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REVERSED WAVELET FUNCTIONS AND SUBSPACES

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Abstract. Let the operators D and T be the dilation-by-2 and translation-by-1 on $\mathcal{L}^2(\mathbb{R})$, which are both bilateral shifts of infinite multiplicity. If $\psi(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$ is a wavelet, then $\{D^m T^n\psi(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$ is an orthonormal basis for the Hilbert space $\mathcal{L}^2(\mathbb{R})$ but the reversed set $\{T^n D^m\psi(\cdot)\}_{(n,m)\in\mathbb{Z}^2}$ is not. In this paper we investigate the role of the reversed functions $T^n D^m\psi(\cdot)$ in wavelet theory. As a consequence, we exhibit an orthogonal decomposition of $\mathcal{L}^2(\mathbb{R})$ into T-reducing subspaces upon which part of the bilateral shift T consists of a countably infinite direct sum of bilateral shifts of multiplicity one, which mirrors a well-known decomposition of the bilateral shift D.

1. Introduction

In the following we will be dealing with the function space $\mathcal{L}^2(\mathbb{R})$, with the usual inner product and norm denoted by $\langle\ ;\ \rangle$ and $\|\ \|$, respectively, as well as with two unitary operators defined on that space, more precisely, two bilateral shifts of infinite multiplicity [3]: the dilation-by-2 operator D on $\mathcal{L}^2(\mathbb{R})$ defined by

$$Df(\cdot) = g(\cdot) \quad \text{with} \quad g(\cdot) = \sqrt{2}\, f\big(2(\cdot)\big),$$

and the translation-by-1 operator T on $\mathcal{L}^2(\mathbb{R})$ defined by

$$Tf(\cdot) = g(\cdot) \quad \text{with} \quad g(\cdot) = f\big((\cdot) - 1\big).$$

Let $\mathbb{Z}$ denote the set of all integers. A unit function $\psi(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$ (i.e., a function $\psi(\cdot)\in\mathcal{L}^2(\mathbb{R})$ such that $\|\psi(\cdot)\| = 1$) is a wavelet if the unit functions $\psi_{m,n}(\cdot)$ — called wavelet functions — generated by $\psi(\cdot)$ as

$$\psi_{m,n}(\cdot) = \sqrt{2}^{\,m}\psi\big(2^m(\cdot) - n\big) = D^m T^n\psi(\cdot) \quad \text{for every} \quad (m,n)\in\mathbb{Z}^2$$

constitute an orthonormal basis for the Hilbert space $\mathcal{L}^2(\mathbb{R})$ [7, 10]. However, it was shown in [2] that there does not exist a unit function $\varphi(\cdot)$ such that the set of reversed functions $\{T^n D^m\varphi(\cdot)\}_{(n,m)\in\mathbb{Z}^2}$ forms an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$.

In this paper we study the role played by the reversed functions $T^n D^m\psi(\cdot)$ — called reversed wavelet functions — where $\psi(\cdot)$ is a wavelet.

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2. Reversed Wavelets

Let D and T be the dilation-by-2 and the translation-by-1 operators on $\mathcal{L}^2(\mathbb{R})$. For each $(m,n) \in \mathbb{Z}^2$ consider the wavelet functions $\psi_{m,n}(\cdot) = D^m T^n \psi(\cdot)$ generated by a wavelet $\psi(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$ so that $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$. For each $m \in \mathbb{Z}$ define the *scale subspaces* as follows [10] (also see [1], [4], [5], [6], [8], [9] and [11]).

$$\mathcal{W}_m = D^m \mathcal{W}_0 = D^m \bigvee_{n\in\mathbb{Z}} T^n \psi(\cdot) = \bigvee_{n\in\mathbb{Z}} D^m T^n \psi(\cdot) \quad \text{with} \quad \mathcal{W}_0 = \bigvee_{n\in\mathbb{Z}} T^n \psi(\cdot). \tag{1}$$

