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A CONCISE INTRODUCTION TO TENSOR PRODUCT

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Abstract. A brief introduction to tensor product in Hilbert space is organized into five sections. The inner product space $\mathcal{H}\otimes\mathcal{K}$ of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, as well as the tensor product $A\otimes B$ of operators A and B are discussed in § 1. Their completion, leading to the concept of tensor product space, is the topic of § 2. Kronecker product comprises the subject of § 3, and commutativity of the tensor product of operators is focused in § 4. Finally, tensor product for classes of operators closes this expository paper in § 5.

1. The Inner Product Space $\mathcal{H}\otimes\mathcal{K}$

Let $\mathcal{H}$ and $\mathcal{K}$ be nonzero complex Hilbert spaces with $\langle\ ;\ \rangle_{\mathcal{H}}$ and $\langle\ ;\ \rangle_{\mathcal{K}}$ denoting their respective inner products (defined to be linear in the first argument). We shall define the concept of tensor product space in terms of the single tensor product of vectors as a conjugate bilinear functional on the Cartesian product of $\mathcal{H}$ and $\mathcal{K}$. (see e.g., [6] and [13]). For an abstract approach see e.g., [3] and [17].

Definition 1. The *single tensor product* of $x\in\mathcal{H}$ and $y\in\mathcal{K}$ is a conjugate bilinear functional

$$x\otimes y\colon \mathcal{H}\times\mathcal{K}\to\mathbb{C}$$

defined by

$$(x\otimes y)\,(u,v)=\langle x\,;u\rangle_{\mathcal{H}}\,\langle y\,;v\rangle_{\mathcal{K}}\quad\text{for every}\quad(u,v)\in\mathcal{H}\times\mathcal{K},$$

Remark 1. Recall that the *outer product* of a pair of vectors $x\in\mathcal{H}$ and $y\in\mathcal{K}$ is a rank-one bounded linear transformation of $\mathcal{K}$ into $\mathcal{H}$

$$x\odot y\colon\mathcal{K}\to\mathcal{H}$$

defined by

$$(x\odot y)v=\langle v\,;y\rangle_{\mathcal{K}}\,x\quad\text{for every}\quad v\in\mathcal{K}.$$

Thus $x\odot y$ has range equal to $\operatorname{span}\{x\}$, lies in $\mathcal{B}[\mathcal{H},\mathcal{K}]$, and

$$\|x\odot y\|=\|x\|_{\mathcal{H}}\,\|y\|_{\mathcal{K}}$$

for $\|x\odot y\|=\sup_{\|v\|_{\mathcal{K}}=1}\|(x\odot y)v\|=\|x\|\,\sup_{\|v\|_{\mathcal{K}}=1}|\langle v\,;y\rangle_{\mathcal{K}}|=\|x\|_{\mathcal{H}}\,\|y\|_{\mathcal{K}}$. Also,

$$(x\odot y)^*=y\odot x$$

as $\langle (x\odot y)v\,;u\rangle_{\mathcal{H}}=\langle v\,;y\rangle_{\mathcal{K}}\,\langle x\,;u\rangle_{\mathcal{H}}=\langle v\,;(y\odot x)u\rangle_{\mathcal{H}}$ for (u,v) in $\mathcal{H}\times\mathcal{K}$. Moreover,

$$\langle (y\odot u)x\,;v\rangle_{\mathcal{K}}=\langle x\,;u\rangle_{\mathcal{H}}\langle y\,;v\rangle_{\mathcal{K}}=(x\otimes y)(u,v)\quad\text{for every}\quad(u,v)\in\mathcal{H}\times\mathcal{K}$$

so that single tensor products and outer products are related as follows.

$$(x\otimes y)(\,\cdot\,,\,\cdot\,)=\langle (y\odot\,\cdot\,)x\,;\,\cdot\,\rangle_{\mathcal{K}}\quad\text{on}\quad\mathcal{H}\times\mathcal{K}.$$

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It is worth noticing that, since $\mathcal{H}$ and $\mathcal{K}$ are linear spaces over the same field, the Cartesian product $\mathcal{H}\times\mathcal{K}$ can also be made a linear space, still over the same scalar field, if vector addition and scalar multiplication are coordinatewise defined:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2\,,\, y_1 + y_2) \quad\text{ and }\quad \alpha(x, y) = (\alpha x, \alpha y)$$

for every (x_1, y_1), (x_2, y_2) and (x, y) in $\mathcal{H}\times\mathcal{K}$ and every α in $\mathbb{C}$.

Remark 2. With scalar multiplication of functionals on $\mathcal{H}\times\mathcal{K}$ defined pointwise, and since $\mathcal{H}$ and $\mathcal{K}$ are linear spaces (over the same scalar field), it follows that scalar multiplication of a single tensor is again a single tensor. Indeed,

$$\alpha\beta(x\otimes y) = \alpha x\otimes\beta y = \beta x\otimes\alpha y \ \text{ for every }\ \alpha,\beta\in\mathbb{C}\ \text{ and every }\ (x, y)\in\mathcal{H}\times\mathcal{K},$$

where the above expressions, namely, $\alpha\beta(x\otimes y)$, $\alpha x\otimes\beta y$ and $\beta x\otimes\alpha y$, are different notations for the same conjugate bilinear functional $(u, v)\mapsto\alpha\beta\,\langle x\,;u\rangle_{\mathcal{H}}\,\langle y\,;v\rangle_{\mathcal{K}}$. (This is similar to the fact that the function that shifts every real number γ by the unity can be equivalently expressed either by $\gamma\mapsto\frac{1}{2}(2\gamma-2)$ or simply by $\gamma\mapsto\gamma-1$.) In other words, pointwise equality holds for the functionals $\alpha x\otimes\beta y$ and $\beta x\otimes\alpha y$ on the same domain $\mathcal{H}\times\mathcal{K}$, and hence this unique single tensor product may be represented by many distinct pairs of vectors from $\mathcal{H}\times\mathcal{K}$ (such as $(\alpha x, \beta y)$, $(\beta x, \alpha y)$, $(\alpha\beta x, y)$, $(x, \alpha\beta y)$, and so on) — the natural map $(x, y)\mapsto x\otimes y$ from the Cartesian product $\mathcal{H}\times\mathcal{K}$ to the collection of all conjugate bilinear functionals of the single tensor type is not injective.

The single tensor product operation deserves its name; it is distributive with respect to addition (i.e., with respect to pointwise defined addition of the associated conjugate bilinear functionals). Indeed, for every x, w in $\mathcal{H}$ and y, z in $\mathcal{K}$,

$$x\otimes(y+z) = x\otimes y + x\otimes z \qquad\text{ and }\qquad (x+w)\otimes y = x\otimes y + w\otimes y \tag{1}$$

so that

$$(x+w)\otimes(y+z) = x\otimes y + w\otimes y + x\otimes z + w\otimes z,$$

and hence addition of single tensors is not necessarily a single tensor:

$$x\otimes y + w\otimes z = (x+w)\otimes(y+z) - x\otimes z - w\otimes y.$$

Thus consider the collection

$$\mathcal{H}\otimes\mathcal{K} = \Big\{\sum_{i=1}^{n}\alpha_i(x_i\otimes y_i):\ \alpha_i\in\mathbb{C},\ \ (x_i, y_i)\in\mathcal{H}\times\mathcal{K},\ \ n\in\mathbb{N}\Big\} \tag{2.a}$$

of all (finite) linear combinations of single tensors on $\mathcal{H}\times\mathcal{K}$, which is clearly a linear space (over the same complex filed $\mathbb{C}$) — addition of a couple of elements from $\mathcal{H}\otimes\mathcal{K}$ is an element of $\mathcal{H}\otimes\mathcal{K}$. The origin of the linear space $\mathcal{H}\otimes\mathcal{K}$ is the single tensor product $0\otimes 0$, which coincides with $x\otimes 0$ or $0\otimes y$ according to Remark 2. Actually, we saw in Remark 2 that scalar multiplication of single tensors are again single tensors, and also that $\alpha_i(x_i\otimes y_i)$ in (2.a) can be written as $\alpha_i x_i\otimes y_i$, or $x_i\otimes\alpha_i y_i$, or even as ${\alpha_i}^{\frac{1}{2}}x_i\otimes{\alpha_i}^{\frac{1}{2}}y_i$. Then the definition of the linear space $\mathcal{H}\otimes\mathcal{K}$ in (2.a) can be equivalently written as

$$\mathcal{H}\otimes\mathcal{K} = \Big\{\sum_{i=1}^{n}x_i\otimes y_i:\ (x_i, y_i)\in\mathcal{H}\times\mathcal{K},\ \ n\in\mathbb{N}\Big\}. \tag{2.b}$$

Again, with addition and scalar multiplication defined pointwise we can show that, for arbitrary α_i, β_j in $\mathbb{C}$, (x_i, y_j) in $\mathcal{H}\times\mathcal{K}$, and n, m in $\mathbb{N}$,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j\,(x_i\otimes y_j)\,(u,v) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j\,\langle x_i\,;u\rangle_{\mathcal{H}}\,\langle y_j\,;v\rangle_{\mathcal{K}}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\langle\alpha_i x_i\,;u\rangle_{\mathcal{H}}\,\langle\beta_j y_j\,;v\rangle_{\mathcal{K}} = \Big\langle\sum_{i=1}^{n}\alpha_i x_i\,;u\Big\rangle_{\mathcal{H}}\Big\langle\sum_{j=1}^{m}\beta_j y_j\,;v\Big\rangle_{\mathcal{K}}\\ &= \Big[\Big(\sum_{i=1}^{n}\alpha_i x_i\Big)\otimes\Big(\sum_{j=1}^{m}\beta_j y_j\Big)\Big](u,v) \quad \text{for every} \quad (u,v)\in\mathcal{H}\times\mathcal{K}.\end{aligned}$$

Therefore,

$$\Big(\sum_{i=1}^{n}\alpha_i x_i\Big)\otimes\Big(\sum_{j=1}^{m}\beta_j y_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j\,(x_i\otimes y_j) \tag{3}$$

for every α_i, β_j in $\mathbb{C}$, every (x_i, y_j) in $\mathcal{H}\times\mathcal{K}$, and every n, m in $\mathbb{N}$. Both sides of the above equation clearly lie in $\mathcal{H}\otimes\mathcal{K}$ since they are linear combination of single tensors; the one at the left-hand side being a trivial linear combination of just one single tensor $\big(\sum_{i=1}^{n}\alpha_i x_i\big)\otimes\big(\sum_{j=1}^{m}\beta_j y_j\big)$ and the other at the right-hand side being a linear combination of possibly many single tensors $\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j(x_i\otimes y_j)$.

Now consider the functional

$$\langle\ ;\ \rangle\colon(\mathcal{H}\otimes\mathcal{K})\times(\mathcal{H}\otimes\mathcal{K})\to\mathbb{C}$$

defined, on the Cartesian product $(\mathcal{H}\otimes\mathcal{K})\times(\mathcal{H}\otimes\mathcal{K})$ of the linear space $\mathcal{H}\otimes\mathcal{K}$ with itself, as follows.

$$\Big\langle\sum_{i=1}^{n}\alpha_i(x_i\otimes y_i)\,;\sum_{j=1}^{m}\beta_j(w_j\otimes z_j)\Big\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\overline{\beta}_j\,\langle x_i\,;w_j\rangle_{\mathcal{H}}\,\langle y_i\,;z_j\rangle_{\mathcal{K}} \tag{4}$$

for arbitrary $\sum_{i=1}^{n}\alpha_i(x_i\otimes y_i)$ and $\sum_{j=1}^{m}\beta_j(w_j\otimes z_j)$ in $\mathcal{H}\otimes\mathcal{K}$. In particular,

$$\langle x\otimes y\,;w\otimes z\rangle = \langle x\,;w\rangle_{\mathcal{H}}\langle y\,;z\rangle_{\mathcal{K}} = (x\otimes y)\,(w,z)$$

for every x, w in $\mathcal{H}$ and y, z in $\mathcal{K}$. Since single tensors are *conjugate* bilinear functionals, it is easy to show that the functional $\langle\ ;\ \rangle$ is a Hermitian sesquilinear form on $(\mathcal{H}\otimes\mathcal{K})\times(\mathcal{H}\otimes\mathcal{K})$. To verify that it is an inner product on the linear space $\mathcal{H}\otimes\mathcal{K}$, we need to show that it induces a positive quadratic form.

Proposition 1. *The sesquilinear form defined in* (4) *is an inner product on* $\mathcal{H}\otimes\mathcal{K}$.

