Applied Mathematics Letters **22** (2009) 1286–1291

ORTHOGONAL DECOMPOSITIONS FOR WAVELETS

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Abstract. Decompositions of Hilbert spaces in terms of reducing subspaces for wavelets operators, as well as decompositions of these operators themselves, are investigated. In particular, it is shown on which reducing subspaces these operators act as bilateral shifts of multiplicity 1. We also exhibit the unitary transformation that performs the unitary equivalence between restrictions of them to appropriate reducing subspaces.

1. Introduction

Let $\mathcal{H}$ be a separable (infinite-dimensional) Hilbert space. By an operator on $\mathcal{H}$ we mean a bounded linear (i.e., a continuous linear) transformation of $\mathcal{H}$ into itself. Recall the following definitions [9]. Let D and T be bilateral shifts of (countably) infinite multiplicity [6] acting on $\mathcal{H}$ such that

$$DT^2 = TD.$$

This ensures that [9, Proposition 3]

$$D^m T^{n2^m} = T^n D^m \tag{1}$$

for every $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers. Any nonzero vector ψ in $\mathcal{H}$ that makes $\{D^m T^n \psi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ into an orthonormal basis for $\mathcal{H}$ is a wavelet, and the vectors $\psi_{m,n} = D^m T^n \psi$ are the wavelet vectors generated by ψ.

For instance, with $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$, the operators D and T on $\mathcal{L}^2(\mathbb{R})$ defined by

$$y = Dx \quad \text{with} \quad y(t) = \sqrt{2}\, x(2t)$$

and

$$y = Tx \quad \text{with} \quad y(t) = x(t-1)$$

(for almost all t in $\mathbb{R}$ with respect to Lebesgue measure) are bilateral shifts of infinite multiplicity satisfying (1) for which there is a function ψ (e.g., the Haar wavelet) that makes $\{D^m T^n \psi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ into an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$. In this case, D and T are referred to as dilation-by-2 and translation-by-1, respectively.

Orthogonal decompositions of Hilbert spaces in terms of reducing subspaces for the operators D and T are revisited in Section 2. Orthogonal decompositions of the operators D and T themselves are considered in Section 3, where it is investigated on which subspaces of $\mathcal{H}$ these operators still act as bilateral shifts; in particular, as bilateral shifts of multiplicity 1. The closing Section 4 shows which restrictions of D and T are unitary equivalent, and it is also exhibited the unitary transformation that performs the unitary equivalence between restrictions of D and restrictions of T on appropriate reducing subspaces.

Date: September 27, 2007.
2000 *Mathematics Subject Classification.* Primary 47A15; Secondary 42C40.
Keywords. Bilateral shifts, decomposition, reducing subspaces, unitary equivalence, wavelets.

2. Wavelets Subspaces

For each $m \in \mathbb{Z}$ we associate with a wavelet $\psi \in \mathcal{H}$ the following infinite-dimensional subspaces of $\mathcal{H}$ [14] (also see [1], [2] [4], [5], [8], [9], [10], [11], [12] and [15]).

$$\mathcal{W}_m = D^m \bigvee_{n\in\mathbb{Z}} T^n\psi = \bigvee_{n\in\mathbb{Z}} D^m T^n\psi, \qquad m \in \mathbb{Z}.$$

(Recall: invertible operators, in particular, unitary operators, can be moved inside the "closed span" symbol — cf. [9, Corollary 3].) Since $\{D^m T^n\psi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is a double-indexed orthonormal basis for $\mathcal{H}$, we get the next wavelet expansion for $\mathcal{H}$.

$$\mathcal{H} = \bigvee_{m\in\mathbb{Z}} \mathcal{W}_m = \bigvee_{m\in\mathbb{Z}} \bigvee_{n\in\mathbb{Z}} D^m T^n\psi \;=\; \bigvee_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} D^m T^n\psi,$$

where $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ is a family of pairwise orthogonal subspaces of $\mathcal{H}$ that spans $\mathcal{H}$, thus yielding an orthogonal direct sum decomposition of $\mathcal{H}$ into $\{\mathcal{W}_m\}$:

$$\mathcal{H} = \bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m. \tag{2}$$