(The third identity holds since the D is unitary [6, Corollary 3].) $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ is an orthogonal subspace basis for $\mathcal{L}^2(\mathbb{R})$ in the sense that it is an orthogonal family of subspaces (i.e., of closed linear manifolds) of $\mathcal{L}^2(\mathbb{R})$ that span $\mathcal{L}^2(\mathbb{R})$,

$$\mathcal{L}^2(\mathbb{R}) = \bigvee_{m\in\mathbb{Z}} \mathcal{W}_m = \bigvee_{m\in\mathbb{Z}} D^m \mathcal{W}_0,$$

with $\bigvee_{m\in\mathbb{Z}} \mathcal{W}_m = \big(\operatorname{span} \bigcup_{m\in\mathbb{Z}} \mathcal{W}_m\big)^-$; the closure of the span of $\bigcup_{m\in\mathbb{Z}} \mathcal{W}_m$. The above expression can be equivalently written as an orthogonal direct sum,

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m = \bigoplus_{m\in\mathbb{Z}} D^m \mathcal{W}_0,$$

since $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ is an orthogonal family of subspaces. This characterize $\mathcal{W}_0$ as a generating subspace for D and, together with the fact that

$$D^k \mathcal{W}_0 \perp D^{k'} \mathcal{W}_0 \quad \text{whenever} \quad k \neq k' \text{ in } \mathbb{Z},$$

$\mathcal{W}_0$ is also a wandering subspace for D. Thus $\mathcal{W}_0$ is a generating wandering subspace for D, confirming that D is a bilateral shift of infinite multiplicity because the multiplicity of D is the dimension of the generating wandering subspace $\mathcal{W}_0$, which is not finite [3, 12]). Observe that, since D is unitary, each $\mathcal{W}_m$ is also D-wandering ($D^k \mathcal{W}_m \perp D^{k'} \mathcal{W}_m$) and generating ($\mathcal{L}^2(\mathbb{R}) = \bigvee_{k\in\mathbb{Z}} D^{k+m} \mathcal{W}_0 = \bigvee_{k\in\mathbb{Z}} D^k \mathcal{W}_m$).

We now associate with a given wavelet $\psi(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$ the *reversed functions* $r_{n,m}(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$, which are defined by

$$r_{n,m}(\cdot) = T^n D^m \psi(\cdot) \quad \text{for every} \quad (n,m) \in \mathbb{Z}^2.$$

It is readily verified (see e.g., [6, Proposition 3]) that, for each $(n,m) \in \mathbb{Z}^2$,

$$T^n D^m = D^m T^{n(2^m)} \quad \text{or, equivalently,} \quad D^{*m} T^n D^m = T^{n(2^m)} \tag{2}$$

since D is unitary (i.e., $D^{-1} = D^*$). Therefore,

$$r_{n,m}(\cdot) = D^m T^{n(2^m)} \psi(\cdot) \quad \text{for every} \quad (n,m) \in \mathbb{Z}^2.$$

This, however, does not imply that $r_{n,m}(\cdot)$ is a wavelet function, unless m is nonnegative, since in that case $n(2^m)$ are integers. Therefore, $r_{n,m}(\cdot)$ are actually the wavelet functions $\psi_{m,n(2^m)}(\cdot)$ for every $m \in \mathbb{N}_0$ and every $n \in \mathbb{Z}$, where $\mathbb{N}_0$ stands for the set of all nonnegative integers (i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). That is,

$$r_{n,m}(\cdot) = D^m T^{n(2^m)} \psi(\cdot) = \psi_{m,n(2^m)}(\cdot) \quad \text{for every} \quad (n,m) \in \mathbb{Z} \times \mathbb{N}_0. \tag{3}$$

Definition 1. Let $\psi(\cdot)$ be a wavelet with respect to the bilateral shifts D and T. The orthonormal functions

$$r_{n,m}(\cdot) = T^n D^m \psi(\cdot) \quad \text{for every} \quad (n,m) \in \mathbb{Z} \times \mathbb{N}_0$$

are called *reversed wavelet functions* generated from the wavelet $\psi(\cdot)$.