Proof. Let $\sum_{i=1}^{n}x_i\otimes y_i$ be an arbitrary element from $\mathcal{H}\otimes\mathcal{K}$ (cf. (2.b)), and let $\{e_k\}_{k=1}^{n'}$ and $\{f_l\}_{l=1}^{n''}$ be orthonormal basis for the subspaces of $\mathcal{H}$ and $\mathcal{K}$ spanned by $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$, respectively. Thus consider the Fourier series expansions

$$x_i = \sum_{k=1}^{n'}\langle x_i\,;e_k\rangle_{\mathcal{H}}e_k \quad\text{and}\quad y_i = \sum_{l=1}^{n''}\langle y_i\,;f_l\rangle_{\mathcal{K}}f_l$$

of each x_i and y_i in the finite-dimensional spaces $\operatorname{span}\{e_k\}_{k=1}^{n'} = \operatorname{span}\{x_i\}_{i=1}^{n}\subseteq\mathcal{H}$ and $\operatorname{span}\{f_l\}_{l=1}^{n''} = \operatorname{span}\{y_i\}_{i=1}^{n}\subseteq\mathcal{K}$ so that, according to (3),

$$x_i \otimes y_i = \Big(\sum_{k=1}^{n'} \langle x_i\,;e_k\rangle_{\mathcal{H}}\, e_k\Big) \otimes \Big(\sum_{l=1}^{n''} \langle y_i\,;f_l\rangle_{\mathcal{K}}\, f_l\Big) = \sum_{k=1}^{n'}\sum_{l=1}^{n''} \langle x_i\,;e_k\rangle_{\mathcal{H}}\,\langle y_i\,;f_l\rangle_{\mathcal{K}}\,(e_k \otimes f_l).$$

Put $\gamma_{k,l} = \sum_{i=1}^{n} \langle x_i\,;e_k\rangle_{\mathcal{H}}\,\langle y_i\,;f_l\rangle_{\mathcal{K}}$ for each pair of positive integers $\{k,l\}$. Thus

$$\sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n}\sum_{k=1}^{n'}\sum_{l=1}^{n''} \langle x_i\,;e_k\rangle_{\mathcal{H}}\,\langle y_i\,;f_l\rangle_{\mathcal{K}}\,(e_k \otimes f_l) = \sum_{k=1}^{n'}\sum_{l=1}^{n''} \gamma_{k,l}\,(e_k \otimes f_l).$$

Since all sums are finite, we may reorder the final sum (and reindex the vectors in each single tensor product) to get

$$\sum_{i=1}^{n} x_i \otimes y_i = \sum_{\ell=1}^{\widehat{n}} \gamma'_\ell\,(e_\ell \otimes f_\ell) \tag{5}$$

with $\widehat{n} = n' \cdot n''$. However, according to (4),

$$\begin{aligned}\Big\|\sum_{i=1}^{n} x_i \otimes y_i\Big\|^2 &= \Big\langle \sum_{i=1}^{n} x_i \otimes y_i\,; \sum_{i=1}^{n} x_i \otimes y_i \Big\rangle = \Big\langle \sum_{l=1}^{\widehat{n}} \gamma'_l\,(e_l \otimes f_l)\,; \sum_{k=1}^{\widehat{n}} \gamma'_k\,(e_k \otimes f_k) \Big\rangle \\ &= \sum_{l=1}^{\widehat{n}}\sum_{k=1}^{\widehat{n}} \gamma'_l\,\overline{\gamma}'_k\,\langle e_l\,;e_k\rangle_{\mathcal{H}}\,\langle f_l\,;f_k\rangle_{\mathcal{K}} = \sum_{k=1}^{\widehat{n}} |\gamma'_k|^2 = \sum_{k=1}^{n'}\sum_{l=1}^{n''} |\gamma_{k,l}|^2. \end{aligned} \tag{6}$$

Therefore $\big\langle \sum_{i=1}^{n} x_i \otimes y_i\,; \sum_{i=1}^{n} x_i \otimes y_i \big\rangle$ is always nonnegative, and it is zero only if $\gamma_{l,k} = 0$ for all $\{k,l\}$, which implies $\sum_{i=1}^{n} x_i \otimes y_i = 0$ (the origin of the linear space $\mathcal{H} \otimes \mathcal{K}$). Outcome: $\langle\ ;\ \rangle$ in (4) induces a positive quadratic form $\|\ \ \|$ (the induced norm on $\mathcal{H} \otimes \mathcal{K}$) so that it is an inner product on $\mathcal{H} \otimes \mathcal{K}$. $\square$

Consider the inner product spaces $(\mathcal{H} \otimes \mathcal{K}, \langle\ ;\ \rangle)$ and $(\mathcal{K} \otimes \mathcal{H}, \langle\ ;\ \rangle)$. From (4),

$$\Big\|\sum_{i=1}^{n} x_i \otimes y_i\Big\|^2 = \Big\langle \sum_{i=1}^{n} x_i \otimes y_i\,; \sum_{j=1}^{n} x_j \otimes y_j \Big\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle x_i\,;x_j\rangle_{\mathcal{H}}\,\langle y_i\,;y_j\rangle_{\mathcal{K}}, \tag{7.a}$$

and so

$$\Big\|\sum_{i=1}^{n} x_i \otimes y_i\Big\| = \Big\|\sum_{i=1}^{n} y_i \otimes x_i\Big\|, \tag{7.b}$$

for every $\sum_{i=1}^{n} x_i \otimes y_i$ in $\mathcal{H} \otimes \mathcal{K}$. In particular,

$$\|x \otimes y\| = \|x\|_{\mathcal{H}}\,\|y\|_{\mathcal{K}}. \tag{7.c}$$

Indeed, $\|x \otimes y\|^2 = \langle x \otimes y\,; x \otimes y\rangle = \langle x\,;x\rangle_{\mathcal{H}}\,\langle y\,;y\rangle_{\mathcal{K}} = \|x\|^2_{\mathcal{H}}\,\|y\|^2_{\mathcal{K}}$ for any $x \otimes y$ in $\mathcal{H} \otimes \mathcal{K}$. Now let $\mathcal{B}[\mathcal{H}]$, $\mathcal{B}[\mathcal{K}]$ and $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ be the normed algebras of all (bounded linear) operators on $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{H} \otimes \mathcal{K}$, respectively.

Definition 2. The *tensor product* on the inner product space $\mathcal{H} \otimes \mathcal{K}$ of two operators $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ is the transformation

$$A \otimes B \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$$

of $\mathcal{H} \otimes \mathcal{K}$ into itself such that

$$(A \otimes B) \sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} Ax_i \otimes By_i \quad \text{for every} \quad \sum_{i=1}^{n} x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{K}.$$

Observe that $I\otimes I=\alpha I\otimes\alpha^{-1}I$, for any nonzero scalar α, is the identity operator on $\mathcal{H}\otimes\mathcal{K}$, where the left-hand side I is the identity on $\mathcal{H}$ and that on the right-hand side is the identity on $\mathcal{K}$. Similarly, $O\otimes O=A\otimes O=O\otimes B$, for every A on $\mathcal{H}$ and every B on $\mathcal{K}$, is the null operator on $\mathcal{H}\otimes\mathcal{K}$, where the left-hand side O is the null operator on $\mathcal{H}$ and that on the right-hand side is the null operator on $\mathcal{K}$. Thus $A\otimes B$ is null if and only if one of A or B is null.

Proposition 2. *For every $\alpha,\beta\in\mathbb{C}$, $A,A_1,A_2\in\mathcal{B}[\mathcal{H}]$ and $B,B_1,B_2\in\mathcal{B}[\mathcal{K}]$,*

(a) $\alpha\,\beta\,(A\otimes B)=\alpha A\otimes\beta B$,

(b$_1$) $A\otimes(B_1+B_2)=A\otimes B_1+A\otimes B_2$,

(b2) $(A_1+A_2)\otimes B=A_1\otimes B+A_2\otimes B$,

(b$_3$) $(A_1+A_2)\otimes(B_1+B_2)=A_1\otimes B_1+A_2\otimes B_1+A_1\otimes B_2+A_2\otimes B_2$,

(c) $(A_1\otimes B_1)\,(A_2\otimes B_2)=A_1A_2\otimes B_1B_2$.

If A and B are invertible, then so is $A\otimes B$ and

(d) $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$.

Proof. Take an arbitrary $\sum_{i=1}^n x_i\otimes y_i$ in $\mathcal{H}\otimes\mathcal{K}$. Recalling Remark 2 we have

$$\alpha\beta\,(A\otimes B)\sum_{i=1}^n x_i\otimes y_i=\alpha\beta\sum_{i=1}^n Ax_i\otimes By_i=\sum_{i=1}^n \alpha Ax_i\otimes\beta By_i=(\alpha A\otimes\beta B)\sum_{i=1}^n x_i\otimes y_i,$$

which proves (a). According to (1) we get

$$\big[A\otimes(B_1+B_2)\big]\sum_{i=1}^n x_i\otimes y_i=\sum_{i=1}^n Ax_i\otimes(B_1+B_2)y_i=\sum_{i=1}^n Ax_i\otimes(B_1y_i+B_2y_i)$$
$$=\sum_{i=1}^n\big[Ax_i\otimes B_1y_i+Ax_i\otimes B_2y_i\big]=\sum_{i=1}^n Ax_i\otimes B_1y_i+\sum_{i=1}^n Ax_i\otimes B_2y_i$$
$$=(A\otimes B_1)\sum_{i=1}^n x_i\otimes y_i+(A\otimes B_2)\sum_{i=1}^n x_i\otimes y_i=\big[A\otimes B_1+A\otimes B_2\big]\sum_{i=1}^n x_i\otimes y_i$$

and, similarly,

$$\big[(A_1+A_2)\otimes B\big]\sum_{i=1}^n x_i\otimes y_i=\big[A_1\otimes B+A_2\otimes B\big]\sum_{i=1}^n x_i\otimes y_i,$$

which proves (b$_1$) and (b$_2$). From (b$_1$) and (b$_2$) we get (b$_3$) at once:

$$\begin{aligned}(A_1+A_2)\otimes(B_1+B_2)&=(A_1\otimes A_2)\otimes B_1+(A_1+A_2)\otimes B_2\\&=A_1\otimes B_1+A_2\otimes B_1+A_1\otimes B_2+A_2\otimes B_2.\end{aligned}$$

Next we get (c) as follows.

$$\big[(A_1\otimes B_1)\,(A_2\otimes B_2)\big]\sum_{i=1}^n x_i\otimes y_i=(A_1\otimes B_1)\Big[(A_2\otimes B_2)\sum_{i=1}^n x_i\otimes y_i\Big]$$
$$=(A_1\otimes B_1)\sum_{i=1}^n A_2x_i\otimes B_2y_i=\sum_{i=1}^n A_1A_2x_i\otimes B_1B_2y_i=(A_1A_2\otimes B_1B_2)\sum_{i=1}^n x_i\otimes y_i;$$

and (c) implies (d): $(A\otimes B)\,(A^{-1}\otimes B^{-1})=I\otimes I=(A^{-1}\otimes B^{-1})\,(A\otimes B)$. $\square$

Tensor product of bounded linear transformations is again bounded and linear. (We shall use the same notation $\|\ \|$ for the norms in $\mathcal{B}[\mathcal{H}]$, $\mathcal{B}[\mathcal{K}]$ and $\mathcal{B}[\mathcal{H}\otimes\mathcal{K}]$.)

Proposition 3. *If* $A\in\mathcal{B}[\mathcal{H}]$ *and* $B\in\mathcal{B}[\mathcal{K}]$*, then* $A\otimes B\in\mathcal{B}[\mathcal{H}\otimes\mathcal{K}]$ *and*

$$\|A\otimes B\| = \|A\|\,\|B\|.$$

Proof. Linearity is verified as follows. Take $\sum_{i=1}^n x_i\otimes y_i$ and $\sum_{i=1}^m w_i\otimes z_i$ in $\mathcal{H}\otimes\mathcal{K}$, and take α and β in $\mathbb{C}$. Put $\alpha_i=\alpha$ for each $i=1,\ldots,n$, and put $\alpha_i=\beta$, $x_i=w_{i-n}$ and $y_i=z_{i-n}$ for each $i=n{+}1,\ldots,n{+}m$. Observe that

$$(A\otimes B)\Big(\alpha\sum_{i=1}^n x_i\otimes y_i+\beta\sum_{i=1}^m w_i\otimes z_i\Big)=(A\otimes B)\sum_{i=1}^{m+n}\alpha_i\,(x_i\otimes y_i)$$
$$=\sum_{i=1}^{m+n}A\alpha_i{}^{\frac{1}{2}}x_i\otimes B\alpha_i{}^{\frac{1}{2}}y_i=\alpha\Big(\sum_{i=1}^n Ax_i\otimes By_i\Big)+\beta\Big(\sum_{i=1}^m Aw_i\otimes Bz_i\Big)$$
$$=\alpha\,(A\otimes B)\sum_{i=1}^n x_i\otimes y_i+\beta\,(A\otimes B)\sum_{i=1}^m w_i\otimes z_i.$$

For boundedness and the norm identity, proceed as follows. Take arbitrary vectors $x\in\mathcal{H}$ and $y\in\mathcal{K}$ and note from (7.c) that

$$\|Ax\otimes By\|=\|Ax\|_{\mathcal{H}}\,\|By\|_{\mathcal{K}}. \tag{8}$$

Now take any $\sum_{i=1}^n x_i\otimes y_i$ in $\mathcal{H}\otimes\mathcal{K}$, consider the setup of the proof of Proposition 1, and observe that $\{A_ke_k\otimes f_k\}$ is an orthogonal set. Indeed, by (4),

$$\langle Ae_k\otimes f_k\,;Ae_l\otimes f_l\rangle=\langle Ae_k\,;Ae_l\rangle_{\mathcal{H}}\,\langle f_k\,;f_l\rangle_{\mathcal{K}}=\|Ae_k\|^2_{\mathcal{H}}\,\delta_{k,l}$$

where $\delta_{k,l}$ is the Kronecker function ($\delta_{l,k}=0$ if $k\neq l$ and $\delta_{k,k}=1$). Thus by (5), Remark 2, Definition 2, Proposition 2(a) and the above relation followed by the Pythagorean Theorem, and also (8), (6),

$$\Big\|(A\otimes I)\sum_{i=1}^n x_i\otimes y_i\Big\|^2=\Big\|(A\otimes I)\sum_{k=1}^{\widehat{n}}\gamma'_k\,(e_k\otimes f_k)\Big\|^2=\Big\|(A\otimes I)\sum_{k=1}^{\widehat{n}}\gamma'^{\frac{1}{2}}_k\,e_k\otimes\gamma'^{\frac{1}{2}}_k\,f_k\Big\|^2$$
$$=\Big\|\sum_{k=1}^{\widehat{n}}\gamma'^{\frac{1}{2}}_k\,Ae_k\otimes\gamma'^{\frac{1}{2}}_k\,f_k\Big\|^2=\Big\|\sum_{k=1}^{\widehat{n}}\gamma'_k\,(Ae_k\otimes f_k)\Big\|^2=\sum_{k=1}^{\widehat{n}}|\gamma'_k|^2\,\|Ae_k\otimes f_k\|^2$$
$$=\sum_{k=1}^{\widehat{n}}|\gamma'_k|^2\,\|Ae_k\|^2_{\mathcal{H}}\le\|A\|^2\sum_{k=1}^{\widehat{n}}|\gamma'_k|^2=\|A\|^2\,\Big\|\sum_{i=1}^n x_i\otimes y_i\Big\|^2,$$

with I being the identity on $\mathcal{K}$. Then $A\otimes I$ is a bounded operator and

$$\|A\otimes I\|\le\|A\|.$$

Analogously (applying exactly the same argument, and with I now being the identity on $\mathcal{H}$), $I\otimes B$ also is bounded and

$$\|I\otimes B\|\le\|B\|.$$

Observe from Proposition 2(c) that

$$A\otimes B=(A\otimes I)\,(I\otimes B)=(I\otimes B)\,(A\otimes I), \tag{9}$$

where the identity on $\mathcal{K}$ makes the tensor product with A and the identity on $\mathcal{H}$ makes the tensor product with B. Recall that the product of bounded operators is again a bounded operator with a bound not greater than the product of the bounds of each factor. Thus $A \otimes B$ is a bounded operator according to (9) and

$$\|A \otimes B\| = \|(A \otimes I)\,(I \otimes B)\| \leq \|(A \otimes I)\|\,\|(I \otimes B) \leq \|A\|\,\|B\|.$$