It will be convenient to split the family $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ into two orthogonal subfamilies of pairwise orthogonal subspaces $\{\mathcal{W}_m^-\}_{m<0}$ and $\{\mathcal{W}_m^+\}_{m\geq 0}$, a negatively and a nonnegatively indexed,

$$\mathcal{W}_m^- = D^m \bigvee_{n\in\mathbb{Z}} T^n\psi = \bigvee_{n\in\mathbb{Z}} D^m T^n\psi, \qquad m < 0,$$

$$\mathcal{W}_m^+ = D^m \bigvee_{n\in\mathbb{Z}} T^n\psi = \bigvee_{n\in\mathbb{Z}} D^m T^n\psi, \qquad m \geq 0,$$

so that the above orthogonal direct sum decomposition of $\mathcal{H}$ can be written as

$$\mathcal{H} = \bigoplus_{m<0} \mathcal{W}_m^- \oplus \bigoplus_{m\geq 0} \mathcal{W}_m^+.$$

Now applying (1) we put

$$\mathcal{R}_m = \bigvee_{n\in\mathbb{Z}} T^n D^m\psi = \bigvee_{n\in\mathbb{Z}} D^m T^{n2^m}\psi = D^m \bigvee_{n\in\mathbb{Z}} T^{n2^m}\psi, \qquad m \geq 0.$$

This is defined just for $m \geq 0$ because $\{D^m T^{n2^m}\psi\}_{m\geq 0}$ is a subset of the orthonormal basis $\{D^m T^n\psi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ only if $n2^m$ is an integer for every $n \in \mathbb{Z}$ and, therefore, only for nonnegative integers m. It is clear (by their own definition) that each $\mathcal{R}_m$ is a subspace of $\mathcal{W}_m$,

$$\mathcal{R}_m \subset \mathcal{W}_m^+ = \mathcal{W}_m, \qquad m \geq 0,$$

and so $\{\mathcal{R}_m\}_{m\geq 0}$ is a family of pairwise orthogonal subspaces. Next consider the orthogonal complement of each $\mathcal{R}_m$ in $\mathcal{W}_m^+$. For each nonnegative integer $m \geq 0$, take the set $\mathbb{Z}_m = \{k \in \mathbb{Z}\colon\; k \neq n2^m \text{ for every } n \in \mathbb{Z}\}$. Recalling that $\{D^m T^n\psi\}$, with $(m,n) \in \mathbb{Z}\times\mathbb{Z}$, makes an orthogonal set, put

$$\mathcal{R}'_m = \mathcal{W}_m^+ \ominus \mathcal{R}_m = D^m\Big(\bigvee_{n\in\mathbb{Z}} T^n\psi \ominus \bigvee_{n\in\mathbb{Z}} T^{n2^m}\psi\Big) = \bigvee_{k\in\mathbb{Z}_m} D^m T^k\psi, \qquad m \geq 0,$$

so that

$$\mathcal{W}_m^+ = \mathcal{R}_m \oplus \mathcal{R}'_m, \qquad m \geq 0,$$

where $\{\mathcal{R}'_m\}_{m\geq 0}$ is a family of pairwise orthogonal subspaces (since $\{\mathcal{W}^+_m\}_{m\geq 0}$ is). Hence the previous orthogonal direct sum decomposition of $\mathcal{H}$ can be rewritten as

$$\mathcal{H} = \bigoplus_{m<0} \mathcal{W}^-_m \,\oplus\, \bigoplus_{m\geq 0} \mathcal{R}_m \,\oplus\, \bigoplus_{m\geq 0} \mathcal{R}'_m. \tag{3}$$