To proceed, we define, for each $m \in \mathbb{N}_0$, the *reversed scale subspaces*

$$\mathcal{R}_m = \bigvee_{n\in\mathbb{Z}} T^n D^m \psi(\cdot) = \bigvee_{n\in\mathbb{Z}} D^m T^{n(2^m)} \psi(\cdot) = D^m \bigvee_{n\in\mathbb{Z}} T^{n(2^m)} \psi(\cdot), \tag{4}$$

where the first identity defines each $\mathcal{R}_n$ while the second and third follow from (2) and the fact that D is unitary [6, Corollary 3], respectively. Thus it is plain from (1) that $\mathcal{R}_m$ is a subspace of the scale subspace $\mathcal{W}_m$:

$$\mathcal{R}_0 = \mathcal{W}_0 \qquad \text{and} \qquad \mathcal{R}_m \subset \mathcal{W}_m \quad \text{for every} \quad m \in \mathbb{N}$$

so that $\{\mathcal{R}_m\}_{m\in\mathbb{N}_0}$ is a sequence of orthogonal subspaces,

$$\mathcal{R}_m \perp \mathcal{R}_{m'} \quad \text{whenever} \quad m \neq m'.$$

Since $\bigvee_{n\in\mathbb{Z}} T^{n(2^m)} \psi(\cdot) \subset \mathcal{W}_0$ for each $m \in \mathbb{N}_0$, set

$$\mathcal{W}_{0,(m)} = \bigvee_{n\in\mathbb{Z}} T^{n(2^m)} \psi(\cdot) \subset \mathcal{W}_0$$

for each $m \in \mathbb{N}_0$ so that $\mathcal{R}_m$ can be expressed as

$$\mathcal{R}_m = D^m \mathcal{W}_{0,(m)} \quad \text{for each} \quad m \in \mathbb{N}_0.$$

Next we show that each $\mathcal{W}_{0,(m)}$ is also D-wandering. Actually, we shall show more than this. For each pair of integers $m \in \mathbb{N}_0$ and $k \in \mathbb{Z}$ define the subspaces

$$\mathcal{W}_{k,(m)} = D^k \mathcal{W}_{0,(m)} = D^k \bigvee_{n\in\mathbb{Z}} T^{n(2^m)} \psi(\cdot) = \bigvee_{n\in\mathbb{Z}} D^k T^{n(2^m)} \psi(\cdot). \tag{5}$$

Note that

$$\mathcal{W}_{0,(0)} = \mathcal{R}_0 = \mathcal{W}_0 \qquad \text{and} \qquad \mathcal{W}_{m,(m)} = \mathcal{R}_m \quad \text{for each} \quad m \in \mathbb{N}_0.$$

Lemma 1. *Any pair of subspaces with distinct indices k taken from the family $\{\mathcal{W}_{k,(m)}\}_{(k,m)\in\mathbb{Z}\times\mathbb{N}_0}$ is an orthogonal pair. For each $m \in \mathbb{N}_0$, $\{\mathcal{W}_{k,(m)}\}_{k\in\mathbb{Z}}$ is a subfamily of D-wandering subspaces included in the family $\{\mathcal{W}_k\}_{k\in\mathbb{Z}}$ of D-wandering scale subspaces. Moreover,*

(i) $\mathcal{W}_{k,(m+1)} \subset \mathcal{W}_{k,(m)}$ for every $(k,m) \in \mathbb{Z} \times \mathbb{N}_0$,
(ii) $\bigcap_{m\in\mathbb{N}_0} \mathcal{W}_{k,(m)} = \operatorname{span}\{D^k \psi(\cdot)\}$ for each $k \in \mathbb{Z}$,
(iii) $\bigcup_{m\in\mathbb{N}_0} \mathcal{W}_{k,(m)} = \mathcal{W}_k$ for each $k \in \mathbb{Z}$.