On the other hand, since $A \otimes B$ is bounded, we also get from (8) that

$$\begin{aligned} \|Ax\|_{\mathcal{H}}\,\|By\|_{\mathcal{K}} &= \|Ax \otimes By\| = \|(A \otimes B)(x \otimes y)\| \\ &\leq \|(A \otimes B)\|\,\|(x \otimes y)\| = \|(A \otimes B)\|\,\|x\|_{\mathcal{H}}\,\|y\|_{\mathcal{K}}, \end{aligned}$$

and hence

$$\|A\|\,\|B\| = \sup_{\|x\|_{\mathcal{H}}=1} \|Ax\|_{\mathcal{H}} \sup_{\|y\|_{\mathcal{K}}=1} \|By\|_{\mathcal{K}} \leq \|(A \otimes B)\|.$$

Therefore, $\|A \otimes B\| = \|A\|\,\|B\|$. □

2. Tensor Product Space

The inner product space $\mathcal{H} \otimes \mathcal{K}$ is not necessarily complete.

Definition 3. The *tensor product* $\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}$ of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is the completion of the inner product space $\mathcal{H} \otimes \mathcal{K}$. The *tensor product* $A \mathbin{\widehat{\otimes}} B$ in $\mathcal{B}[\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}]$ of two operators $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ is the extension of the tensor product $A \otimes B$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ of Definition 2 over the completion $\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}$ of $\mathcal{H} \otimes \mathcal{K}$.

Remark 3. In the above definition, the tensor product $A \otimes B$ of two operators $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ on $\mathcal{H} \otimes \mathcal{K}$ is extended by (uniform) continuity over $\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}$ according to a standard result concerning extensions of operators over completions (see e.g., [10, Theorem 5.23]):

If $\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}$ is the completion of $\mathcal{H} \otimes \mathcal{K}$, then $A \otimes B$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ has an extension $A \mathbin{\widehat{\otimes}} B$ in $\mathcal{B}[\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}]$. Moreover, $A \mathbin{\widehat{\otimes}} B$ is unique up to unitary transformations and

$$\|A \mathbin{\widehat{\otimes}} B\| = \|A \otimes B\|.$$

Proposition 4. *For every $\alpha, \beta \in \mathbb{C}$, $A, A_1, A_2 \in \mathcal{B}[\mathcal{H}]$ and $B, B_1, B_2 \in \mathcal{B}[\mathcal{K}]$,*

(a) $\alpha\,\beta\,(A \mathbin{\widehat{\otimes}} B) = \alpha A \mathbin{\widehat{\otimes}} \beta B$,

(b_1) $A \mathbin{\widehat{\otimes}} (B_1 + B_2) = A \mathbin{\widehat{\otimes}} B_1 + A \mathbin{\widehat{\otimes}} B_2$,

(b2) $(A_1 + A_2) \mathbin{\widehat{\otimes}} B = A_1 \mathbin{\widehat{\otimes}} B + A_2 \mathbin{\widehat{\otimes}} B$,

(b_3) $(A_1 + A_2) \mathbin{\widehat{\otimes}} (B_1 + B_2) = A_1 \mathbin{\widehat{\otimes}} B_1 + A_2 \mathbin{\widehat{\otimes}} B_1 + A_1 \mathbin{\widehat{\otimes}} B_2 + A_2 \mathbin{\widehat{\otimes}} B_2$,

(c) $(A_1 \mathbin{\widehat{\otimes}} B_1)\,(A_2 \mathbin{\widehat{\otimes}} B_2) = A_1 A_2 \mathbin{\widehat{\otimes}} B_1 B_2$,

(d) $(A \mathbin{\widehat{\otimes}} B)^* = A^* \mathbin{\widehat{\otimes}} B^*$.

If A and B are invertible, then so is $A \otimes B$ and

(e) $(A \mathbin{\widehat{\otimes}} B)^{-1} = A^{-1} \mathbin{\widehat{\otimes}} B^{-1}$.

Proof. Recall that if $\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$ is the completion of $\mathcal{H}\otimes\mathcal{K}$ and $A\,\widehat{\otimes}\,B$ in $\mathcal{B}[\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}]$ is the extension of $A\otimes B$ in $\mathcal{B}[\mathcal{H}\otimes\mathcal{K}]$, then there exists a unitary transformation

$$U\colon \mathcal{H}\otimes\mathcal{K}\to\mathcal{R}(U)\ \bar{\subset}\ \mathcal{H}\,\widehat{\otimes}\,\mathcal{K},$$

where $\bar{\subset}$ means *densely included*, such that

$$(A\,\widehat{\otimes}\,B)|_{\mathcal{R}(U)}=U(A\otimes B)\,U^*. \tag{10}$$

That is, the range $\mathcal{R}(U)$ of U is a dense linear manifold of the Hilbert space $\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$, and the restriction of $A\,\widehat{\otimes}\,B$ to $\mathcal{R}(U)$ is unitarily equivalent to $A\otimes B$. Note that, according to (10) and Proposition 2(a,b$_1$,b$_2$,c),

$$\begin{aligned}\big[\alpha\,\beta\,(A\,\widehat{\otimes}\,B)\big]\big|_{\mathcal{R}(U)}&=\alpha\,\beta\,\big[(A\,\widehat{\otimes}\,B)|_{\mathcal{R}(U)}\big]=\alpha\,\beta\,\big[U(A\otimes B)\,U^*\big]\\&=U\big[\alpha\,\beta\,(A\otimes B)\big]\,U^*=U(\alpha A\otimes\beta B)\,U^*=(\alpha A\,\widehat{\otimes}\,\beta B)|_{\mathcal{R}(U)},\end{aligned}$$

$$\begin{aligned}\big[A\,\widehat{\otimes}\,(B_1\!+\!B_2)\big]\big|_{\mathcal{R}(U)}&=U\big[A\otimes(B_1\!+\!B_2)\big]\,U^*=U(A\otimes B_1\!+\!A\otimes B_2)\,U^*\\&=U(A\otimes B_1)\,U^*\!+U(A\otimes B_2)\,U^*\\&=\big[A\,\widehat{\otimes}\,B_1\big]\big|_{\mathcal{R}(U)}\!+\big[A\,\widehat{\otimes}\,B_2\big]\big|_{\mathcal{R}(U)}\!=\big[A\,\widehat{\otimes}\,B_1\!+\!A\,\widehat{\otimes}\,B_2\big]\big|_{\mathcal{R}(U)},\end{aligned}$$

similarly,

$$\big[(A_1+A_2)\,\widehat{\otimes}\,B\big]\big|_{\mathcal{R}(U)}=\big[A_1\,\widehat{\otimes}\,B+A_2\,\widehat{\otimes}\,B\big]\big|_{\mathcal{R}(U)}$$

and, since $\mathcal{R}(U)$ is $A\,\widehat{\otimes}\,B$-invariant for every $A\,\widehat{\otimes}\,B$ in $\mathcal{B}[\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}]$ by (10),

$$\begin{aligned}&\big[(A_1\,\widehat{\otimes}\,B_1)\,(A_2\,\widehat{\otimes}\,B_2)\big]\big|_{\mathcal{R}(U)}=(A_1\,\widehat{\otimes}\,B_1)|_{\mathcal{R}(U)}\,(A_2\,\widehat{\otimes}\,B_2)|_{\mathcal{R}(U)}\\&=U(A_1\otimes B_1)\,U^*U(A_2\otimes B_2)\,U^*=U(A_1\otimes B_1)\,(A_2\otimes B_2)\,U^*\\&=U(A_1A_2\otimes B_1B_2)\,U^*=(A_1A_2\,\widehat{\otimes}\,B_1B_2)|_{\mathcal{R}(U)}.\end{aligned}$$

Therefore, since $\mathcal{R}(U)$ is dense in $\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$ and the above tensor products restricted to $\mathcal{R}(U)$ are (uniformly) continuous (because they are unitarily equivalent to tensors products on $\mathcal{H}\otimes\mathcal{K}$ by (10), which are (uniformly) continuous by Proposition 3), we get the results in (a), (b$_1$), (b$_2$) and (c) by extension by continuity (see e.g., [10, Theorem 3.45]). From (b$_1$) and (b$_2$) we get (b$_3$) at once:

$$\begin{aligned}(A_1+A_2)\,\widehat{\otimes}\,(B_1+B_2)&=(A_1\,\widehat{\otimes}\,A_2)\,\widehat{\otimes}\,B_1+(A_1+A_2)\,\widehat{\otimes}\,B_2\\&=A_1\,\widehat{\otimes}\,B_1+A_2\,\widehat{\otimes}\,B_1+A_1\,\widehat{\otimes}\,B_2+A_2\,\widehat{\otimes}\,B_2.\end{aligned}$$

Now, to verify (d), consider the tensor product $A\otimes B$ of $A\in\mathcal{B}[\mathcal{H}]$ and $B\in\mathcal{B}[\mathcal{K}]$ on the inner product space $\mathcal{H}\otimes\mathcal{K}$, take arbitrary vectors in $\mathcal{R}(U)\ \bar{\subset}\ \mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$, say $\sum_i x_i\,\widehat{\otimes}\,y_i$ and $\sum_j w_j\,\widehat{\otimes}\,z_j$, put $\sum_{i=1}^n x_i\otimes y_i=U^*\big(\sum_i x_i\,\widehat{\otimes}\,y_i\big)$ and $\sum_{j=1}^m w_j\otimes z_j=U^*\big(\sum_j w_j\,\widehat{\otimes}\,z_j\big)$ in $\mathcal{H}\otimes\mathcal{K}$, and observe by (10) and (4) that

$$\begin{aligned}&\Big\langle(A\,\widehat{\otimes}\,B)|_{\mathcal{R}(U)}\sum_i x_i\,\widehat{\otimes}\,y_i\,;\sum_j w_j\,\widehat{\otimes}\,z_j\Big\rangle=\Big\langle U(A\otimes B)\,U^*\sum_i x_i\,\widehat{\otimes}\,y_i\,;\sum_j w_j\,\widehat{\otimes}\,z_j\Big\rangle\\&=\Big\langle(A\otimes B)\,U^*\sum_i x_i\,\widehat{\otimes}\,y_i\,;\,U^*\sum_j w_j\,\widehat{\otimes}\,z_j\Big\rangle=\Big\langle(A\otimes B)\sum_{i=1}^n x_i\otimes y_i\,;\sum_{j=1}^m w_j\otimes z_j\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&= \Big\langle \sum_{i=1}^{n} Ax_i \otimes By_i \,;\, \sum_{j=1}^{m} w_j \otimes z_j \Big\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle Ax_i \,;\, w_j\rangle_{\mathcal{H}}\, \langle By_i \,;\, z_j\rangle_{\mathcal{K}} \qquad (11)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i \,;\, A^*w_j\rangle_{\mathcal{H}}\, \langle y_i \,;\, B^*z_j\rangle_{\mathcal{K}} = \Big\langle \sum_{i=1}^{n} x_i \otimes y_i \,;\, \sum_{j=1}^{m} A^*w_j \otimes B^*z_j \Big\rangle = \\
&\Big\langle \sum_{i=1}^{n} x_i \otimes y_i \,;\, (A^* \otimes B^*) \sum_{j=1}^{m} w_j \otimes z_j \Big\rangle = \Big\langle U^* \sum_i x_i \,\widehat{\otimes}\, y_i \,;\, (A^* \otimes B^*)\, U^* \sum_j w_j \,\widehat{\otimes}\, z_j \Big\rangle = \\
&\Big\langle \sum_i x_i \,\widehat{\otimes}\, y_i \,;\, U(A^* \otimes B^*)\, U^* \sum_j w_j \,\widehat{\otimes}\, z_j \Big\rangle = \Big\langle \sum_i x_i \,\widehat{\otimes}\, y_i \,;\, (A^* \,\widehat{\otimes}\, B^*)|_{\mathcal{R}(U)} \sum_j w_j \,\widehat{\otimes}\, z_j \Big\rangle,
\end{aligned}
$$

which extends by continuity to $A \,\widehat{\otimes}\, B$ on $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$ by the same argument of the previous paragraph (i.e., $\mathcal{R}(U)$ is dense in $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$, $A \otimes B$ and $A^* \otimes B^*$ are (uniformly) continuous, as well as the inner product, and composition of uniformly continuous mappings is again uniformly continuous (see e.g., [10, Problem 3.28]). Therefore, with $\sum_i x_i \,\widehat{\otimes}\, y_i$ and $\sum_j w_j \,\widehat{\otimes}\, z_j$ denoting arbitrary elements of $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$, the identity

$$
\Big\langle (A \,\widehat{\otimes}\, B) \sum_i x_i \,\widehat{\otimes}\, y_i \,;\, \sum_j w_j \,\widehat{\otimes}\, z_j \Big\rangle = \Big\langle \sum_i x_i \,\widehat{\otimes}\, y_i \,;\, (A^* \,\widehat{\otimes}\, B^*) \sum_j w_j \,\widehat{\otimes}\, z_j \Big\rangle
$$

holds in the Hilbert space $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$. Then, since existence and uniqueness of the adjoint is ensured for Hilbert space operators, the above identity ensures the result in (d): the adjoint $(A \,\widehat{\otimes}\, B)^*$ of $A \,\widehat{\otimes}\, B$ is $A^* \,\widehat{\otimes}\, B^*$. Finally, assertion (e) follows directly from (c): $(A \,\widehat{\otimes}\, B)\,(A^{-1} \,\widehat{\otimes}\, B^{-1}) = I \,\widehat{\otimes}\, I = (A^{-1} \,\widehat{\otimes}\, B^{-1})\,(A \,\widehat{\otimes}\, B)$. □

