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# AN INTRODUCTION TO TIME-SHIFT EQUATIONS

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Abstract. Dilation Equations play an important role in Wavelet Multiresolution Approximation. A dilation equation is a difference equation for fixed scale summing over all time-shifts. In this note we introduce a class of equations called Time-Shift Equations. A time-shift equation is a difference equation for fixed time-shift summing over all scales. Analysis leading to derivation of these equations will occupy the rest of the note.

## 1. Introduction

This note is a brief introduction to a class of equations — called Time-Shift Equations (TiEq) — which is an analog of the class of Dilation Equations (DiEq).

Dilation Equations "live" in Wavelet Multiresolution Approximation (MRA), see for instance the paper of Strang [12] and the book of Keinert [4, Chapter 1]. The basic DiEq, also called "refinement equation", is a two-scale difference equation, of the form

$$\phi(\cdot) = \sum_{n\in\mathbb{Z}} c_n \phi\big(2(\cdot) - n\big) = \sum_{n\in\mathbb{Z}} c_n D T^n \phi(\cdot), \tag{1.1}$$

where $\phi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ called scaling function — generating a MRA $\mathcal{L}^2(\mathbb{R})$-subspaces $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ — is normalized so that [12]

$$\int_{-\infty}^{\infty} \phi(t)\,dt = 1 \quad \Longrightarrow \quad \sum_{n\in\mathbb{Z}} c_n = 2. \tag{1.2}$$

Here and in what follows $D$ and $T$ are, respectively, dyadic-scale and unit-time-shift $\mathcal{L}^2(\mathbb{R})$-operators defined by

$$Df(\cdot) = \sqrt{2} f\big(2(\cdot)\big) \quad \text{and} \quad Tf(\cdot) := f\big((\cdot) - 1\big). \tag{1.3}$$

The basic Time-Shift Equation to be derived is the difference equation of the form (for some $\varphi(\cdot) \in \mathcal{L}^2(\mathbb{R})$)

$$\varphi(\cdot) = \sum_{m\in\mathbb{Z}} \beta_m \sqrt{2}^{\,m} \varphi\big(2^m(\cdot) - 1\big) = \sum_{m\in\mathbb{Z}} \beta_m D^m T \varphi(\cdot). \tag{1.4}$$

We note that $m$ always stands for scale level and $2^m$ is scale while $n$ is time-shift. Thus the right hand side of equation (1.1) is for scale level $m = 1$ or for scale $2^1$, over all time-shifts, while that of equation (1.4) is at time-shift 1 over all scale levels.

Background and steps leading to TiEq will be presented in Section 2. The Connection between TiEq and DiEq is shown in Section 3.

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## 2. Why Time-Shift Equations?

Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a wavelet and let $\psi_{m,n}(\cdot)$, $(m,n) \in \mathbb{Z}^2$, be the wavelet functions generated from $\psi(\cdot)$ by time-shiftings followed by dyadic-scalings, that is,

$$\psi_{m,n}(\cdot) := \sqrt{2}^{\,m}\psi\big(2^m(\cdot)-n\big) = D^mT^n\psi(\cdot) = D^m\psi\big((\cdot)-n\big), \quad (m,n) \in \mathbb{Z}^2. \tag{2.1}$$

Then, by definition, $\{\psi_{m,n}(\cdot)\}$ is a double-indexed $\mathcal{L}^2(\mathbb{R})$-orthonormal basis (o.n.b.) [5, 9]. As a consequence the *"scale detail subspaces"* $\mathcal{W}_m(\psi)$ constructed from $\psi_{m,n}(\cdot)$ as

$$\mathcal{W}_m(\psi) := \bigvee_{n\in\mathbb{Z}} D^m\psi\big((\cdot)-n\big), \quad m \in \mathbb{Z}, \tag{2.2}$$

constitute an $\mathcal{L}^2(\mathbb{R})$-orthogonal subspaces basis (o.s.b.) [7]. Therefore the space $\mathcal{L}^2(\mathbb{R})$ admits an orthogonal scale detail subspaces decomposition

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m(\psi). \tag{2.3}$$

Moreover, since $D$ has a bounded inverse, we also have

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} D^m\mathcal{W}_0(\psi), \tag{2.4}$$

where

$$\mathcal{W}_0(\psi) := \bigvee_{n\in\mathbb{Z}} \psi\big((\cdot)-n\big), \tag{2.5}$$

is a generating — because of (2.4) — and $D$-wandering subspace — because

$$D^m\mathcal{W}_0(\psi) \perp D^{m'}\mathcal{W}_0(\psi), \quad \text{whenever} \quad m \neq m'. \tag{2.6}$$

What is interesting is the fact that (2.4) actually defines $D$ as a bilateral shift whose multiplicity is the dimension of its generating wandering subspace $\mathcal{W}_0(\psi)$ [3]. This fact will be used in Section 3.