Next we turn to another kind of subspace, where the spans run over m, rather than over n, as follows. For each $n \in \mathbb{Z}$ we now associate with a wavelet $\psi \in \mathcal{H}$ another infinite-dimensional subspace of $\mathcal{H}$, namely,

$$\mathcal{H}_n = \bigvee_{m\in\mathbb{Z}} D^m T^n \psi \qquad n \in \mathbb{Z}.$$

Since $\{D^m T^n \psi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is a double-indexed orthonormal basis for $\mathcal{H}$, we get another wavelet expansion for $\mathcal{H}$,

$$\mathcal{H} = \bigvee_{n\in\mathbb{Z}} \mathcal{H}_n = \bigvee_{n\in\mathbb{Z}} \bigvee_{m\in\mathbb{Z}} D^m T^n \psi \;=\; \bigvee_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} D^m T^n \psi,$$

where $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a family of orthogonal subspaces of $\mathcal{H}$ that spans $\mathcal{H}$, thus yielding another orthogonal direct sum decomposition of $\mathcal{H}$, now into $\{\mathcal{H}_n\}$:

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n. \tag{4}$$

Finally, consider the following subspace of $\mathcal{H}$ (thus a Hilbert space itself).

$$\mathcal{M} \;=\; \bigvee_{(m,n)\in\mathbb{N}_0\times\mathbb{Z}} T^n D^m \psi \;=\; \bigvee_{m\in\mathbb{N}_0} \bigvee_{n\in\mathbb{Z}} T^n D^m \psi = \bigvee_{n\in\mathbb{Z}} \bigvee_{m\in\mathbb{N}_0} T^n D^m \psi \;=\; \bigvee_{m\in\mathbb{N}_0} \mathcal{R}_m.$$

(Here $\mathbb{N}_0$ denotes the set of all nonnegative integers.) Once again, recalling that $\{D^m T^n \psi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is a double-indexed orthonormal basis for $\mathcal{H}$, it follows that $\{\mathcal{R}_m\}_{m\geq 0}$ is a family of orthogonal subspaces of $\mathcal{M}$ that span $\mathcal{M}$, thus yielding an orthogonal direct sum decomposition of $\mathcal{M}$ into $\{\mathcal{R}_m\}$:

$$\mathcal{M} \;=\; \bigoplus_{m\geq 0} \mathcal{R}_m \;\subset\; \bigoplus_{m\geq 0} \mathcal{W}^+_m. \tag{5}$$

3. Wavelets Decompositions

To begin with we recall some known decompositions for the bilateral shifts D and T. The first one exhibits a decomposition for the bilateral shift D.

Lemma 1. *Each $\mathcal{H}_n$ reduces D so that*

$$D = \bigoplus_{n\in\mathbb{Z}} D|_{\mathcal{H}_n},$$

where each $D|_{\mathcal{H}_n}$ is a bilateral shift of multiplicity 1 on each Hilbert space $\mathcal{H}_n$.

Proof. See [9, Theorem 1]. □

A counterpart of the above result, now exhibiting a similar decomposition for the bilateral shift T, goes as follows. (Here $\mathcal{M}^\perp$ stands for the orthogonal complement of $\mathcal{M}$; that is, $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$.)

Lemma 2. *Each $\mathcal{R}_m$ reduces T, and so does $\mathcal{M}$. Hence*

$$T = \bigoplus_{m\geq 0} T|_{\mathcal{R}_m} \;\oplus\; T|_{\mathcal{M}^\perp},$$

where each $T|_{\mathcal{R}_m}$ is a bilateral shift of multiplicity 1 on each Hilbert space $\mathcal{R}_m$.

Proof. See [13, Theorem 1]. □

We show next that $T|_{\mathcal{M}^\perp}$ is a bilateral shift.