Since there exists a unitary transformation $U \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{R}(U) \mathrel{\bar{\subset}} \mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$, for every $\sum_{i=1}^{n} x_i \otimes y_i$ in $\mathcal{H} \otimes \mathcal{K}$ put $\sum_{i=1}^{n} x_i \,\widehat{\otimes}\, y_i = U \sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} U(x_i \otimes y_i)$ in $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$. We shall identify these and write

$$
\sum_{i=1}^{n} x_i \,\widehat{\otimes}\, y_i = \sum_{i=1}^{n} x_i \otimes y_i.
$$

In particular,

$$
x \,\widehat{\otimes}\, y = x \otimes y.
$$

With $\sum_i x_i \,\widehat{\otimes}\, y_i$ and $\sum_j w_j \,\widehat{\otimes}\, z_j$ denoting arbitrary elements of $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$ and putting $\sum_{i=1}^{n} x_i \otimes y_i = U^*\big(\sum_i x_i \,\widehat{\otimes}\, y_i\big)$ and $\sum_{j=1}^{m} w_j \otimes z_j = U^*\big(\sum_j w_j \,\widehat{\otimes}\, z_j\big)$ in $\mathcal{H} \otimes \mathcal{K}$, we get from 4 and 7 (since U is unitary)

$$
\Big\langle \sum_i x_i \,\widehat{\otimes}\, y_i \,;\, \sum_j w_j \,\widehat{\otimes}\, z_j \Big\rangle = \sum_i \sum_j \langle x_i \,;\, w_j\rangle_{\mathcal{H}}\, \langle y_i \,;\, z_j\rangle_{\mathcal{K}} \tag{12.a}
$$

so that

$$
\Big\| \sum_i x_i \,\widehat{\otimes}\, y_i \Big\|^2 = \sum_i \sum_j \langle x_i \,;\, x_j\rangle_{\mathcal{H}}\, \langle y_i \,;\, y_j\rangle_{\mathcal{K}}, \tag{12.b}
$$

and hence

$$
\Big\| \sum_i x_i \,\widehat{\otimes}\, y_i \Big\| = \Big\| \sum_i y_i \,\widehat{\otimes}\, x_i \Big\|. \tag{12.c}
$$

Moreover, from Definition 2 and (10), extending (11) by continuity from $\mathcal{R}(U)$ to $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$, and using the same argument applied to prove part (d) above, we also get

$$A \,\widehat{\otimes}\, B\Big(\sum_i x_i \,\widehat{\otimes}\, y_i\Big) = \sum_i Ax_i \,\widehat{\otimes}\, By_i \tag{12.d}$$

and

$$\Big\langle (A \,\widehat{\otimes}\, B) \sum_i x_i \,\widehat{\otimes}\, y_i \,;\, \sum_j w_j \,\widehat{\otimes}\, z_j \Big\rangle = \sum_i \sum_j \langle Ax_i \,;\, w_j\rangle_{\mathcal{H}} \, \langle By_i \,;\, z_j\rangle_{\mathcal{K}}. \tag{12.e}$$

Furthermore, note that according to Proposition 4(a),

$$A \,\widehat{\otimes}\, B = A_\alpha \,\widehat{\otimes}\, B_\alpha = \alpha\, A \,\widehat{\otimes}\, \alpha^{-1} B, \tag{12.f}$$

with $A_\alpha = \alpha\, A$ and $B_\alpha = \alpha^{-1} B$ for every nonzero scalar α so that

$$\|A \,\widehat{\otimes}\, B\| = \|A_\alpha\|\, \|B_\alpha\| \tag{12.g}$$

for every $0 \neq \alpha \in \mathbb{C}$ by Remark 3 and Proposition 3.

3. Kronecker Product

Remark 4. Suppose $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{K} = \mathbb{C}^m$. Take vectors $x = (\xi_1, \dots, \xi_n)$ in $\mathbb{C}^n$ and $y = (\upsilon_1, \dots, \upsilon_m)$ in $\mathbb{C}^m$. The *Kronecker product* $x \circledast y$ of x and y is the vector

$$x \circledast y = (\xi_1 y, \dots, \xi_n y) = (\xi_1 \upsilon_1, \dots, \xi_1 \upsilon_m, \cdots, \xi_n \upsilon_1, \dots, \xi_n \upsilon_m) \;\text{ in }\; \textstyle\bigoplus_{i=1}^n \mathbb{C}^m = \mathbb{C}^{nm}.$$

With scalar multiplication and addition in $\mathbb{C}^k$ defined as usual (i.e., coordinatewise), it follows that scalar multiplication of a Kronecker product is again a Kronecker product (but not uniquely determined — as in Remark 2). However, there are vectors in $\mathbb{C}^{nm}$ that are not Kronecker products (e.g., $(0,1,1,0)$ in $\mathbb{C}^4$ is not a Kronecker product for any pair of vectors (x,y) in $\mathbb{C}^2 \times \mathbb{C}^2$). Moreover, the sum of Kronecker products is not necessarily a Kronecker product. Indeed, with x_1, x_2 in $\mathbb{C}^n$ and y_1, y_2 in $\mathbb{C}^m$, it is readily verified that

$$x_1 \circledast y_1 + x_2 \circledast y_2 = (x_1 + x_2) \circledast (y_1 + y_2) - x_1 \circledast y_2 - x_2 \circledast y_1.$$

Thus consider the collection $\mathbb{C}^n \circledast \mathbb{C}^m$ of all (finite) linear combinations or, equivalently, of all (finite) sums of Kronecker products in $\mathbb{C}^{nm}$:

$$\mathbb{C}^n \circledast \mathbb{C}^m = \Big\{ \sum_{i=1}^N x_i \circledast y_i :\; (x_i, y_i) \in \mathbb{C}^n \times \mathbb{C}^m, \;\; N \in \mathbb{N} \Big\},$$

which is clearly a (complex) linear space — as in (2.a) and (2.b). We claim that

$$\mathbb{C}^{nm} = \mathbb{C}^n \circledast \mathbb{C}^m.$$

Indeed, take an arbitrary vector $z = (\zeta_{1,1}, \dots, \zeta_{1,m}, \cdots, \zeta_{n,1}, \dots, \zeta_{n,m})$ in $\mathbb{C}^{nm}$, and observe that $z = \sum_{i=1}^n e_i \circledast z_i$, where $\{e_i\}_{i=1}^n$ is the canonical basis for $\mathbb{C}^n$ and each z_i in $\mathbb{C}^m$ is given by $z_i = (\zeta_{i,1}, \dots, \zeta_{i,m})$ so that $\mathbb{C}^{nm} \subseteq \mathbb{C}^n \circledast \mathbb{C}^m$. The converse is clear: $\mathbb{C}^n \circledast \mathbb{C}^m \subseteq \mathbb{C}^{nm}$. Thus $\mathbb{C}^{nm} = \mathbb{C}^n \circledast \mathbb{C}^m$. Also note that

$$\begin{aligned}
&\big\langle x \circledast y \,;\, w \circledast z \big\rangle_{\mathbb{C}^{nm}} = \big\langle \xi_1 y, \dots, \xi_n y \;;\; \omega_1 z, \dots, \omega_n z \big\rangle_{\mathbb{C}^{nm}} \\
&= \big\langle \xi_1(\upsilon_1, \dots, \upsilon_m), \dots, \xi_n(\upsilon_1, \dots, \upsilon_m) \;;\; \omega_1(\zeta_1, \dots, \zeta_m), \dots, \omega_n(\zeta_1, \dots, \zeta_m) \big\rangle_{\mathbb{C}^{nm}} \\
&= \big\langle \xi_1 \upsilon_1, \dots, \xi_1 \upsilon_m, \cdots, \xi_n \upsilon_1, \dots, \xi_n \upsilon_m \;;\; \omega_1 \zeta_1, \dots, \omega_1 \zeta_m, \cdots, \omega_n \zeta_1, \dots, \omega_n \zeta_m \big\rangle_{\mathbb{C}^{nm}} \\
&= \xi_1 \upsilon_1 \omega_1 \zeta_1 + \dots + \xi_1 \upsilon_m \omega_1 \zeta_m + \cdots + \xi_n \upsilon_1 \omega_n \zeta_1 + \dots + \xi_n \upsilon_m \omega_n \zeta_m \\
&= \xi_1 \omega_1 (\upsilon_1 \zeta_1 + \dots + \upsilon_m \zeta_m) + \dots + \xi_n \omega_n (\upsilon_1 \zeta_1 + \dots + \upsilon_m \zeta_m) \;=\; \langle x \,;\, w\rangle_{\mathbb{C}^n} \langle y \,;\, z\rangle_{\mathbb{C}^m}
\end{aligned}$$

for every x, w in $\mathbb{C}^n$ and every y, z in $\mathbb{C}^m$. Applying the same argument of Proposition 1, it can be shown that the expression below defines an inner product on the linear space $\mathbb{C}^n \circledast \mathbb{C}^m$,

$$\Big\langle \sum_{i=1}^N x_i \circledast y_i \,;\sum_{j=1}^M w_j \circledast z_j \Big\rangle = \sum_{i=1}^N \sum_{j=1}^M \big\langle x_i \circledast y_i \,; w_j \circledast z_j \big\rangle_{\mathbb{C}^{nm}} = \sum_{i=1}^N \sum_{j=1}^M \langle x_i \,; w_j\rangle_{\mathbb{C}^n} \langle y_i \,; z_j\rangle_{\mathbb{C}^m}$$

for arbitrary $\sum_{i=1}^N x_i \circledast y_i$ and $\sum_{j=1}^M w_j \circledast z_j$ in $\mathbb{C}^n \circledast \mathbb{C}^m$, inducing the norm

$$\Big\| \sum_{i=1}^N x_i \circledast y_i \Big\| = \Big(\sum_{i=1}^N \sum_{j=1}^N \big\langle x_i \circledast y_i \,; x_j \circledast y_j \big\rangle_{\mathbb{C}^{nm}} \Big)^{\frac{1}{2}} = \Big(\sum_{i=1}^N \sum_{j=1}^N \langle x_i \,; x_j\rangle_{\mathbb{C}^n} \langle y_i \,; y_j\rangle_{\mathbb{C}^m} \Big)^{\frac{1}{2}}.$$

In particular,

$$\|x \circledast y\| = \|x \circledast y\|_{\mathbb{C}^{nm}} = \|x\|_{\mathbb{C}^n} \|y\|_{\mathbb{C}^m}.$$

Now observe that the natural mapping $\Phi\colon \mathbb{C}^n \circledast \mathbb{C}^m \to \mathbb{C}^n \otimes \mathbb{C}^m$ defined by

$$\Phi\Big(\sum_{i=1}^N x_i \circledast y_i \Big) = \sum_{i=1}^N \langle x_i \,; \cdot\,\rangle_{\mathbb{C}^n} \langle y_i \,; \cdot\,\rangle_{\mathbb{C}^m} = \sum_{i=1}^N x_i \otimes y_i$$

is clearly linear, injective, surjective and, moreover, preserves inner product:

$$\Big\langle \Phi\Big(\sum_{i=1}^N x_i \circledast y_i \Big) ; \Phi\Big(\sum_{j=1}^M w_j \circledast z_j \Big) \Big\rangle = \Big\langle \sum_{i=1}^N x_i \otimes y_i \,; \sum_{j=1}^M w_j \otimes z_j \Big\rangle$$
$$= \sum_{i=1}^N \sum_{j=1}^M \langle x_i \,; w_j\rangle_{\mathbb{C}^n} \langle y_i \,; z_j\rangle_{\mathbb{C}^m} = \Big\langle \sum_{i=1}^N x_i \circledast y_i \,; \sum_{j=1}^M w_j \circledast z_j \Big\rangle$$

for every $\sum_{i=1}^N x_i \circledast y_i$ and $\sum_{j=1}^M w_j \circledast z_j$ in $\mathbb{C}^n \circledast \mathbb{C}^m$. Thus $\mathbb{C}^n \circledast \mathbb{C}^m$ and $\mathbb{C}^n \otimes \mathbb{C}^m$ are unitarily equivalent inner product spaces (Hilbert spaces, actually, once $\mathbb{C}^n$ and $\mathbb{C}^m$ are finite-dimensional):

$$\mathbb{C}^n \circledast \mathbb{C}^m \cong \mathbb{C}^n \otimes \mathbb{C}^m,$$

and therefore $\mathbb{C}^{nm} \cong \mathbb{C}^n \otimes \mathbb{C}^m$. Thus $\mathbb{C}^{nm}$ and $\mathbb{C}^n \otimes \mathbb{C}^m$ are identified with each other and such an identification will, as usual, be written with $=$ instead of $\cong$:

$$\mathbb{C}^{nm} = \mathbb{C}^n \otimes \mathbb{C}^m.$$

Finally, take arbitrary linear transformations A in $\mathcal{B}[\mathbb{C}^n]$ and B in $\mathcal{B}[\mathbb{C}^m]$, and let them be represented with respect to the canonical basis for $\mathbb{C}^n$ and $\mathbb{C}^m$ by the square ($n \times n$ and $m \times m$) matrices

$$A = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & & \vdots \\ \beta_{m1} & \dots & \beta_{mm} \end{pmatrix}.$$