Proof. Since $\{\mathcal{W}_k\}_{k\in\mathbb{Z}}$ are mutually orthogonal and since

$$\mathcal{W}_{k,(m)} \subset \mathcal{W}_k \quad \text{for every} \quad (k,m) \in \mathbb{Z} \times \mathbb{N}_0,$$

it follows that

$$\mathcal{W}_{k,(m)} \perp \mathcal{W}_{k',(m')} \quad \text{whenever } k \neq k' \text{ in } \mathbb{Z} \quad \text{for every} \quad (m,m') \in \mathbb{N}_0^2.$$

In particular, $\mathcal{W}_{k,(m)} \perp \mathcal{W}_{k',(m)}$ if $k \neq k'$ for any m so that

$$D^k \mathcal{W}_{0,(m)} \perp D^{k'} \mathcal{W}_{0,(m)} \quad \text{whenever } k \neq k' \text{ in } \mathbb{Z} \quad \text{for each} \quad m \in \mathbb{N}_0.$$

Therefore the subspaces $\{\mathcal{W}_{0,(m)}\}_{m\in\mathbb{N}_0}$ are indeed D-wandering. Hence, so are the subspaces $\{\mathcal{W}_{k,(m)}\}_{m\in\mathbb{N}_0}$ for every $k\in\mathbb{Z}$. Now observe that $n(2^{m+1})=2n(2^m)$ for any integers $m\in\mathbb{N}_0$ and $n\in\mathbb{Z}$. Thus we have from (5) with $k=0$ that

$$\mathcal{W}_{0,(m+1)} \;=\; \bigvee_{n\in\mathbb{Z}} T^{2n(2^m)}\psi(\cdot) \quad \text{for each} \quad m\in\mathbb{N}_0$$

so that $\mathcal{W}_{0,(m+1)}\subset\mathcal{W}_{0,(m)}$ for every $m\in\mathbb{N}_0$. Similarly, in general,

$$\mathcal{W}_{k,(m+1)}\subset\mathcal{W}_{k,(m)} \quad \text{for every} \quad (k,m)\in\mathbb{Z}\times\mathbb{N}_0,$$

which shows (i). To verify (ii) take an arbitrary $\psi(\cdot)\in\mathcal{L}^2(\mathbb{R})$. Note that

$$\operatorname{span}\bigcup_{n\in\mathbb{Z}}\big\{T^{n(2^m)}\psi(\cdot)\big\} \;=\; \operatorname{span}\Big\{\{\psi(\cdot)\}\cup\bigcup_{n\in\mathbb{Z}\backslash\{0\}}\big\{T^{n(2^m)}\psi(\cdot)\big\}\Big\}$$

for every $m\in\mathbb{N}_0$, and hence

$$\operatorname{span}\{\psi(\cdot)\} \;\subseteq\; \Big(\operatorname{span}\bigcup_{n\in\mathbb{Z}}\{T^{n(2^m)}\psi(\cdot)\}\Big)^{-} \;=\; \bigvee_{n\in\mathbb{Z}} T^{n(2^m)}\psi(\cdot) \quad \text{for all} \quad m\in\mathbb{N}_0.$$

Moreover, if a nonzero $\varphi(\cdot)$ lies in $\operatorname{span}\bigcup_{n\in\mathbb{Z}\backslash\{0\}}\{T^{n(2^m)}\psi(\cdot)\}$ for all $m\in\mathbb{N}_0$ then $\varphi\notin\operatorname{span}\{T^{n(2^m)}\psi(\cdot)\}$ for any pair of integers $n\in\mathbb{Z}\backslash\{0\}$ and $m\in\mathbb{N}_0$, which is a contradiction. (Reason: $T^{n(2^m)}\psi(\cdot)=D^{-m}r_{n,m}(\cdot)$ for each $(n,m)\in\mathbb{Z}\times\mathbb{N}_0$ by (3) so that $\{T^{n(2^m)}\psi(\cdot)\}_{(n,m)\in\mathbb{Z}\times\mathbb{N}_0}$ is an orthonormal set by Definition 1 because D^{-m} is unitary for every $m\in\mathbb{Z}$). Thus

$$\operatorname{span}\bigcup_{n\in\mathbb{Z}\backslash\{0\}}\big\{T^{n(2^m)}\psi(\cdot)\big\}=\{0\}.$$