Now, if a wavelet $\psi(\cdot)$ is "derived" from a scaling function $\phi(\cdot)$ — generating a MRA $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ — that is, the subspaces $\mathcal{V}_m$ — called scaling approximation subspaces — satisfy the following properties [9, 10, 11]:

(o) $\mathcal{V}_0 := \bigvee_{n\in\mathbb{Z}} \phi\big((\cdot)-n\big)$ and $\mathcal{V}_{m+1} = D\mathcal{V}_m$, $(m,n) \in \mathbb{Z}^2$,
(i) $\mathcal{V}_m \subset \mathcal{V}_{m+1}$, $m \in \mathbb{Z}$,
(ii) $\bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \{0\}$,
(iii) $\overline{\bigcup_{m\in\mathbb{Z}}} \mathcal{V}_m = \mathcal{L}^2(\mathbb{R})$.

Then we also have

$$\mathcal{V}_{m+1} = \mathcal{V}_m \oplus \mathcal{W}_m(\psi), \quad m \in \mathbb{Z}. \tag{2.7}$$

This establishes relationship between the scaling function $\phi(\cdot)$ and the associated wavelet $\psi(\cdot)$. Note that in property (o), the functions $\phi\big((\cdot)-n\big)$, $n \in \mathbb{Z}$, are taken to be orthonormal.

The place where DiEq were born is in the "nested" property (i).

It follows from property (o) that

$$\mathcal{V}_m := \bigvee_{n\in\mathbb{Z}} D^m\phi\big((\cdot)-n\big), \quad m \in \mathbb{Z}. \tag{2.8}$$

Moreover, each $\mathcal{V}_m$ can also be "represented" in terms of the scale detail subspaces $\mathcal{W}_{m'}(\psi),\ -\infty < m' \leq m-1$, as

$$\mathcal{V}_m = \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}(\psi), \quad m \in \mathbb{Z}. \tag{2.9}$$

This is a consequence of (2.7).

To distinguish between $\mathcal{V}_m$ in (2.8) and its representation in (2.9) we denote the right hand side of (2.9) by $\mathcal{V}_m(\psi)$

$$\mathcal{V}_m(\psi) := \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}(\psi), \quad m \in \mathbb{Z}, \tag{2.10}$$

to indicate that, since it depends on the wavelet functions $\psi_{m,n}(\cdot)$, it depends on the wavelet $\psi(\cdot)$ as well. Moreover, $\mathcal{V}_m(\psi),\ m \in \mathbb{Z}$, automatically satisfy the MRA properties (i), (ii) and (iii).

It is evident that dyadic-scaling plays a central role while time-shifting is somewhat neglected! In other words, Wavelets and MRA are basically "scale-based" theories.

An important fact which we feel has been "overlooked" is the fact that one can also construct *"time-shift detail subspaces"* $\mathcal{H}_n(\psi),\ n \in \mathbb{Z}$, from the wavelet functions $\psi_{m,n}(\cdot)$, $(m,n) \in \mathbb{Z}^2$, as [6],

$$\mathcal{H}_n(\psi) := \bigvee_{m\in\mathbb{Z}} D^m \psi\big((\cdot) - n\big), \quad n \in \mathbb{Z}. \tag{2.11}$$

Moreover, these $\mathcal{H}_n(\psi)$ are also orthogonal and form a second o.s.b. for $\mathcal{L}^2(\mathbb{R})$. Consequently, we now have, in addition to (2.3), the time-shift orthogonal decomposition

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n(\psi). \tag{2.12}$$

This suggests that one ought to explore "time-shift-based" approach to Wavelets and MRA. Preliminary results along these lines were reported in [7, 8].

To derive the proposed TiEq we begin with the time-shift detail subspaces $\mathcal{G}_n(\psi)$ defined by

$$\mathcal{G}_n(\psi) := \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n'}(\psi), \quad n \in \mathbb{Z}. \tag{2.13}$$

These are simply a time-shift analog of the scale subspaces $\mathcal{V}_m(\psi)$. Moreover, it is a simple matter to verify that $\{\mathcal{G}_n(\psi)\}$ also satisfies the MRA properties (i), (ii) and (iii).