The *Kronecker product* $A \circledast B$ of the matrices A in $\mathcal{B}[\mathbb{C}^n]$ and B in $\mathcal{B}[\mathbb{C}^m]$ is

$$A \circledast B = \begin{pmatrix} \alpha_{11}B & \dots & \alpha_{1n}B \\ \vdots & & \vdots \\ \alpha_{n1}B & \dots & \alpha_{nn}B \end{pmatrix} \quad \text{in} \quad \mathcal{B}\big[\textstyle\bigoplus_{i=1}^n \mathbb{C}^m\big] = \mathcal{B}[\mathbb{C}^{mn}],$$

a square ($mn \times mn$) matrix. Observe that

$$(A\circledast B)\,(x\circledast y)=\Big(\sum_{i=1}^{N}\alpha_{1,i}\,\xi_i\,By,\ldots,\sum_{i=1}^{N}\alpha_{n,i}\,\xi_i\,By\Big)=Ax\circledast By,$$

and hence

$$(A\circledast B)\sum_{i=1}^{N}x_i\circledast y_i=\sum_{i=1}^{N}(A\circledast B)\,(x_i\circledast y_i)=\sum_{i=1}^{N}Ax_i\circledast By_i$$

for every $\sum_{i=1}^{N}x_i\circledast y_i\in\mathbb{C}^n\circledast\mathbb{C}^m=\mathbb{C}^{nm}$. Since Φ is a unitary transformation, it follows that tensor products $\otimes$ and Kronecker products $\circledast$, of both vectors and matrices, are unitarily equivalent. That is,

$$x\circledast y\;\cong\;x\otimes y\quad\text{ and }\quad A\circledast B\;\cong\;A\otimes B,$$

which means that

$$\Phi(x\circledast y)\;=\;x\otimes y\quad\text{ and }\quad\Phi(A\circledast B)\;=\;(A\otimes B)\Phi,$$

so that these products are identified with each other. Such an identification is usually expressed by writing $=$ for $\cong$:

$$x\circledast y=x\otimes y\quad\text{ and }\quad A\circledast B=A\otimes B.$$

Indeed,

$$\begin{aligned}\Phi^{-1}(A\otimes B)\Phi\Big(\sum_{i=1}^{N}x_i\circledast y_i\Big)\;&=\;\Phi^{-1}(A\otimes B)\sum_{i=1}^{N}x_i\otimes y_i=\Phi^{-1}\Big(\sum_{i=1}^{N}Ax_i\otimes By_i\Big)\\&=\sum_{i=1}^{N}Ax_i\circledast By_i=(A\circledast B)\sum_{i=1}^{N}x_i\circledast y_i\end{aligned}$$

for every $\sum_{i=1}^{N}x_i\circledast y_i\in\mathbb{C}^n\circledast\mathbb{C}^m=\mathbb{C}^{nm}$. Thus all properties in Propositions 2, 3, 4(d), including equations (8) and (9), hold if the tensor product $\otimes$ is replaced with the Kronecker product $\circledast$. For more properties of Kronecker products, see e.g., [1].

Remark 5. The same discussion of Remark 4 can be readily extended from finite-dimensional to separable spaces, as follows. Suppose $\mathcal{H}$ is any infinite-dimensional separable Hilbert space and let $\mathcal{K}$ be an arbitrary Hilbert space. Since $\mathcal{H}$ is separable, it is unitarily equivalent to $\ell_+^2=\ell_+^2(\mathbb{C})$ so that we can assume the identification $\mathcal{H}=\ell_+^2$. Put

$$x\circledast y=(\xi_1y,\xi_2y,\ldots)\quad\text{in}\quad\textstyle\bigoplus_{k=1}^{\infty}\mathcal{K}=\ell_+^2(\mathcal{K})$$

for arbitrary $x=(\xi_1,\xi_2,\ldots)$ in $\mathcal{H}=\ell_+^2$ and y in $\mathcal{K}$ so that

$$\|x\circledast y\|_{\ell_+^2(\mathcal{K})}^2=\sum_{k=1}^{\infty}\|\xi_ky\|_{\mathcal{K}}^2=\|y\|_{\mathcal{K}}^2\sum_{k=1}^{\infty}|\xi_k|^2=\|y\|_{\mathcal{K}}^2\,\|x\|_{\ell_+^2}^2=\|x\|_{\mathcal{H}}^2\,\|y\|_{\mathcal{K}}^2,$$

and consider the collection

$$\mathcal{H}\circledast\mathcal{K}=\Big\{\sum_{i=1}^{N}x_i\circledast y_i:\;(x_i,y_i)\in\mathcal{H}\times\mathcal{K},\;\;N\in\mathbb{N}\Big\}$$

which is a linear space. We claim that $\mathcal{H}\circledast\mathcal{K}$ is densely included in $\ell_+^2(\mathcal{K})$:

$$\mathcal{H}\circledast\mathcal{K}\;\bar{\subset}\;\ell_+^2(\mathcal{K}).$$

Indeed, $\mathcal{H}\circledast\mathcal{K}\subseteq\ell^2_+(\mathcal{K})$ trivially. Moreover, take any $z=(z_1,z_2,\dots)$ in $\ell^2_+(\mathcal{K})$. Since $z=\sum_{k=1}^\infty e_k\circledast z_k$, where $\{e_k\}_{k=1}^\infty$ is the canonical orthonormal basis for $\mathcal{H}$,

$$\Big\|z-\sum_{k=1}^N e_k\circledast z_k\Big\|^2_{\ell^2_+(\mathcal{K})}=\Big\|\sum_{k=N+1}^\infty e_k\circledast z_k\Big\|^2_{\ell^2_+(\mathcal{K})}\le\sum_{k=N+1}^\infty\|z_k\|^2_{\mathcal{K}}\to 0.$$

Thus $\mathcal{H}\circledast\mathcal{K}$ is dense in $\ell^2_+(\mathcal{K})$ (i.e., $(\mathcal{H}\circledast\mathcal{K})^-=\ell^2_+(\mathcal{K})$). Hence

$$\mathcal{H}\,\widehat{\circledast}\,\mathcal{K}=\ell^2_+(\mathcal{K}),$$

that is, $\ell^2_+(\mathcal{K})$ is the completion of $\mathcal{H}\circledast\mathcal{K}$. Now, as in Remark 4, the expression

$$\Big\langle\sum_{i=1}^N x_i\circledast y_i\,;\sum_{j=1}^M w_j\circledast z_j\Big\rangle=\sum_{i=1}^N\sum_{j=1}^M\big\langle x_i\circledast y_i\,;w_j\circledast z_j\big\rangle_{\ell^2_+(\mathcal{K})}=\sum_{i=1}^N\sum_{j=1}^M\langle x_i\,;w_j\rangle_{\mathcal{H}}\langle y_i\,;z_j\rangle_{\mathcal{K}}$$

for every $\sum_{i=1}^N x_i\circledast y_i$ and $\sum_{j=1}^M w_j\circledast z_j$ in $\mathcal{H}\circledast\mathcal{K}$, defines an inner product on the linear space $\mathcal{H}\circledast\mathcal{K}$. Again the natural mapping $\Phi\colon\mathcal{H}\circledast\mathcal{K}\to\mathcal{H}\otimes\mathcal{K}$ defined by

$$\Phi\Big(\sum_{i=1}^N x_i\circledast y_i\Big)=\sum_{i=1}^N\langle x_i\,;\cdot\,\rangle_{\mathcal{H}}\,\langle y_i\,;\cdot\,\rangle_{\mathcal{K}}=\sum_{i=1}^N x_i\otimes y_i$$

is unitary so that $\mathcal{H}\circledast\mathcal{K}$ and $\mathcal{H}\otimes\mathcal{K}$ are unitarily equivalent inner product spaces:

$$\mathcal{H}\circledast\mathcal{K}\;\cong\;\mathcal{H}\otimes\mathcal{K},$$

and so (see e.g., [10, Corollary 4.38]) there exists a unitary $\widehat{\Phi}\colon\mathcal{H}\,\widehat{\circledast}\,\mathcal{K}\to\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$ of the completion $\mathcal{H}\,\widehat{\circledast}\,\mathcal{K}$ of $\mathcal{H}\circledast\mathcal{K}$ onto the completion $\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$ of $\mathcal{H}\otimes\mathcal{K}$. Therefore $\mathcal{H}\,\widehat{\circledast}\,\mathcal{K}$ and $\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$ are unitarily equivalent Hilbert spaces. Then

$$\ell^2_+(\mathcal{K})\;\cong\;\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}.$$

Since $\ell^2_+(\mathcal{K})$ and $\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}$ are unitarily equivalent, they are identified with each other and such an identification is, as usual, written with $=$ instead of $\cong$ so that

$$\ell^2_+(\mathcal{K})=\mathcal{H}\,\widehat{\otimes}\,\mathcal{K}:$$

The tensor product of a separable $\mathcal{H}$ and any $\mathcal{K}$ is the Hilbert space $\ell^2_+(\mathcal{K})=\bigoplus_{k=1}^\infty\mathcal{K}$ consisting of the orthogonal direct sum of countably many copies of $\mathcal{K}$. Now take any A in $\mathcal{B}[\mathcal{H}]$ and any B in $\mathcal{B}[\mathcal{K}]$, and consider the matrix representation of A with respect to the canonical orthonormal basis for $\mathcal{H}=\ell^2_+=\ell^2_+(\mathbb{C})$,

$$A=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \cdots\\ \alpha_{21} & \alpha_{22} & \cdots\\ \vdots & \vdots & \end{pmatrix}\quad\text{in}\quad\mathcal{B}[\mathcal{H}],$$

and define

$$A\,\widehat{\circledast}\,B=\begin{pmatrix}\alpha_{11}B & \alpha_{12}B & \cdots\\ \alpha_{21}B & \alpha_{22}B & \cdots\\ \vdots & \vdots & \end{pmatrix}\quad\text{in}\quad\mathcal{B}\big[\textstyle\bigoplus_{k=1}^\infty\mathcal{K}\big]=\mathcal{B}[\ell^2_+(\mathcal{K})].$$

Observe that

$$(A\,\widehat{\circledast}\,B)\,(x\circledast y)=\Big(\sum_{i=1}^\infty\alpha_{1,i}\,\xi_i\,By,\sum_{i=1}^\infty\alpha_{2,i}\,\xi_i\,By,\,\dots\Big)=Ax\,\widehat{\circledast}\,By,$$

and hence

$$(A \,\widehat{\circledast}\, B)\sum_{i=1}^{N} x_i \circledast y_i = \sum_{i=1}^{N} (A \,\widehat{\circledast}\, B)\,(x_i \circledast y_i) = \sum_{i=1}^{N} Ax_i \circledast By_i$$

for every $\sum_{i=1}^{N} x_i \circledast y_i \in \mathcal{H} \circledast \mathcal{K}$ so that

$$(A \,\widehat{\circledast}\, B)\sum_{i} x_i \,\widehat{\circledast}\, y_i = \sum_{i} Ax_i \,\widehat{\circledast}\, By_i$$

for every $\sum_i x_i \,\widehat{\circledast}\, y_i \in (\mathcal{H} \circledast \mathcal{K})^- \;=\; \ell^2_+(\mathcal{K})$ because $(A \,\widehat{\circledast}\, B)\colon \ell^2_+(\mathcal{K}) \to \ell^2_+(\mathcal{K})$ is continuous. Since $\widehat{\Phi}\colon \mathcal{H} \,\widehat{\circledast}\, \mathcal{K} \to \mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$ is a unitary transformation, it follows that

$$x \circledast y \;\cong\; x \otimes y \quad \text{ and } \quad A \,\widehat{\circledast}\, B \;\cong\; A \,\widehat{\otimes}\, B,$$

which means that

$$\widehat{\Phi}(x \circledast y) \;=\; x \otimes y \quad \text{ and } \quad \widehat{\Phi}(A \,\widehat{\circledast}\, B) \;=\; (A \,\widehat{\otimes}\, B)\widehat{\Phi},$$

so that these products are identified with each other. Such an identification is usually expressed by writing $=$ for $\cong$:

$$x \circledast y = x \otimes y \quad \text{ and } \quad A \,\widehat{\circledast}\, B = A \,\widehat{\otimes}\, B.$$

Indeed,

$$\begin{aligned} \widehat{\Phi}^{-1}(A \,\widehat{\otimes}\, B)\widehat{\Phi}\Big(\sum_i x_i \,\widehat{\circledast}\, y_i\Big) \;&=\; \widehat{\Phi}^{-1}(A \,\widehat{\otimes}\, B)\sum_i x_i \,\widehat{\otimes}\, y_i = \widehat{\Phi}^{-1}\Big(\sum_i Ax_i \,\widehat{\otimes}\, By_i\Big) \\ &= \sum_i Ax_i \,\widehat{\circledast}\, By_i = (A \circledast B)\sum_i x_i \,\widehat{\circledast}\, y_i \end{aligned}$$

for every $\sum_i x_i \,\widehat{\circledast}\, y_i \in (\mathcal{H} \circledast \mathcal{K})^- = \ell^2_+(\mathcal{K})$.

