcal{K}}\Big(\bigoplus_k B^n\Big)\Phi_{\mathcal{K}}^* z\Big\| \\ &\le \Big\|\bigoplus_k A^n\Big\|\,\Big\|\Big(\bigoplus_k B^n\Big)\,\Phi_{\mathcal{K}}^* z\Big\| \le \Big\|\Big(\bigoplus_k B^n\Big)\,\Phi_{\mathcal{K}}^* z\Big\| \le \|\Phi_{\mathcal{K}}^* z\|,\end{aligned}$$

and so there exists a nonzero vector $w = \Phi_{\mathcal{K}}^* z \in \ell_+^2(\mathcal{K})$ such that

$$\|w\| = \|z\| = \|(A\otimes B)^n z\| \le \Big\|\Big(\bigoplus_k B^n\Big)w\Big\| \le \|w\|,$$

and hence

$$\Big\|\Big(\bigoplus_k B\Big)^n w\Big\| = \Big\|\Big(\bigoplus_k B^n\Big)w\Big\| = \|w\|$$

for every positive integer n. Dually (since $\|(A\otimes B)^{*n}z\| = \|(A^{*n}\otimes B^{*n})z\|$) we get $\big\|\big(\bigoplus_k B\big)^{*n} w\big\| = \big\|\big(\bigoplus_k B^{*n}\big)w\big\| = \|w\|$, and therefore

$$\Big\|\Big(\bigoplus_k B\Big)^n w\Big\| = \Big\|\Big(\bigoplus_k B\Big)^{*n} w\Big\| = \|w\|$$

for every positive integer n. But this means that the contraction $\bigoplus_k B$ is not completely nonunitary. If $\bigoplus_k B$ is not completely nonunitary, then B is not completely nonunitary (because a countable direct sum of completely nonunitary contractions is a completely nonunitary contraction). Thus, if B in $\mathcal{B}[\mathcal{K}]$ is completely nonunitary, then $A \otimes B$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ is completely nonunitary. Similarly, since $A \otimes B$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ and $B \otimes A$ in $\mathcal{B}[\mathcal{K} \otimes \mathcal{H}]$ are unitarily equivalent (cf. Proof of Corollary 1 again), and since every contraction unitarily equivalent to a completely nonunitary contraction is itself completely nonunitary, it follows if A in $\mathcal{B}[\mathcal{H}]$ is completely nonunitary, then $A \otimes B$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ is completely nonunitary. □

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Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: carlos@ele.puc-rio.br