Theorem 1. *$T|_{\mathcal{M}^\perp}$ is a bilateral shift acting on $\mathcal{M}^\perp$.*

Proof. First we need the following general result on orthogonal direct summands of bilateral shifts on Hilbert spaces. Let A be an operator acting on a Hilbert space.

Claim 1: If $A = B \oplus C$, where A and B are bilateral shifts, then C is a bilateral shift as well.

Indeed, recall that every unitary operator U can be decomposed as $U = S \oplus W$, where S is a bilateral shift and W is a reductive unitary operator (see e.g., [3, p.18] — an operator is reductive if all its invariant subspaces are reducing). Note that a bilateral shift has no reductive direct summand. In fact, if U is a bilateral shift and if $\mathcal{N} \neq \{0\}$ is W-invariant, then $W(\mathcal{N}) = \mathcal{N}$ (because W is a reductive unitary) so that $U(\mathcal{N}) = \mathcal{N}$, which is a contradiction (because a bilateral shift has no eigenvalue). Since a bilateral shift is nonreductive, a unitary operator is non-reductive if and only if it has a bilateral shift as a direct summand. Also recall that any direct sum of bilateral shifts is again a bilateral shift. Now suppose

$$A = B \,\oplus\, C,$$

where A and B are bilateral shifts. Since direct summands of a unitary operator are again unitary, it follows that C is unitary. If C is not a bilateral shift, then it has a (nonzero) reductive direct summand W so that $C = S \oplus W$, where S is a bilateral shift. Thus $A = B \oplus S \oplus W$, with $B \oplus S$ being a bilateral shift, has the same reductive direct summand W, and therefore A is not a bilateral shift, which is a contradiction. Outcome: C must be a bilateral shift, thus concluding the proof of the above claimed result.

Therefore, since T is a bilateral shift such that (cf. Lemma 2)

$$T = \bigoplus_{m\geq 0} T|_{\mathcal{R}_m} \;\oplus\; T|_{\mathcal{M}^\perp},$$

where $\bigoplus_{m\geq 0} T|_{\mathcal{R}_m}$ is a bilateral shift, $T|_{\mathcal{M}^\perp}$ also is a bilateral shift by Claim 1. □

Since $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ is a family of pairwise orthogonal subspaces of $\mathcal{H}$, it follows by (3) and (5) that $\mathcal{M}^\perp$ admits the orthogonal direct sum decomposition

$$\mathcal{M}^\perp = \mathcal{M}_1^\perp \;\oplus\; \mathcal{M}_2^\perp,$$

with

$$\mathcal{M}_1^\perp = \bigoplus_{m\geq 0} \mathcal{R}'_m \quad \text{and} \quad \mathcal{M}_2^\perp = \bigoplus_{m<0} \mathcal{W}_m^- \tag{6}$$

which are subspaces of $\mathcal{H}$, thus Hilbert spaces themselves.

Theorem 2. *Each $\mathcal{R}'_m$ reduces T, and hence both $\mathcal{M}_1^\perp$ and $\mathcal{M}_2^\perp$ reduce T so that*

$$T|_{\mathcal{M}^\perp} = T|_{\mathcal{M}_1^\perp} \oplus T|_{\mathcal{M}_2^\perp}.$$

Proof. Take an arbitrary nonnegative integer $m \geq 0$ so that $n \pm 2^m \in \mathbb{Z}$ for every $n \in \mathbb{Z}$. Consider the set $\mathbb{Z}_m = \{k \in \mathbb{Z}\colon\ k \neq n2^m \text{ for every } n \in \mathbb{Z}\}$, and note that

$$k \in \mathbb{Z}_m \quad \Longrightarrow \quad k \pm 2^m \in \mathbb{Z}_m.$$

Indeed, suppose $k \neq n2^m$ for every $n \in \mathbb{Z}$. If $k \pm 2^m = n2^m$ for some $n \in \mathbb{Z}$, then $k = n2^m \mp 2^m = (n \mp 1)2^m$, which is a contradiction. Conversely, suppose $k \pm 2^m \neq n2^m$ for every$n \in \mathbb{Z}$. If $k = n2^m$ for some $n \in \mathbb{Z}$, then $k \pm 2^m = n2^m \pm 2^m = (n \pm 1)2^m$, which is again a contradiction. Thus