Therefore,

$$\bigcap_{m\in\mathbb{N}_0}\bigvee_{n\in\mathbb{Z}} T^{n(2^m)}\psi(\cdot)=\bigcap_{m\in\mathbb{N}_0}\Big(\operatorname{span}\bigcup_{n\in\mathbb{Z}}\big\{T^{n(2^m)}\psi(\cdot)\big\}\Big)^{-}=\operatorname{span}\{\psi(\cdot)\}.$$

Since each D^k is invertible, it follows from (5) that

$$\bigcap_{m\in\mathbb{N}_0}\mathcal{W}_{k,(m)}=\bigcap_{m\in\mathbb{N}_0}D^k\bigvee_{n\in\mathbb{Z}}T^{n(2^m)}\psi(\cdot)=D^k\bigcap_{m\in\mathbb{N}_0}\bigvee_{n\in\mathbb{Z}}T^{n(2^m)}\psi(\cdot)=D^k\operatorname{span}\{\psi(\cdot)\}.$$

for every $k\in\mathbb{Z}$, which proves (ii). Furthermore, according to (1) and (5),

$$\mathcal{W}_{k,(m)}\subset\mathcal{W}_{k,(0)}=\mathcal{W}_k \quad \text{for every} \quad (k,m)\in\mathbb{Z}\times\mathbb{N}_0$$

so that (iii) holds true. □

3. Reversed Time-Shifts

We now turn to the time-shift subspaces analog of the scale subspaces $\mathcal{W}_m$ [8]:

$$\mathcal{H}_n \;=\; \bigvee_{m\in\mathbb{Z}} D^m\,T^n\psi(\cdot) \quad \text{for every} \quad n\in\mathbb{Z}, \tag{6}$$

so that $\mathcal{H}_0=\bigvee_{m\in\mathbb{Z}}D^m\psi(\cdot)$. The family $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ also constitute an orthogonal subspace basis for $\mathcal{L}^2(\mathbb{R})$ [8]. Then, as before, the *reversed time-shift subspaces* are defined, for every $n\in\mathbb{Z}$, by

$$\mathcal{S}_n \;=\; T^n\bigvee_{m\in\mathbb{N}_0}D^m\psi(\cdot) \;=\; \bigvee_{m\in\mathbb{N}_0}T^nD^m\psi(\cdot) \;=\; \bigvee_{m\in\mathbb{N}_0}D^m\,T^{n(2^m)}\psi(\cdot). \tag{7}$$

Again, the first identity defines each $\mathcal{S}_n$, while the second follows from the fact that T is also unitary [6, Corollary 3] and the third follows from (2). Therefore

$$\mathcal{S}_n = T^n\mathcal{S}_0 \quad \text{for every} \quad n\in\mathbb{Z}, \qquad \text{where} \qquad S_0 \;=\; \bigvee_{m\in\mathbb{N}_0} D^m\psi(\cdot)\;\subset\;\mathcal{H}_0.$$

Lemma 2. *The reversed time-shift subspaces $\mathcal{S}_n$ generated by the reversed wavelet functions $r_{n,m}(\cdot) = T^nD^m\psi(\cdot)$ for each $(n,m)\in\mathbb{Z}\times\mathbb{N}_0$ are mutually orthogonal. Moreover, $\mathcal{S}_0$ is a wandering subspace for the translation-by-1 operator T.*