4. Commutativity

Remark 6. Suppose $\mathcal{H}$ and $\mathcal{K}$ are infinite-dimensional separable Hilbert spaces so that we can identify $\mathcal{H} = \mathcal{K} = \ell^2_+$. Let $\{e_k\}$ be the canonical orthonormal basis for ℓ^2_+. Throughout this remark all matrix representations of operators in $\mathcal{B}[\ell^2_+]$ will be with respect to the orthonormal basis $\{e_k\}$. For each pair of positive integers $\{k, l\}$ consider the outer product

$$e_k \odot e_l \;\in\; \mathcal{B}[\ell^2_+].$$

The matrix representation of each $e_k \odot e_l$ has 1 as the (k, l) entry and zeros elsewhere. Now consider the tensor product (cf. Remark 5)

$$(e_k \odot e_l) \,\widehat{\otimes}\, (e_l \odot e_k) = (e_k \odot e_l) \,\widehat{\circledast}\, (e_l \odot e_k) \;\in\; \mathcal{B}[\ell^2_+ \,\widehat{\otimes}\, \ell^2_+].$$

Again, the matrix representation of each $(e_k \odot e_l) \,\widehat{\otimes}\, (e_l \odot e_k)$ has 1 as the (l, k) entry of the (k, l) block and zeros elsewhere. Define the *permutation operator*

$$\widehat{\Pi} = \sum_{k,l} (e_k \odot e_l) \,\widehat{\otimes}\, (e_l \odot e_k) \;\in\; \mathcal{B}[\ell^2_+ \,\widehat{\otimes}\, \ell^2_+],$$

whose matrix representation has precisely a single 1 in each row and in each column, which implies that $\widehat{\Pi}$ is invertible. In fact, $\widehat{\Pi}$ is a symmetry; that is, $\widehat{\Pi}$ is a unitary self-adjoint. Indeed, take arbitrary $x=(\xi_1,\xi_2,\dots)$ and $y=(\upsilon_1,\upsilon_2,\dots)$ in ℓ^2_+. Then

$$\begin{aligned}\widehat{\Pi}(x\otimes y) \;&=\; \Big(\sum_{k,l}(e_k\odot e_l)\,\widehat{\otimes}\,(e_l\odot e_k)\Big)(x\otimes y)=\sum_{k,l}(e_k\odot e_l)x\,\widehat{\otimes}\,(e_l\odot e_k)y\\ &=\sum_{k,l}\langle x\,;e_l\rangle_{\ell^2_+}e_k\,\widehat{\otimes}\,\langle y\,;e_k\rangle_{\ell^2_+}e_l=\sum_k\langle y\,;e_k\rangle_{\ell^2_+}e_k\otimes\sum_l\langle x\,;e_l\rangle_{\ell^2_+}e_l\;=\;y\otimes x\end{aligned}$$

according to (3) and Definition (2) — extended by continuity. Thus, by (7.c),

$$\|\widehat{\Pi}(x\otimes y)\|=\|y\otimes x\|=\|x\otimes y\|.$$

Since $\widehat{\Pi}$ is linear and continuous and $\big\|\sum_i x_i\,\widehat{\otimes}\,y_i\big\|=\big\|\sum_i y_i\,\widehat{\otimes}\,x_i\big\|$ by (12.c),

$$\widehat{\Pi}\Big(\sum_i x_i\,\widehat{\otimes}\,y_i\Big)=\sum_i y_i\,\widehat{\otimes}\,x_i\quad\text{ and }\quad\Big\|\widehat{\Pi}\Big(\sum_i x_i\,\widehat{\otimes}\,y_i\Big)\Big\|=\Big\|\sum_i x_i\,\widehat{\otimes}\,y_i\Big\|$$

for every $\sum_i x_i\,\widehat{\otimes}\,y_i$ in $\ell^2_+\,\widehat{\otimes}\,\ell^2_+$ so that $\widehat{\Pi}$ is an isometry. Then $\widehat{\Pi}$ is an invertible isometry, which means that $\widehat{\Pi}$ is unitary. Moreover, since $(e_k\odot e_l)^*=e_l\odot e_k$,

$$\widehat{\Pi}^*=\sum_{k,l}(e_k\odot e_l)^*\,\widehat{\otimes}\,(e_l\odot e_k)^*=\sum_{k,l}(e_l\odot e_k)\,\widehat{\otimes}\,(e_k\odot e_l)=\widehat{\Pi}$$

and $\widehat{\Pi}$ is self-adjoint as well. Hence $\widehat{\Pi}$ is a symmetry and so an involution. That is,

$$\widehat{\Pi}^{-1}=\widehat{\Pi}^*=\widehat{\Pi}\quad\text{ so that }\quad\widehat{\Pi}^2=I.$$

Therefore, for every pair A and B of operators in $\mathcal{B}[\ell^2_+]$,

$$\begin{aligned}\widehat{\Pi}(A\,\widehat{\otimes}\,B)\Big(\sum_i x_i\,\widehat{\otimes}\,y_i\Big)\;&=\;\widehat{\Pi}\Big(\sum_i Ax_i\,\widehat{\otimes}\,By_i\Big)=\Big(\sum_i By_i\,\widehat{\otimes}\,Ax_i\Big)\\ &=\;(B\,\widehat{\otimes}\,A)\Big(\sum_i y_i\,\widehat{\otimes}\,x_i\Big)=(B\,\widehat{\otimes}\,A)\,\widehat{\Pi}\Big(\sum_i x_i\,\widehat{\otimes}\,y_i\Big)\end{aligned}$$

for every $\sum_i x_i\,\widehat{\otimes}\,y_i$ in $\ell^2_+\,\widehat{\otimes}\,\ell^2_+$ so that

$$\widehat{\Pi}(A\,\widehat{\otimes}\,B)=(B\,\widehat{\otimes}\,A)\widehat{\Pi}.$$

In other words, the tensor products $A\,\widehat{\otimes}\,B$ and $B\,\widehat{\otimes}\,A$ are unitarily equivalent.

This can be generalized without assuming that $\mathcal{H}$ and $\mathcal{K}$ are necessarily infinite-dimensional and separable as follows. Let $\mathcal{H}$ and $\mathcal{K}$ be arbitrary Hilbert spaces and define the mapping $\Pi\colon\mathcal{H}\otimes\mathcal{K}\to\mathcal{K}\otimes\mathcal{H}$ by

$$\Pi\Big(\sum_{i=1}^n x_i\otimes y_i\Big)=\sum_{i=1}^n y_i\otimes x_i$$

for every $\sum_{i=1}^n x_i\otimes y_i$ in $\mathcal{H}\otimes\mathcal{K}$. This is clearly invertible: $\Pi^{-1}\big(\sum_{i=1}^n y_i\otimes x_i\big)=\sum_{i=1}^n x_i\otimes y_i$ for every $\sum_{i=1}^n y_i\otimes x_i$ in $\mathcal{K}\otimes\mathcal{H}$. It is readily verified that

$$\Pi\Big(\alpha\sum_{i=1}^n x_i\otimes y_i+\beta\sum_{i=1}^m w_i\otimes z_i\Big)=\alpha\,\Pi\Big(\sum_{i=1}^n x_i\otimes y_i\Big)+\beta\,\Pi\Big(\sum_{i=1}^m w_i\otimes z_i\Big)$$

for every α, β in $\mathbb{C}$ and every $\sum_{i=1}^n x_i\otimes y_i$ and $\sum_{i=1}^m w_i\otimes z_i$ in $\mathcal{H}\otimes\mathcal{K}$. Moreover,

$$\Big\|\Pi\Big(\sum_{i=1}^{n}x_i\otimes y_i\Big)\Big\| = \Big\|\sum_{i=1}^{n}y_i\otimes x_i\Big\| = \Big\|\sum_{i=1}^{n}x_i\otimes y_i\Big\|$$

for every $\sum_{i=1}^{n}x_i\otimes y_i$ by the very definition of Π and according to (7.b). Thus Π is an invertible linear isometry (i.e., a unitary transformation) from the inner product space $\mathcal{H}\otimes\mathcal{K}$ to the inner product space $\mathcal{K}\otimes\mathcal{H}$, and so is its extension $\widehat{\Pi}$ from the completion $\mathcal{H}\mathbin{\widehat{\otimes}}\mathcal{K}$ to the completion $\mathcal{K}\mathbin{\widehat{\otimes}}\mathcal{H}$. Therefore, applying the same argument of Remark 6, for every pair of operators $A\in\mathcal{B}[\mathcal{H}]$ and $B\in\mathcal{B}[\mathcal{K}]$,

$$\widehat{\Pi}(A\mathbin{\widehat{\otimes}}B)\Big(\sum_i x_i\mathbin{\widehat{\otimes}}y_i\Big) = (B\mathbin{\widehat{\otimes}}A)\,\widehat{\Pi}\Big(\sum_i x_i\mathbin{\widehat{\otimes}}y_i\Big)$$

for every $\sum_i x_i\mathbin{\widehat{\otimes}}y_i$ in $\mathcal{H}\mathbin{\widehat{\otimes}}\mathcal{K}$ so that

$$\widehat{\Pi}(A\mathbin{\widehat{\otimes}}B) = (B\mathbin{\widehat{\otimes}}A)\widehat{\Pi}.$$

Outcome: $A\mathbin{\widehat{\otimes}}B$ and $B\mathbin{\widehat{\otimes}}A$ are unitarily equivalent. That is,

$$A\mathbin{\widehat{\otimes}}B \,\cong\, B\mathbin{\widehat{\otimes}}A. \tag{13}$$

We refer to this fact by saying that *the tensor product of a pair of operators is unitarily equivalent commutative.*

Observe that, according to Remark 5 (and up to unitary equivalence),

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}\mathbin{\widehat{\otimes}}E = \begin{pmatrix} A\mathbin{\widehat{\otimes}}E & B\mathbin{\widehat{\otimes}}E\\ C\mathbin{\widehat{\otimes}}E & D\mathbin{\widehat{\otimes}}E\end{pmatrix}\ \text{ in }\ \mathcal{B}[\mathcal{H}\mathbin{\widehat{\otimes}}\mathcal{K}] \tag{14}$$

for any $A\in\mathcal{B}[\mathcal{H}']$, $B\in\mathcal{B}[\mathcal{H}'',\mathcal{H}']$, $C\in\mathcal{B}[\mathcal{H}',\mathcal{H}'']$, $D\in\mathcal{B}[\mathcal{H}'']$ and $E\in\mathcal{B}[\mathcal{K}]$, where the above operator matrices lie in $\mathcal{B}[\mathcal{H}]$ and $\mathcal{B}[\mathcal{H}\otimes\mathcal{K}]$, with $\mathcal{H}=\mathcal{H}'\oplus\mathcal{H}''$. (Note: the tensor products $B\mathbin{\widehat{\otimes}}E$ and $C\mathbin{\widehat{\otimes}}E$ are defined exactly as the definition $A\mathbin{\widehat{\otimes}}E$ and $D\mathbin{\widehat{\otimes}}E$ in Definition 2.) The identity in (14) can be proved in the general case (without the assumption of Remark 5 that $\mathcal{H}$ is separable) as follows. Let $\mathcal{H}'$, $\mathcal{H}''$ and $\mathcal{K}$ be Hilbert spaces, and consider the Hilbert space $\mathcal{H}=\mathcal{H}'\oplus\mathcal{H}''$, where $\oplus$ stands for (orthogonal) direct sum. Take the tensor product spaces $\mathcal{H}'\mathbin{\widehat{\otimes}}\mathcal{K}$ and $\mathcal{H}''\mathbin{\widehat{\otimes}}\mathcal{K}$, their direct sum $(\mathcal{H}'\mathbin{\widehat{\otimes}}\mathcal{K})\oplus(\mathcal{H}''\mathbin{\widehat{\otimes}}\mathcal{K})$, and consider the natural mapping $\widehat{\Psi}\colon\mathcal{H}\mathbin{\widehat{\otimes}}\mathcal{K}\to(\mathcal{H}'\mathbin{\widehat{\otimes}}\mathcal{K})\oplus(\mathcal{H}''\mathbin{\widehat{\otimes}}\mathcal{K})$ given by

$$\widehat{\Psi}\Big(\sum_i x_i\mathbin{\widehat{\otimes}}y_i\Big) = \Big(\sum_i x_i'\mathbin{\widehat{\otimes}}y_i\,,\,\sum_i x_i''\mathbin{\widehat{\otimes}}y_i\Big)$$

with each $x_i\in\mathcal{H}$ written as $x_i=(x_i'\,,\,x_i'')\in\mathcal{H}'\oplus\mathcal{H}''$ where $x_i'\in\mathcal{H}'$ and $x_i''\in\mathcal{H}''$. This is clearly linear and invertible. Moreover, it is also an isometry by (12.b):

$$\begin{aligned}
\Big\|\widehat{\Psi}\Big(\sum_i x_i\mathbin{\widehat{\otimes}}y_i\Big)\Big\|^2 &= \Big\|\Big(\sum_i x_i'\mathbin{\widehat{\otimes}}y_i,\sum_i x_i''\mathbin{\widehat{\otimes}}y_i\Big)\Big\|^2 = \Big\|\sum_i x_i'\mathbin{\widehat{\otimes}}y_i\Big\|^2+\Big\|\sum_i x_i''\mathbin{\widehat{\otimes}}y_i\Big\|^2\\
&= \sum_{i,j}\langle x_i'\,;x_j'\rangle_{\mathcal{H}'}\,\langle y_i\,;y_j\rangle_{\mathcal{K}} + \sum_{i,j}\langle x_i''\,;x_j''\rangle_{\mathcal{H}''}\,\langle y_i\,;y_j\rangle_{\mathcal{K}}\\
&= \sum_{i,j}\big(\langle x_i'\,;x_j'\rangle_{\mathcal{H}'}+\langle x_i''\,;x_j''\rangle_{\mathcal{H}''}\big)\,\langle y_i\,;y_j\rangle_{\mathcal{K}} = \sum_{i,j}\big\langle(x_i'\,,\,x_i'')\,;(x_j'\,,\,x_j'')\big\rangle_{\mathcal{H}}\,\big\langle y_i\,;y_j\big\rangle_{\mathcal{K}}\\
&= \sum_{i,j}\langle x_i\,;x_j\rangle_{\mathcal{H}}\,\langle y_i\,;y_j\rangle_{\mathcal{K}} = \Big\|\sum_i x_i\mathbin{\widehat{\otimes}}y_i\Big\|^2
\end{aligned}$$

for every $\sum_i x_i \,\widehat{\otimes}\, y_i$. Thus $\widehat{\Psi}$ is a unitary transformation. Then $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$ and the direct sum $(\mathcal{H}' \,\widehat{\otimes}\, \mathcal{K}) \oplus (\mathcal{H}'' \,\widehat{\otimes}\, \mathcal{K})$ are unitarily equivalent, and we write

$$\mathcal{H} \,\widehat{\otimes}\, \mathcal{K} = (\mathcal{H}' \,\widehat{\otimes}\, \mathcal{K}) \oplus (\mathcal{H}'' \,\widehat{\otimes}\, \mathcal{K}). \tag{15.a}$$