$$k \pm 2^m \in \mathbb{Z}_m \quad \Longrightarrow \quad k \in \mathbb{Z}_m.$$

Outcome:

$$k \in \mathbb{Z}_m \quad \Longleftrightarrow \quad k \pm 2^m \in \mathbb{Z}_m.$$

Therefore, according to (1),

$$\begin{aligned} T(\mathcal{R}'_m) &= \bigvee_{k\in\mathbb{Z}_m} TD^mT^k\psi = \bigvee_{k\in\mathbb{Z}_m} D^mT^{2^m}T^k\psi \\ &= \bigvee_{k\in\mathbb{Z}_m} D^mT^{(k+2^m)}\psi = \bigvee_{k\in\mathbb{Z}_m} D^mT^k\psi = \mathcal{R}'_m. \end{aligned}$$

Since T is unitary (i.e., since $T^* = T^{-1}$), this implies that

$$T(\mathcal{R}'_m) = \mathcal{R}'_m = T^*(\mathcal{R}'_m)$$

and so $\mathcal{R}'_m$ reduces T, which in turn implies that $\mathcal{M}_1^\perp = \bigoplus_{m\geq 0}\mathcal{R}'_m$ also reduces T. Since $\mathcal{M}_2^\perp$ is the orthogonal complement of $\mathcal{M}_1^\perp$ in $\mathcal{M}^\perp$ (i.e., $\mathcal{M}_2^\perp = \mathcal{M}^\perp \ominus \mathcal{M}_1^\perp$), and since both $\mathcal{M}^\perp$ and $\mathcal{M}_1^\perp$ reduce T, it follows that $\mathcal{M}_2^\perp$ reduces T as well. $\square$

Recall from the proof of Theorem 2 that $k \pm 2^m \in \mathbb{Z}_m$ if and only if $k \in \mathbb{Z}_m$, and so a trivial induction ensures that $k + j2^m \in \mathbb{Z}_m$, for each $j \in \mathbb{Z}$, if and only if $k \in \mathbb{Z}_m$. Thus, for each $m \geq 0$ and each $k \in \mathbb{Z}_m$, consider the subspace

$$\mathcal{R}'_{m,k} = \bigvee_{j\in\mathbb{Z}} \psi_{m,k+j2^m} \subset \bigvee_{k\in\mathbb{Z}_m} \psi_{m,k} = \mathcal{R}'_m,$$

Note by the above argument that

$$\bigvee_{k\in\mathbb{Z}_m} \mathcal{R}'_{m,k} = \bigvee_{k\in\mathbb{Z}_m}\bigvee_{j\in\mathbb{Z}} \psi_{m,k+j2^m} = \bigvee_{(j,k)\in\mathbb{Z}\times\mathbb{Z}_m} \psi_{m,k+j2^m} = \bigvee_{k\in\mathbb{Z}_m} \psi_{m,k} = \mathcal{R}'_m.$$

However, it is worth noticing that $\{\mathcal{R}'_{m,k}\}_{k\in\mathbb{Z}_m}$ is *not* a family of pairwise orthogonal subspaces; that is, for each $m \geq 0$ the subspace $\mathcal{R}'_{m,k}$ is not orthogonal to $\mathcal{R}'_{m,k'}$ if $k \neq k'$. Indeed, for any $k \in \mathbb{Z}_m$, it follows that $\psi_{m,k+2^m} \in \mathcal{R}'_{m,k}$ and $\psi_{m,k'} \in \mathcal{R}'_{m,k'}$ coincide if $k' = k + 2^m \in \mathbb{Z}_m$. Thus $\mathcal{R}'_m$ is *not* the *orthogonal* direct sum of $\mathcal{R}'_{m,k}$ over all $k \in \mathbb{Z}_m$ (in fact, it does not make sense talking about the orthogonal direct sum $\bigoplus_{k\in\mathbb{Z}_m}\mathcal{R}'_{m,k}$ because $\mathcal{R}'_{m,k} \not\perp \mathcal{R}'_{m,k'}$ whenever $k \neq k'$).