The difference between $\{\mathcal{V}_m(\psi)\}_{m\in\mathbb{Z}}$ and $\{\mathcal{G}_n(\psi)\}_{n\in\mathbb{Z}}$ is that the former consists of scale detail subspaces and it represents the original MRA $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$, while the latter consists of time-shift detail subspaces, hence it cannot represent the scaling subspaces $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$. This is also due in part to the fact that $\mathcal{G}_n(\psi)$ is $D$-reducing while $\mathcal{V}_m$ is only $D^*$-invariant.

Suppose now that there is a function $\varphi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ such that

$$\varphi\big((\cdot) - n\big) \perp D^m \varphi\big((\cdot) - n\big), \quad \forall\, (m,n) \in \mathbb{Z}^2. \tag{2.14}$$

Let the subspaces $\mathcal{T}_n,\ n \in \mathbb{Z}$, be defined by

$$\mathcal{T}_n := \bigvee_m D^m \varphi\big((\cdot) - n\big), \quad n \in \mathbb{Z}, \tag{2.15}$$

and satisfy the following properties:

(i) $\mathcal{T}_n \subset \mathcal{T}_{n+1},\ n \in \mathbb{Z}$,
(ii) $\bigcap_{n\in\mathbb{Z}} \mathcal{T}_n = \{0\}$,
(iii) $\overline{\bigcup_{m\in\mathbb{Z}} \mathcal{T}_n} = \mathcal{L}^2(\mathbb{R})$.

Moreover,

$$\mathcal{T}_{n+1} = \mathcal{T}_n \oplus \mathcal{H}_n(\psi), \quad n \in \mathbb{Z}, \tag{2.16}$$

where $\mathcal{H}_n(\psi)$ is as previously defined. Then clearly $\mathcal{T}_n$ also admits the representation

$$\mathcal{T}_n = \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n'}(\psi) := \mathcal{G}_n(\psi), \quad n \in \mathbb{Z}. \tag{2.17}$$

We note that $\mathcal{T}_n$ as defined by (2.15) is $D$-reducing and so is $\mathcal{G}_n(\psi)$.

It follows from (2.15) and property (i) that

$$\mathcal{T}_0 := \bigvee_{m\in\mathbb{Z}} D^m \varphi(\cdot) \ \subset\ \mathcal{T}_1 := \bigvee_{m\in\mathbb{Z}} D^m \varphi\big((\cdot) - 1\big). \tag{2.18}$$

Therefore, since $\varphi(\cdot)$ also lives in $\mathcal{T}_1$ it can be represented in terms of the orthonormal functions $D^m\varphi\big((\cdot)-1\big),\ m \in \mathbb{Z}$ — spanning $\mathcal{T}_1$ — as

$$\varphi(\cdot) = \sum_{m\in\mathbb{Z}} \beta_m D^m \varphi\big((\cdot) - 1\big). \tag{2.19}$$

This is the basic Time-Shift Equation announced in (1.4). We also have, as in the case of the DiEq (1.1),

$$\int_{-\infty}^{\infty} \varphi(t)\, dt = 1 \quad \Longrightarrow \quad \sum_{m\in\mathbb{Z}} \frac{\beta_m}{\sqrt{2}^{\,m}} = 1. \tag{2.20}$$

In general we have the TiEq

$$\varphi\big((\cdot) - n\big) = \sum_{m\in\mathbb{Z}} \beta_{m,n}\, D^m \varphi\big((\cdot) - (n+1)\big), \quad n \in \mathbb{Z}. \tag{2.21}$$

## 3. A Connection Between TiEq and DiEq

We close by showing that a TiEq can be converted — up to a unitary operator — into a DiEq and vice versa.

First, given a wavelet $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$, then the space $\mathcal{L}^2(\mathbb{R})$ automatically admits the scale detail subspaces orthogonal decomposition (2.4)

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m(\psi) = \bigoplus_{m\in\mathbb{Z}} D^m \mathcal{W}_0(\psi) := \mathcal{L}^2(\mathbb{R}; D), \tag{3.1}$$

where $\mathcal{W}_0(\psi)$ defined by (2.5)

$$\mathcal{W}_0(\psi) := \bigvee_{n\in\mathbb{Z}} \psi\big((\cdot) - n\big),$$

is a $D$-wandering generating subspace, and $\mathcal{L}^2(\mathbb{R}; D)$ indicates that (3.1) is a $D$-wandering subspaces representation of $\mathcal{L}^2(\mathbb{R})$.