Proof. Recall that $\{r_{n,m}(\cdot)\}_{(n,m)\in\mathbb{Z}\times\mathbb{N}_0}$ is a double indexed orthonormal set of functions (Definition 1) so that $r_{n,m}(\cdot)\perp r_{n',m'}(\cdot)$ whenever $(n,m)\neq(n',m')$; in particular, $r_{n,m}(\cdot)\perp r_{n',m}(\cdot)$ for every $m\in\mathbb{N}_0$ whenever $n\neq n'$ in $\mathbb{Z}$. Thus we get from (7) that $\{S_n\}_{n\in\mathbb{Z}}$ is an orthogonal family of subspaces so that

$$T^n\mathcal{S}_0 = \mathcal{S}_n = \bigvee_{m\in\mathbb{N}_0} r_{n,m}(\cdot) \;\;\perp\;\; \bigvee_{m\in\mathbb{N}_0} r_{n',m}(\cdot) = \mathcal{S}_{n'} = T^{n'}\mathcal{S}_0 \quad \text{whenever} \quad n\neq n',$$

and hence $\mathcal{S}_0$ is a wandering subspace for T. □

As we saw above, if $\psi(\cdot)$ is a wavelet in $\mathcal{L}^2(\mathbb{R})$ with respect to the bilateral shifts D and T on $\mathcal{L}^2(\mathbb{R})$, then $\{D^m\,T^n\psi(\cdot)\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is a double indexed orthonormal basis for the Hilbert space $\mathcal{L}^2(\mathbb{R})$ so that, according to (1) and (6),

$$\mathcal{L}^2(\mathbb{R}) \;=\; \bigvee_{(m,n)\in\mathbb{Z}^2} D^mT^n\psi(\cdot) = \bigvee_{m\in\mathbb{Z}}\mathcal{W}_m \;=\; \bigvee_{n\in\mathbb{Z}}\mathcal{H}_n$$

and, since $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ and $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ are both families of orthogonal subspaces of $\mathcal{L}^2(\mathbb{R})$, this can be rewritten as orthogonal direct sums [8],

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}}\mathcal{W}_m = \bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n.$$

Now consider the following subspace of $\mathcal{L}^2(\mathbb{R})$.

$$\mathcal{M} \;=\; \bigvee_{(n,m)\in\mathbb{Z}\times\mathbb{N}_0} T^nD^m\psi(\cdot).$$

Since $\{T^nD^m\psi\}_{(n,m)\in\mathbb{N}_0\times\mathbb{Z}}$ also is an orthonormal set (Definition 1), it follows that it is an orthonormal basis for the Hilbert space $\mathcal{M}$. Put, as in (4) and (7),

$$\mathcal{R}_m = \bigvee_{n\in\mathbb{Z}} T^nD^m\psi(\cdot) \quad \text{and} \quad \mathcal{S}_n = \bigvee_{m\in\mathbb{N}_0} T^nD^m\psi(\cdot).$$

Theorem 1 in [6] exhibited the following decomposition of the bilateral shift D. *If $\psi(\cdot)$ is a wavelet in $\mathcal{L}^2(\mathbb{R})$, then $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a family of pairwise orthogonal subspaces of $\mathcal{L}^2(\mathbb{R})$ that spans $\mathcal{L}^2(\mathbb{R})$:*

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n.$$

Moreover, each $\mathcal{H}_n$ reduces D on $\mathcal{L}^2(\mathbb{R})$ so that

$$D = \bigoplus_{n\in\mathbb{Z}} D|_{\mathcal{H}_n},$$

with $D|_{\mathcal{H}_n}$ being a bilateral shift of multiplicity one acting on each subspace $\mathcal{H}_n$.

We establish next its counterpart, by exhibiting a similar decomposition of the bilateral shift T.