Theorem 3. *Each $\mathcal{R}'_{m,k}$ reduces T and each $T|_{\mathcal{R}'_{m,k}}$ is a bilateral shift of multiplicity 1 on $\mathcal{R}'_{m,k}$.*

Proof. Take any $m \geq 0$, any $k \in \mathbb{Z}_m$, and observe that

$$T(\mathcal{R}'_{m,k}) = \bigvee_{j\in\mathbb{Z}} TD^mT^{k+j2^m} = \bigvee_{j\in\mathbb{Z}} D^mT^{2m}T^{k+j2^m} = \bigvee_{j\in\mathbb{Z}} D^mT^{k+(j+1)2^m} = \mathcal{R}'_{m,k}.$$

Since T is unitary (i.e., since $T^* = T^{-1}$), this implies that

$$T(\mathcal{R}'_{m,k}) = \mathcal{R}'_{m,k} = T^*(\mathcal{R}'_{m,k})$$

and so each subspace $\mathcal{R}'_{m,k}$ reduces T. Moreover, for an arbitrary $j \in \mathbb{Z}$,

$$T\psi_{m,k+j2^m} = TD^mT^{k+j2^m}\psi = D^mT^{2^m}T^{k+j2^m}\psi = D^mT^{k+(j+1)2^m}\psi = \psi_{m,k+(j+1)2^m}.$$

Since for fixed $m \geq 0$ and fixed $k \in \mathbb{Z}_m$ the set $\{\psi_{m,k+j2^m}\}_{j\in\mathbb{Z}}$ consists of orthogonal vectors, it is an orthonormal basis for each subspace $\mathcal{R}'_{m,k}$. Thus each $T|_{\mathcal{R}'_{m,k}}$ shifts (sequentially) a $\mathbb{Z}$-indexed orthonormal basis for each $\mathcal{R}'_{m,k}$, which means that $T|_{\mathcal{R}'_{m,k}}$ is a bilateral shift of multiplicity 1 acting on $\mathcal{R}'_{m,k}$. □

Lemma 2 ensures that the restriction of T to some subspaces of $\mathcal{W}_m$ (i.e., the restriction of T to each $\mathcal{R}_m$) is a bilateral shift of multiplicity 1. The next theorem shows that some powers (integer or fractional) of T in fact are bilateral shifts of multiplicity 1 when restricted to every $\mathcal{W}_m$.

Theorem 4. *Each $\mathcal{W}_m$ reduces $T^{\frac{1}{2^m}}$ and each $T^{\frac{1}{2^m}}|_{\mathcal{W}_m}$ is a bilateral shift of multiplicity 1 on the Hilbert space $\mathcal{W}_m$.*

Proof. Take an arbitrary $m \in \mathbb{Z}$. It follows from [9, Proposition 5(b,b*)] that

$$\mathcal{W}_m = T^{\frac{1}{2^m}}(\mathcal{W}_m) = T^{\frac{1}{2^m}*}(\mathcal{W}_m), \tag{7}$$

and hence $\mathcal{W}_m$ is $T^{\frac{1}{2^m}}$-invariant and also $T^{\frac{1}{2^m}*}$-invariant, which means that $\mathcal{W}_m$ reduces T. Note that $\{\psi_{m,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for the Hilbert space $\mathcal{W}_m$ since it spans $\mathcal{W}_m$. Take any $\psi_{m,n}$ (for some $n \in \mathbb{Z}$). Observe that

$$T^{\frac{1}{2^m}}|_{\mathcal{W}_m}\psi_{m,n} = T^{\frac{1}{2^m}}\psi_{m,n} = T^{\frac{1}{2^m}}D^mT^n\psi.$$