Moreover, since $\big(\sum_i x_i' \,\widehat{\otimes}\, y_i \,, \sum_i x_i'' \,\widehat{\otimes}\, y_i\big)$ is the unitary image of $\sum_i x_i \,\widehat{\otimes}\, y_i$ under $\widehat{\Psi}$, we shall also identify them and write

$$\sum_i x_i \,\widehat{\otimes}\, y_i = \Big(\sum_i x_i' \,\widehat{\otimes}\, y_i \,, \sum_i x_i'' \,\widehat{\otimes}\, y_i\Big). \tag{15.b}$$

In particular

$$x \otimes y = (x' \otimes y \,,\, x'' \otimes y)$$

for every $x \otimes y$ with $x = (x',\, x'') \in \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ and $y \in \mathcal{K}$. Therefore, by taking $A \in \mathcal{B}[\mathcal{H}']$, $B \in \mathcal{B}[\mathcal{H}'', \mathcal{H}']$, $C \in \mathcal{B}[\mathcal{H}', \mathcal{H}'']$, $D \in \mathcal{B}[\mathcal{H}'']$ and $E \in \mathcal{B}[\mathcal{K}]$, we get

$$\begin{aligned}
&\Big[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \widehat{\otimes}\, E\Big](x \otimes y) = \Big[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \widehat{\otimes}\, E\Big]\Big[\begin{pmatrix} x' \\ x'' \end{pmatrix} \otimes y\Big] \\
&= \Big[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x' \\ x'' \end{pmatrix}\Big] \otimes Ey = \begin{pmatrix} Ax' + Bx'' \\ Cx' + Dx'' \end{pmatrix} \otimes Ey = \begin{pmatrix} (Ax' + Bx'') \otimes Ey \\ (Cx' + Dx'') \otimes Ey \end{pmatrix} \\
&= \begin{pmatrix} Ax' \otimes Ey + Bx'' \otimes Ey \\ Cx' \otimes Ey + Dx'' \otimes Ey \end{pmatrix} = \begin{pmatrix} (A \otimes E)(x' \otimes y) + (B \otimes E)(x'' \otimes y) \\ (C \otimes E)(x' \otimes y) + (D \otimes E)(x'' \otimes y) \end{pmatrix} \\
&= \begin{pmatrix} A \,\widehat{\otimes}\, E & B \,\widehat{\otimes}\, E \\ C \,\widehat{\otimes}\, E & D \,\widehat{\otimes}\, E \end{pmatrix}\begin{pmatrix} x' \otimes y \\ x'' \otimes y \end{pmatrix} = \begin{pmatrix} A \,\widehat{\otimes}\, E & B \,\widehat{\otimes}\, E \\ C \,\widehat{\otimes}\, E & D \,\widehat{\otimes}\, E \end{pmatrix}\Big[\begin{pmatrix} x' \\ x'' \end{pmatrix} \otimes y\Big] \\
&= \begin{pmatrix} A \,\widehat{\otimes}\, E & B \,\widehat{\otimes}\, E \\ C \,\widehat{\otimes}\, E & D \,\widehat{\otimes}\, E \end{pmatrix}(x \otimes y).
\end{aligned}$$

for $x \otimes y = (x',\, x'') \otimes y = (x' \otimes y \,,\, x'' \otimes y)$ with $x = (x',\, x'') \in \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ and $y \in \mathcal{K}$. Thus following the result in (14) since

$$\Big[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \widehat{\otimes}\, E\Big] \sum_i x_i \,\widehat{\otimes}\, y_i = \begin{pmatrix} A \,\widehat{\otimes}\, E & B \,\widehat{\otimes}\, E \\ C \,\widehat{\otimes}\, E & D \,\widehat{\otimes}\, E \end{pmatrix} \sum_i x_i \,\widehat{\otimes}\, y_i$$

for every $\sum_i x_i \,\widehat{\otimes}\, y_i = \sum_i (x_i' \,,\, x_i'') \,\widehat{\otimes}\, y_i = \big(\sum_i x_i' \,\widehat{\otimes}\, y_i \,, \sum_i x_i'' \,\widehat{\otimes}\, y_i\big)$ in $\mathcal{H} \,\widehat{\otimes}\, \mathcal{K}$. Note that, since unitary equivalence is indeed an equivalence relation, thus transitive, it follows by (13) and (14) that

$$E \,\widehat{\otimes} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cong \begin{pmatrix} E \,\widehat{\otimes}\, A & E \,\widehat{\otimes}\, B \\ E \,\widehat{\otimes}\, C & E \,\widehat{\otimes}\, D \end{pmatrix} \quad \text{in} \quad \mathcal{B}[\mathcal{K} \,\widehat{\otimes}\, \mathcal{H}] \tag{16}$$

5. Classes of Operators

We shall be dealing with the following well-known classes of operators. An operator T is self-adjoint if $T^* = T$, unitary if $T^* = T^{-1}$, nonnegative (i.e., $O \leq T$) if it is self-adjoint and $0 \leq \langle Tx\,; x\rangle$ for every x, normal if $TT^* = T^*T$, quasinormal if T commutes with T^*T, subnormal if it has a normal extension, hyponormal if $TT^* \leq T^*T$, quasihyponormal if $O \leq T^*(T^*T - TT^*)T$, semi-quasihyponormal if $|T|^2 \leq |T^2|$, paranormal if $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$ for every x, normaloid if $r(T) = \|T\|$ (where $r(T)$ stands for spectral radius), and spectraloid if $r(T) = w(T)$ (where $w(T)$

stands for the numerical radius). It is also well-known that these classes are related by proper inclusion as follows (see e.g., [11]).

$$\text{Nonnegative} \subset \text{Self-Adjoint} \subset \text{Normal} \qquad \text{and} \qquad \text{Unitary} \subset \text{Normal};$$

$$\begin{gathered}\text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal} \subset \text{Quasihyponormal} \\ \subset \text{Semi-Quasihyponormal} \subset \text{Paranormal} \subset \text{Normaloid} \subset \text{Spectraloid}.\end{gathered}$$

Proposition 5. *If $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ are either* (a) *self-adjoint,* (b) *unitary,* (c) *nonnegative,* (d) *normal,* (e) *quasinormal,* (f) *hyponormal,* (g) *quasihyponormal,* (h) *semi-quasihyponormal, or* (i) *normaloid, then so is $A \mathbin{\widehat{\otimes}} B \in \mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}$.*

Proof. Assertion (a) is trivial by Proposition 4(d): If $A^* = A$ and $B^* = B$, then

$$(A \mathbin{\widehat{\otimes}} B)^* = A^* \mathbin{\widehat{\otimes}} B^* = A \mathbin{\widehat{\otimes}} B.$$

Assertion (b) is also trivial by Proposition 4(d,e): If $A^* = A^{-1}$ and $B^* = B^{-1}$, then

$$(A \mathbin{\widehat{\otimes}} B)^* = A^* \mathbin{\widehat{\otimes}} B^* = A^{-1} \mathbin{\widehat{\otimes}} B^{-1} = (A \mathbin{\widehat{\otimes}} B)^{-1}.$$

Assertion (c) follows by Proposition 4(c,d): If $O \le A = A^*$ and $O \le B = B^*$, then take their unique nonnegative square root so that, for every $\sum_i x_i \mathbin{\widehat{\otimes}} y_i$ in $\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}$,

$$\begin{aligned}&\Big\langle (A \mathbin{\widehat{\otimes}} B) \sum_i x_i \mathbin{\widehat{\otimes}} y_i \,;\, \sum_i x_i \mathbin{\widehat{\otimes}} y_i \Big\rangle = \Big\langle (A^{\frac12} \mathbin{\widehat{\otimes}} B^{\frac12})\,(A^{\frac12} \mathbin{\widehat{\otimes}} B^{\frac12}) \sum_i x_i \mathbin{\widehat{\otimes}} y_i \,;\, \sum_i x_i \mathbin{\widehat{\otimes}} y_i \Big\rangle \\ &= \Big\langle (A^{\frac12} \mathbin{\widehat{\otimes}} B^{\frac12}) \sum_i x_i \mathbin{\widehat{\otimes}} y_i \,;\, (A^{\frac12} \mathbin{\widehat{\otimes}} B^{\frac12}) \sum_i x_i \mathbin{\widehat{\otimes}} y_i \Big\rangle = \Big\|(A^{\frac12} \mathbin{\widehat{\otimes}} B^{\frac12}) \sum_i x_i \mathbin{\widehat{\otimes}} y_i \Big\|^2 \ge 0.\end{aligned}$$

Assertion (d) is trivial by Proposition 4(c,d): If $A\,A^* = A^*A$ and $BB^* = B^*B$, then

$$\begin{aligned}(A \mathbin{\widehat{\otimes}} B)\,(A \mathbin{\widehat{\otimes}} B)^* &= (A \mathbin{\widehat{\otimes}} B)\,(A^* \mathbin{\widehat{\otimes}} B^*) = A\,A^* \mathbin{\widehat{\otimes}} BB^* \\ &= A^*A \mathbin{\widehat{\otimes}} B^*B = (A^* \mathbin{\widehat{\otimes}} B^*)\,(A \mathbin{\widehat{\otimes}} B) = (A \mathbin{\widehat{\otimes}} B)^*(A \mathbin{\widehat{\otimes}} B).\end{aligned}$$

Assertion (e) is also trivially verified by Proposition 4(c,d): If $A^*A\,A = A\,A^*A$ and $B^*B\,B = B\,B^*B$, then

$$\begin{aligned}&(A \mathbin{\widehat{\otimes}} B)^*(A \mathbin{\widehat{\otimes}} B)\,(A \mathbin{\widehat{\otimes}} B) = (A^* \mathbin{\widehat{\otimes}} B^*)\,(A \mathbin{\widehat{\otimes}} B)\,(A \mathbin{\widehat{\otimes}} B) = A^*A\,A \mathbin{\widehat{\otimes}} B^*B\,B \\ &= A\,A^*A \mathbin{\widehat{\otimes}} B\,B^*B = (A \mathbin{\widehat{\otimes}} B)\,(A^* \mathbin{\widehat{\otimes}} B^*)\,(A \mathbin{\widehat{\otimes}} B) = (A \mathbin{\widehat{\otimes}} B)\,(A \mathbin{\widehat{\otimes}} B)^*(A \mathbin{\widehat{\otimes}} B).\end{aligned}$$

Assertion (f) needs an auxiliary result. Take arbitrary operators $S, S_1, S_2 \in \mathcal{B}[\mathcal{H}]$ and $T, T_1, T_2 \in \mathcal{B}[\mathcal{K}]$, and note that, if $O \le S$ and $T_1 \le T_2$ (i.e., $O \le T_2 - T_1$), then

$$O \le S \mathbin{\widehat{\otimes}} (T_2 - T_1) = S \mathbin{\widehat{\otimes}} T_2 - S \mathbin{\widehat{\otimes}} T_1$$

by Proposition 4(a,b_1) and assertion (c). Similarly, if $O \le T$ and $S_1 \le S_2$, then

$$O \le (S_2 - S_1) \mathbin{\widehat{\otimes}} T = S_2 \mathbin{\widehat{\otimes}} T - S_1 \mathbin{\widehat{\otimes}} T$$

by Proposition 4(a,b_2) and assertion (c) again. Summing up:

$$O \le S \quad \text{and} \quad T_1 \le T_2 \quad \text{implies} \quad S \mathbin{\widehat{\otimes}} T_1 \le S \mathbin{\widehat{\otimes}} T_2, \tag{17.a}$$

$$O \le T \quad \text{and} \quad S_1 \le S_2 \quad \text{implies} \quad S_1 \mathbin{\widehat{\otimes}} T \le S_2 \mathbin{\widehat{\otimes}} T, \tag{17.b}$$

and therefore,

$$O \le S_1 \le S_2 \quad \text{and} \quad O \le T_1 \le T_2 \quad \text{implies} \quad S_1 \mathbin{\widehat{\otimes}} T_1 \le S_2 \mathbin{\widehat{\otimes}} T_2. \tag{17.c}$$