For the unit time-shift operator $T$, unfortunately, there is no decomposition in terms of the wavelet functions $\psi_{m,n}(\cdot)$ — similar to (3.1). However, since $T$ is also a bilateral shift of countably infinite multiplicity, the space $\mathcal{L}^2(\mathbb{R})$ can admit a $T$-wandering subspaces orthogonal representation of the form

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} T^n \mathcal{W}_{0,T} := \mathcal{L}^2(\mathbb{R}; T), \tag{3.2}$$

where $\mathcal{W}_{0,T}$ is a $T$-wandering generating subspace.

It is well known that bilateral shifts of equal multiplicity are unitarily equivalent [13, Chapter 1]. This for the case of the bilateral shifts $D$ and $T$ can be seen as follows.

Next, let $f(\cdot) \in \mathcal{L}^2(\mathbb{R}; D)$ so we have the dyadic-scale orthogonal representation

$$f(\cdot) = \sum_{m\in\mathbb{Z}} D^m w_m(\cdot), \quad w_m(\cdot) \in \mathcal{W}_0(\psi),\ m \in \mathbb{Z}, \text{ and } \sum_{m\in\mathbb{Z}} \|w_m(\cdot)\|^2 = \|f(\cdot)\|^2. \tag{3.3}$$

Let

$$\omega\colon\ \mathcal{W}_0(\psi) \to \mathcal{W}_{0,T}$$

be a unitary operator sending $\mathcal{W}_0(\psi)$ onto $\mathcal{W}_{0,T}$. Then the operator

$$\Omega\colon \mathcal{L}^2(\mathbb{R}; D) \to \mathcal{L}^2(\mathbb{R}, T),$$

defined by

$$\Omega\, f(\cdot) := \sum_{m\in\mathbb{Z}} T^m\, \omega\, w_m(\cdot), \tag{3.4}$$

is clearly unitary. Moreover, it is easy to see that

$$\Omega\, D = T\, \Omega, \tag{3.5}$$

that is, $D$ and $T$ are unitarily equivalent (both are bilateral shifts of the same cardinality).

We summarize the above in the next proposition.

**Proposition 1.** *Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a wavelet then the space $\mathcal{L}^2(\mathbb{R})$ admits a dyadic-scaling representation $\mathcal{L}^2(\mathbb{R}; D)$ defined by the orthogonal decomposition* (3.1)*, as well as a unit-time-shift representation $\mathcal{L}^2(\mathbb{R}; T)$ defined by* (3.3)*. The former is unique while the latter needs not be.*

Now, return to the basic TiEq (1.4)

$$\varphi(\cdot) = \sum_{m\in\mathbb{Z}} \beta_m D^m \varphi\big((\cdot) - 1\big) = \sum_{m\in\mathbb{Z}} \beta_m D^m T \varphi(\cdot), \tag{3.6}$$

which can be rewritten as

$$D^* \varphi(\cdot) = \sum_{m\in\mathbb{Z}} \beta_m D^{m-1} \varphi\big((\cdot) - 1\big). \tag{3.7}$$

Substituting for $D$ from (3.5) we obtain

$$D^* \varphi(\cdot) = \sum_{n\in\mathbb{Z}} \beta_n \Omega^* T^{n-1} \Omega\, \varphi\big((\cdot) - 1\big), \tag{3.8}$$

where, since the right hand side is summing with respect to time-shifts, we have replaced $m$ by $n$. Then, since $\Omega^*$ is also unitary, we can have

$$\Omega\, D^*\varphi(\cdot) = \sum_{n\in\mathbb{Z}} \beta_n T^{n-1}\Omega\,\varphi\big((\cdot)-1\big). \tag{3.9}$$

This is clearly a Dilation-Type Equation. To see this, let us rewrite the basic DiEq (1.1) as

$$D^*\phi(\cdot) = \sum_{n\in\mathbb{Z}} c_n T^{n-1}\phi\big((\cdot)-1\big), \tag{3.10}$$

which is simply equation (3.9) for $\Omega = I$, and in which $\varphi(\cdot)$ as well as $\beta_n$ are replaced by $\phi(\cdot)$ and $c_n$. We have therefore shown that a TiEq of the form (3.8) is connected to a DiEq of the form (3.9) via the "Dilation-Type Equation" (3.10). In exactly the same way, a DiEq of the form (3.10) is connected to a TiEq of the form (3.7) via the "Time-Shift-Type Equation"

$$\Omega^* D^*\phi(\cdot) = \sum_{n\in\mathbb{Z}} c_n D^{n-1}\Omega^*\phi\big((\cdot)-1\big). \tag{3.11}$$

This note is intended as a brief introduction to TiEq. Further work on these equations as well as on Time-Shift MRA will be reported elsewhere.

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