However, it follows from (1) that

$$T^{\frac{1}{2^m}}D^m = D^mT. \tag{8}$$

Indeed, by setting $n = 1$, replacing m with $-m$, and multiplying by D^m (both from left and from right) the expression in (1) we get the identity in (8). Therefore,

$$T^{\frac{1}{2^m}}|_{\mathcal{W}_m}\psi_{m,n} = D^mTT^nw = D^mT^{n+1}\psi = \psi_{m,n+1}.$$

Thus $T^{\frac{1}{2^m}}|_{\mathcal{W}_m}$ shifts (sequentially) a $\mathbb{Z}$-indexed orthonormal basis for $\mathcal{W}_m$, which means that $T^{\frac{1}{2^m}}|_{\mathcal{W}_m}$ is a bilateral shift of multiplicity 1 acting on $\mathcal{W}_m$. □

Corollary 1. *Take arbitrary integers m and n in $\mathbb{Z}$, and i in $\mathbb{N}_0$. The operators $D|_{\mathcal{H}_n}$, $T|_{\mathcal{R}_i}$, and $T^{\frac{1}{2^m}}|_{\mathcal{W}_m}$ are unitarily equivalent to each other.*

Proof. Shifts of the same multiplicity are unitarily equivalent (e.g., see [7, Chapter 2]). Thus the claimed results follow by Lemmas 1, 2 and Theorem 4. □

The spaces $\mathcal{W}_m$, $\mathcal{R}_m$ and $\mathcal{H}_n$ are trivially unitarily equivalent, since they have the same countably infinite dimension — orthonormal bases with the same cardinality. However, constructing unitary transformations between these spaces that make those bilateral shifts (of multiplicity 1) unitarily equivalent may turn out to be a recalcitrant task. A result along these lines is considered in the next section.

4. Constructing the Unitary Equivalence Between $\mathcal{H}_n$ and $\mathcal{R}_m$

Take an arbitrary vector $h_n \in \mathcal{H}_n$, which can be expressed by

$$h_n = \sum_{k\in\mathbb{Z}} \alpha_k D^k T^n \psi,$$

where $\{\alpha_k\}_{k\in\mathbb{Z}}$ is a square-summable family of scalars:

$$\sum_{k\in\mathbb{Z}} |\alpha_k|^2 = \|h_n\|^2 < \infty.$$

Consider the transformation that assigns, to each h_n, the vector

$$\Phi(n,m)h_n = \sum_{k\in\mathbb{Z}} \alpha_k T^k D^m \psi.$$

This defines a surjective linear transformation between $\mathcal{H}_n$ and $\mathcal{R}_m$,

$$\Phi(n,m)\colon \mathcal{H}_n \to \mathcal{R}_m$$

(for arbitrary $n \in \mathbb{Z}$ and $m \in \mathbb{N}_0$). Indeed, $\Phi(n,m)$ is trivially linear on $\mathcal{H}_n$ and clearly surjective onto $\mathcal{R}_m$.

Theorem 5. *$\Phi(n,m)$ is unitary and intertwines $D|_{\mathcal{H}_n}$ with $T|_{\mathcal{R}_m}$:*

$$\Phi(n,m)D|_{\mathcal{H}_n} = T|_{\mathcal{R}_m}\Phi(n,m).$$

Proof. Take any integer $n \in \mathbb{Z}$, and an arbitrary $h_n \in \mathcal{H}_n$. Since

$$\|\Phi(n,m)h_n\|^2 = \sum_{k\in\mathbb{Z}} |\alpha_k|^2 = \|h_n\|^2,$$

it follows that the surjective linear transformation $\Phi(n,m)$ also is an isometry, and therefore $\Phi(n,m)$ is unitary. Now observe that