Recall that $O \leq A\,A^*$ and $O \leq B\,B^*$ for every $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$. Thus, if $A\,A^* \leq A^*A$ and $BB^* \leq B^*B$ then, by Proposition 4(c,d) and (17.c),

$$\begin{aligned}(A \mathbin{\widehat{\otimes}} B)\,(A \mathbin{\widehat{\otimes}} B)^* &= (A \mathbin{\widehat{\otimes}} B)\,(A^* \mathbin{\widehat{\otimes}} B^*) = A\,A^* \mathbin{\widehat{\otimes}} BB^* \\ &\leq A^*A \mathbin{\widehat{\otimes}} B^*B = (A^* \mathbin{\widehat{\otimes}} B^*)\,(A \mathbin{\widehat{\otimes}} B) = (A \mathbin{\widehat{\otimes}} B)^*(A \mathbin{\widehat{\otimes}} B).\end{aligned}$$

Assertion (g) also follows by the inequalities in (17). Recall that the absolute value $|T|$ of any T in $\mathcal{B}[\mathcal{H}]$ is defined as the unique nonnegative square root of the nonnegative operator $|T|^2 = T^*T$ in $\mathcal{B}[\mathcal{H}]$. Note that, according to Proposition 4(c,d),

$$|A \mathbin{\widehat{\otimes}} B|^2 = (A^* \mathbin{\widehat{\otimes}} B^*)(A \mathbin{\widehat{\otimes}} B) = A^*A \mathbin{\widehat{\otimes}} B^*B = |A|^2 \mathbin{\widehat{\otimes}} |B|^2, \tag{18.a}$$

and hence, by uniqueness of the nonnegative square root of nonnegative operators,

$$|A \mathbin{\widehat{\otimes}} B| = |A| \mathbin{\widehat{\otimes}} |B|. \tag{18.b}$$

Therefore, if $|A|^4 \leq |A^2|^2$ and $|B|^4 \leq |B^2|^2$ then, by the inequality in (17.c),

$$\begin{aligned}|A \mathbin{\widehat{\otimes}} B|^4 &= (|A|^2 \mathbin{\widehat{\otimes}} |B|^2)^2 = |A|^4 \mathbin{\widehat{\otimes}} |B|^4 \\ &\leq |A^2|^2 \mathbin{\widehat{\otimes}} |B^2|^2 = (|A^2| \mathbin{\widehat{\otimes}} |B^2|)^2 = |A^2 \mathbin{\widehat{\otimes}} B^2|^2.\end{aligned}$$

Assertion (h) is similar. If $|A|^2 \leq |A^2|$ and $|B|^2 \leq |B^2|$ then, by (17) and (18),

$$|A \mathbin{\widehat{\otimes}} B|^2 = |A|^2 \mathbin{\widehat{\otimes}} |B|^2 \leq |A^2| \mathbin{\widehat{\otimes}} |B^2| = |A^2 \mathbin{\widehat{\otimes}} B^2|.$$

Assertion (i) is readily verified by Propositions 3 and 4(c). Indeed, a trivial induction shows that, according to Proposition 4(c),

$$(A \mathbin{\widehat{\otimes}} B)^n = A^n \mathbin{\widehat{\otimes}} B^n, \tag{19.a}$$

and hence, by Proposition 3 and Remark 3,

$$\|(A \mathbin{\widehat{\otimes}} B)^n\| = \|A^n\|\,\|B^n\| \tag{19.b}$$

for every nonnegative integer n. Thus if $\|A^n\| = \|A\|^n$ and $\|B^n\| = \|B\|^n$, then

$$\|(A \mathbin{\widehat{\otimes}} B)^n\| = \|A^n\|\,\|B^n\| = \|A\|^n\|B\|^n = \big(\|A\|\,\|B\|\big)^n = \|A \mathbin{\widehat{\otimes}} B\|^n$$

for every nonnegative integer n. $\square$

Proposition 6. *If A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$ are subnormal, then so is their tensor product $A \mathbin{\widehat{\otimes}} B$ in $\mathcal{H} \mathbin{\widehat{\otimes}} \mathcal{K}$.*

Proof. Let A and B be subnormal operators, and let N and M be the normal extensions of A and B:

$$N = \begin{pmatrix} A & X \\ O & Y \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} B & W \\ O & Z \end{pmatrix}.$$

According to (14),

$$N \mathbin{\widehat{\otimes}} M = \begin{pmatrix} A & X \\ O & Y \end{pmatrix} \mathbin{\widehat{\otimes}} M = \begin{pmatrix} A \mathbin{\widehat{\otimes}} M & X' \\ O & Y' \end{pmatrix}.$$

Now recall from (13) that the tensor product of a pair of operators is unitarily equivalent commutative which together with (14) implies (16), and therefore,

$$A \mathbin{\widehat{\otimes}} M = A \mathbin{\widehat{\otimes}} \begin{pmatrix} B & W \\ O & Z \end{pmatrix} \cong \begin{pmatrix} A \mathbin{\widehat{\otimes}} B & W' \\ O & Z' \end{pmatrix}.$$

Hence

$$N \mathbin{\widehat{\otimes}} M \cong \begin{pmatrix} \begin{pmatrix} A \mathbin{\widehat{\otimes}} B & W' \\ O & Z' \end{pmatrix} & X' \\ O & Y' \end{pmatrix},$$

Since, by Proposition 5(d), $N \mathbin{\widehat{\otimes}} M$ is normal, and since normality is preserved under unitary equivalence, it follows that $A \mathbin{\widehat{\otimes}} B$ is subnormal. □

Remark 7. The proof of Proposition 6, ensuring subnormality of the tensor product of subnormal operators, followed a path that is different from that in the proof of Proposition 5 (which relies on the concept of adjoint). The same happens with other properties, not only with tensor products. For instance, it also happens with direct sums of subnormal operators. Indeed, recall that (cf. [4, p. 43]), if

$$\begin{pmatrix} A & X \\ O & Y \end{pmatrix} \ \text{ in } \ \mathcal{B}[\mathcal{H} \oplus \mathcal{H}'] \quad \text{ and } \quad \begin{pmatrix} B & W \\ O & Z \end{pmatrix} \ \text{ in } \ \mathcal{B}[\mathcal{K} \oplus \mathcal{K}']$$

are normal, then $XX^* = A^*A - A\,A^*$, $X^*X = YY^* - Y^*Y$, $XY^* = A^*X$; and similarly, $WW^* = B^*B - BB^*$, $W^*W = ZZ^* - Z^*Z$, $WZ^* = B^*W$. This implies that the operator below, which is made up of direct sums, also is normal:

$$\begin{pmatrix} A & O & X & O \\ O & B & O & W \\ O & O & Y & O \\ O & O & O & X \end{pmatrix} = \begin{pmatrix} A \oplus B & X \oplus W \\ O & Y \oplus Z \end{pmatrix} \ \text{ in } \ \mathcal{B}[\mathcal{H} \oplus \mathcal{K} \oplus \mathcal{H}' \oplus \mathcal{K}'].$$

Thus $A \oplus B$ is subnormal whenever A and B are.

This can be generalized: If each

$$\begin{pmatrix} B_k & W_k \\ O & Z_k \end{pmatrix} \ \text{ in } \ \mathcal{B}[\mathcal{K}_k \oplus \mathcal{K}'_k]$$

is normal, then the operator below is still normal:

$$\begin{pmatrix} \bigoplus_k B_k & \bigoplus_k W_k \\ O & \bigoplus_k Z_k \end{pmatrix} \ \text{ in } \ \mathcal{B}\big[\bigoplus_k \mathcal{K}_k \oplus \bigoplus_k \mathcal{K}'_k\big].$$

Thus a countable direct sum $\bigoplus_k B_k$ of subnormal operators B_k is subnormal.

Now suppose $\mathcal{H}$ is separable (so that, if it is infinite-dimensional, then we may identify $\mathcal{H} = \ell^2_+$). According to Remark 5,

$$I \mathbin{\widehat{\otimes}} B = \begin{pmatrix} B & & \\ & B & \\ & & \ddots \end{pmatrix} = \bigoplus_k B \ \text{ in } \ \mathcal{B}\big[\bigoplus_k \mathcal{K}\big].$$

which is subnormal whenever B is. This can be trivially extended from the identity to any diagonal operator (and hence to any diagonalizable operator). In particular, this can be extended to any orthogonal projection. Indeed, any orthogonal projection P can be decomposed as $P = I \oplus O$ on $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P)$, where $\mathcal{R}(P)$ is the range of P and $\mathcal{N}(P)$ is the null space (kernel) of P, so that $I = P|_{\mathcal{R}(P)}$ and $O = P|_{\mathcal{N}(P)}$. Therefore, by Remark 5 again,

$$P \mathbin{\widehat{\otimes}} B = (I \oplus O) \mathbin{\widehat{\otimes}} B = \begin{pmatrix} B & & & & & \\ & B & & & & \\ & & \ddots & & & \\ & & & O & & \\ & & & & O & \\ & & & & & \ddots \end{pmatrix} = \bigoplus\nolimits_k B \oplus \bigoplus\nolimits_\ell O = \bigoplus\nolimits_k B \oplus O,$$

which is subnormal whenever B is. Thus the above result extends naturally to compact normal operators by the Spectral Theorem (according to Remark 5, and so under the assumption that $\mathcal{H}$ is separable): Let $N = \bigoplus_k \lambda_k P_k$ be the spectral decomposition of a compact normal operator N on $\mathcal{H}$, with $\{\lambda_k\} = \sigma_P(N)$ denoting the point spectrum of N (i.e., the set of all eigenvalues of N), so that

$$N \mathbin{\widehat{\otimes}} B = \Big(\bigoplus\nolimits_k \lambda_k P_k\Big) \mathbin{\widehat{\otimes}} B = \bigoplus\nolimits_k \lambda_k \,(P_k \mathbin{\widehat{\otimes}} B).$$

We may proceed formally to extend the above result from compact normal to plain normal operators, again by the Spectral Theorem (and still according to Remark 5, thus under the assumption that $\mathcal{H}$ is separable): Let $N = \bigoplus_{\lambda\in\sigma(N)} \lambda P_\lambda$ be the spectral decomposition of a normal operator N on $\mathcal{H}$ (with $\sigma(N)$ denoting the spectrum of N) so that

$$N \mathbin{\widehat{\otimes}} B = \Big(\bigoplus\nolimits_{\lambda\in\sigma(N)} \lambda P_\lambda\Big) \mathbin{\widehat{\otimes}} B = \bigoplus\nolimits_{\lambda\in\sigma(N)} \lambda \,(P_\lambda \mathbin{\widehat{\otimes}} B).$$

Thus $N \mathbin{\widehat{\otimes}} B$ is subnormal whenever B is subnormal and N is normal. Therefore, still under the assumption that $\mathcal{H}$ is separable, Remark 5 says that if the subnormal A on $\mathcal{H}$ has a normal extension $N = \left(\begin{smallmatrix} A & X \\ O & Y \end{smallmatrix}\right)$ acting on a separable space, then

$$N \mathbin{\widehat{\otimes}} B = \begin{pmatrix} A & X \\ O & Y \end{pmatrix} \mathbin{\widehat{\otimes}} B = \begin{pmatrix} A \mathbin{\widehat{\otimes}} B & X' \\ O & Y' \end{pmatrix}$$

is subnormal, and hence it has a normal extension, say

$$\begin{pmatrix} N \mathbin{\widehat{\otimes}} B & X'' \\ O & Y'' \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} A \mathbin{\widehat{\otimes}} B & X' \\ O & Y' \end{pmatrix} & X'' \\ O & Y'' \end{pmatrix},$$

which is a normal extension for $A \mathbin{\widehat{\otimes}} B$ too, and so $A \mathbin{\widehat{\otimes}} B$ is subnormal. This leads to an alternate proof for Proposition 6 which is entirely based on Remark 5, and hence restricted to the assumption that $\mathcal{H}$ is separable

Propositions 5 and 6 exhibited some classes of operators that, as it is well known, are preserved when taking the tensor product. The converse to many of those statements also holds true. *If $A \mathbin{\widehat{\otimes}} B$ is either normal, quasinormal, subnormal or hyponormal, then so are both A and B* (if they are nonzero) [16]. Moreover, this has been verified for other classes of close to normal operators. For instance, p-hyponormality [5], quasi-p-hyponormality, w-hyponormality and log-hyponormality [7], as well as posinormality [12], are also preserved when taking tensor products. However, such a preservation may fail for some important classes of operators. Indeed, there exist paranormal or spectraloid operators A and B for which $A \,\widehat{\otimes}\, B$ is not paranormal or spectraloid. That is, the properties of being either paranormal or spectraloid are not preserved when taking tensor products [14, pp. 629,631].

Tensor products of operators comprise a most useful way for providing examples and counterexamples (e.g., see [14, Section 6]). One of the main reasons for that comes from the fact that $\sigma(A \,\widehat{\otimes}\, B) = \sigma(A)\,\sigma(B)$ — the spectrum of $A \,\widehat{\otimes}\, B$ is the product of the spectra of A and B [2] (see also [15]). In particular, the first example of a strongly stable operator that is not similar to any contraction was obtained in [8] by means of the tensor product $S^* \,\widehat{\otimes}\, F$ on $\ell^2_+ \,\widehat{\otimes}\, (\ell^2_+ \oplus \ell^2_+) = \ell^2_+(\ell^2_+ \oplus \ell^2_+)$ of the adjoint of the canonical unilateral shift S^* on ℓ^2_+ with the Foguel operator F on $\ell^2_+ \oplus \ell^2_+$ (see [9, Section 8.2]). For a discussion on stability for tensor products of Hilbert space operators see [5] and [12].

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