Theorem 1. *If $\psi(\cdot)$ is a wavelet in $\mathcal{L}^2(\mathbb{R})$, then $\{\mathcal{R}_m\}_{m\in\mathbb{N}_0}$ and $\{\mathcal{S}_n\}_{n\in\mathbb{Z}}$ are families of pairwise orthogonal subspaces of $\mathcal{M}$ and both span $\mathcal{M}$:*

$$\mathcal{M} = \bigoplus_{m\in\mathbb{N}_0} \mathcal{R}_m = \bigoplus_{n\in\mathbb{Z}} \mathcal{S}_n.$$

Moreover, $\mathcal{M}$ reduces the bilateral shift T on $\mathcal{L}^2(\mathbb{R})$, and so does each $\mathcal{R}_m$. Hence

$$T = \bigoplus_{m\in\mathbb{N}_0} T|_{\mathcal{R}_m} \oplus T|_{\mathcal{M}^\perp},$$

with each $T|_{\mathcal{R}_m}$ being a bilateral shift of multiplicity one acting on each subspace $\mathcal{R}_m$, and $T|_{\mathcal{M}^\perp}$ is a unitary operator acting on $\mathcal{M}^\perp$.

Proof. By unconditional convergence of the Fourier series we may write either

$$\mathcal{M} = \bigvee_{m\in\mathbb{N}_0} \bigvee_{n\in\mathbb{Z}} T^n D^m \psi(\cdot) = \bigvee_{m\in\mathbb{N}_0} \mathcal{R}_m \quad \text{or} \quad \mathcal{M} = \bigvee_{n\in\mathbb{Z}} \bigvee_{m\in\mathbb{N}_0} T^n D^m \psi(\cdot) = \bigvee_{n\in\mathbb{Z}} \mathcal{S}_n.$$

Since $\mathcal{R}_m \perp \mathcal{R}_{m'}$ for $m' \neq m$ ($m' \in \mathbb{N}_0$) and $\mathcal{S}_n \perp \mathcal{S}_{n'}$ for $n' \neq n$ ($n' \in \mathbb{Z}$) (according to Lemmas 1 and 2), we get

$$\mathcal{M} = \bigoplus_{m\in\mathbb{N}_0} \mathcal{R}_m = \bigoplus_{n\in\mathbb{Z}} \mathcal{S}_n.$$

Moreover, since $\mathcal{S}_n = T^n \mathcal{S}_0$ for every $n \in \mathbb{Z}$,

$$\mathcal{M} = \bigvee_{n\in\mathbb{Z}} \mathcal{S}_n = \bigvee_{n\in\mathbb{Z}} T^n \mathcal{S}_0$$

is invariant for both T and T^*. Indeed, T is unitary so that $T^* = T^{-1}$ and

$$T^{\pm 1}(\mathcal{M}) = T^{\pm 1} \bigvee_{n\in\mathbb{Z}} T^n \mathcal{S}_0 = \bigvee_{n\in\mathbb{Z}} T^{\pm 1} T^n \mathcal{S}_0 = \bigvee_{n\in\mathbb{Z}} T^n \mathcal{S}_0 = \mathcal{M}.$$

This means that $\mathcal{M}$ reduces T. Since direct summands of a unitary operator are again unitary operators, it follows that $T|_{\mathcal{M}}$ is unitary, as well as $T|_{\mathcal{M}^\perp}$, where $\mathcal{M}^\perp = \mathcal{L}^2(\mathbb{R}) \ominus \mathcal{M}$ is the orthogonal complement of $\mathcal{M}$. The same argument (T is unitary) shows that each

$$\mathcal{R}_m = \bigvee_{n\in\mathbb{Z}} T^n D^m \psi(\cdot)$$

also is T-reducing. Therefore, since

$$\mathcal{M} = \bigoplus_{m\in\mathbb{N}_0} \mathcal{R}_m,$$

it follows that

$$T|_{\mathcal{M}} = \bigoplus_{m\in\mathbb{N}_0} T|_{\mathcal{R}_m},$$

where each $T|_{\mathcal{R}_m}$ is unitary. Summing up: with respect to the decomposition

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{M} \oplus \mathcal{M}^\perp$$

the bilateral shift T on $\mathcal{L}^2(\mathbb{R})$ can be written as

$$T = T|_{\mathcal{M}} \oplus T|_{\mathcal{M}^\perp} = \bigoplus_{m\in\mathbb{N}_0} T|_{\mathcal{R}_m} \oplus\, T|_{\mathcal{M}^\perp}.$$