$$Dh_n = \sum_{k\in\mathbb{Z}} \alpha_k D^{k+1} T^n \psi = \sum_{k\in\mathbb{Z}} \alpha_{k-1} D^k T^n \psi,$$

$$\Phi(n,m)Dh_m = \sum_{k\in\mathbb{Z}} \alpha_{k-1} T^k D^m \psi,$$

$$T\Phi(n,m)h_n = \sum_{k\in\mathbb{Z}} \alpha_k T^{k+1} D^m \psi = \sum_{k\in\mathbb{Z}} \alpha_{k-1} T^k D^m \psi$$

so that

$$\Phi(n,m)Dh_n = T\Phi(n,m)h_n.$$

Since $\mathcal{H}_n$ reduces D and $\mathcal{R}_m$ reduces T (cf. Lemmas 1 and 2), the above expression ensures that $\Phi(n,m)D|_{\mathcal{H}_n} = T|_{\mathcal{R}_m}\Phi(n,m)$. □

Observe that the operators

$$D|_{\mathcal{H}_n}: \mathcal{H}_n \to \mathcal{H}_n \quad \text{and} \quad T|_{\mathcal{R}_m}: \mathcal{R}_m \to \mathcal{R}_m$$

are clearly unitarily equivalent (since they are bilateral shifts of the same multiplicity 1). The above proposition exhibits the unitary transformation

$$\Phi(n,m): \mathcal{H}_n \to \mathcal{R}_m$$

that carries out such a unitary equivalence, showing *how* these operators are, in fact, unitarily equivalent.

References

1. I. Antoniou and K. Gustafson, *Wavelets and stochastic processes*, Math. Comput. Simulation **49** (1999), 81–104.
2. X. Dai and D.R. Larson, *Wandering Vectors for Unitary Systems and Orthogonal Wavelets*, Mem. Amer. Math. Soc. Vol. 134, no. 640, Providence, 1998.
3. P.A. Fillmore, *Notes on Operator Theory*, Van Nostrand, New York, 1970.
4. T.N.T. Goodman, S.L. Lee and W.S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc. **338** (1993), 639–654.
5. K. Gustafson, *Wavelets and expectations: a different path to wavelets*, Harmonic, Wavelet and p-Adic Analysis (Eds. N.M. Chuong. Yu V. Egorov, A. Khrennikov, Y. Meyer, D. Mumford), World Scientific, Hackensack, NJ (2007), 5–22.
6. P.R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.
7. C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997.
8. C.S. Kubrusly and N. Levan, *On generating wandering subspaces for unitary operators*, Adv. Math. Sci. Appl. **14** (2004), 41–48.
9. C.S. Kubrusly and N. Levan, *Abstract wavelets generated by Hilbert space shift operators*, Adv. Math. Sci. Appl. **16** (2006), 643–660.
10. N. Levan and C.S. Kubrusly, *A wavelet "time-shift-detail" decomposition*, Math. Comput. Simulation **63** (2003), 73–78.
11. N. Levan and C.S. Kubrusly, *Time-shifts generalized multiresolution analysis over dyadic-scaling reducing subspaces*, Int. J. Wavelets Multiresolut. Inf. Process. **2** (2004), 237–248.
12. N. Levan and C.S. Kubrusly, *Multiresolution approximation scale and time-shift subspaces*, Multidimens. Syst. Signal Process. **17** (2006), 343–354.
13. N. Levan and C.S. Kubrusly, *Reversed wavelet functions and subspaces*, Int. J. of Wavelets, Multiresolut. Inf. Process. **5** (2007), 699–707.
14. S.G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$*, Trans. Amer. Math. Soc. **315** (1989), 69–87.
15. J.B. Robertson, *On wandering subspaces for unitary operators*, Proc. Amer. Math. Soc. **16** (1965), 233–236.

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