It remains to check whether or not these unitary direct summands of the bilateral shift T are bilateral shifts themselves. For each $(n,m) \in \mathbb{Z}\times\mathbb{N}_0$ put

$$r_{n,m}(\cdot) = T^n D^m \psi(\cdot)$$

so that $\{r_{n,m}(\cdot)\}_{(n,m)\in\mathbb{Z}\times\mathbb{N}_0}$ as in Definition 1 is an orthonormal basis for

$$\mathcal{M} \;=\; \bigvee_{(n,m)\in\mathbb{Z}\times\mathbb{N}_0} r_{n,m}(\cdot).$$

Similarly, for each $m\in\mathbb{N}_0$, $\{r_{n,m}(\cdot)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for

$$\mathcal{R}_m \;=\; \bigvee_{n\in\mathbb{Z}} r_{n,m}(\cdot).$$

Take an arbitrary $m\in\mathbb{Z}$ and observe that

$$T|_{\mathcal{R}_m} r_{n,m}(\cdot) = T^{n+1}D^m\psi(\cdot) = r_{n+1,m}(\cdot).$$

Hence each $T|_{\mathcal{R}_m}$ shifts the orthonormal basis $\{r_{n,m}(\cdot)\}_{n\in\mathbb{Z}}$ for each Hilbert space $\mathcal{R}_m$, and so each $T|_{\mathcal{R}_m}$ is a bilateral shift of multiplicity one acting on $\mathcal{R}_m$. $\square$

Remark 1. Observe that, according to (4) and (5),

$$\mathcal{M} \;=\; \bigvee_{m\in\mathbb{N}_0} \mathcal{R}_m \;=\; \bigvee_{m\in\mathbb{N}_0} \mathcal{W}_{m,(m)} \;=\; \bigvee_{m\in\mathbb{N}_0} D^m\mathcal{W}_{0,(m)}.$$

Thus $\mathcal{M}$ is not necessarily D invariant because $\mathcal{W}_{0,(m)}$ depends on m. However, since D is unitary, it follows from Lemma 1(i) that

$$\begin{aligned} D(\mathcal{M}) \;&=\; \bigvee_{m\in\mathbb{N}_0} D^{m+1}\mathcal{W}_{0,(m)} \;\subseteq\; D(\mathcal{W}_0)\cup\bigvee_{m\in\mathbb{N}} D^{m+1}\mathcal{W}_{0,(m)} \\ &=\; D(\mathcal{W}_0)\cup D^2\bigvee_{m\in\mathbb{N}} D^{m-1}\mathcal{W}_{0,(m)} \;\subset\; D(\mathcal{W}_0)\cup D^2\bigvee_{m\in\mathbb{N}} D^{m-1}\mathcal{W}_{0,(m-1)} \\ &\subseteq\; D(\mathcal{W}_0)\cup D^2\bigvee_{m\in\mathbb{N}_0} D^{m}\mathcal{W}_{0,(m)} \;=\; D(\mathcal{W}_0)\cup D^2(\mathcal{M}), \end{aligned}$$

and hence

$$\mathcal{M} \;\subset\; \mathcal{W}_0 \cup D(\mathcal{M}).$$

Even though the reversed wavelets do not span the space $\mathcal{L}^2(\mathbb{R})$, the subspace $\mathcal{M}$ spanned by them can be useful for studying details on $\mathcal{M}$ since the translation-by-1 operator T behaves very simply on the subspaces $\mathcal{R}_m$ of $\mathcal{M}$.

Remark 2. It is worth noticing that the results in this paper can be framed in an abstract approach by replacing the concrete function space $\mathcal{L}^2(\mathbb{R})$ with an arbitrary infinite-dimensional separable Hilbert space $\mathcal{H}$, and assuming that D and T are noncommuting bilateral shifts of infinite multiplicity acting on $\mathcal{H}$ and satisfying relation (2) or, equivalently, such that $DT^2 = TD$ — see [6, Proposition 3].

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